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# $M/M/1/1$ QUEUEING-INVENTORY SYSTEM WITH RETRIAL OF UNSATISFIED CUSTOMERS

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**ABSTRACT:** This paper considers an $M/M/1/1$ queueing-inventory system with retrial of unsatisfied customers. Arrivals taking place when server is busy, proceed to an orbit of infinite capacity. From the orbit, the head of the queue alone retires to access the server. Failed attempts to access an idle server with positive inventory results in the retrial customer returning to orbit. The inter-retrial times are independent identically distributed exponential random variables with parameter $\theta$, irrespective of the number of customers in the orbit. Primary customers, who encounter an idle server without stock at its arrival epoch, leave the system for ever. We compute the condition for stability and then employ algorithmic approach for the computation of the system steady-state probability. We derive the expected waiting time of a customer in the orbit; distribution of the time until the first customer goes to orbit and also the probability of no customer going to orbit in a given interval of time. An optimization problem is investigated numerically.

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## 1. INTRODUCTION

Sigman, K and Simchi-Levi [27] introduced the concept of positive service time in to inventory models with arbitrarily distributed service time, exponentially distributed lead time with customer arrival constituting a Poisson process. This was followed by Berman et al. [9] with deterministic service time wherein they formulated the model as a dynamic programming problem. Subsequently there was a spate of research papers on queueing-inventory models (see Saffari et al. ([22], [23]), Krishnamoorthy and Viswanath [15], Schwarz and Daduna [25], Schwarz et al. ([24], [26]), Baek and Moon ([8], [7])). A very recent contribution of interest to inventory models with positive service time involving a random environment is by Krenzler and Daduna [16] wherein the authors established decomposition of the system state in the long run. Another recent paper of Krenzler and Daduna [17] investigates queueing-inventory models in a random environment. They provide a counter example to show that the steady-state distribution of an $M/G/1/\infty$ queueing-inventory system under $(s, S)$ policy and lost sales, need not have a product form. Nevertheless, in general, loss systems in a random environment have a product form steady-state distribution. They also introduce a blocking set where all activities other than replenishment stay suspended whenever the Markov Chain is in that set. This resulted in arriving at a product form solution to the considered system in the steady-state.

Retrial of unsatisfied customers is extensively discussed in queueing literature (see Falin and Templeton [11], Artalejo and Gomez Corral [6]). However, in the context of inventory with retrial of unsatisfied customers, not much work is reported, especially those involving positive service time. The negligible service time case is discussed in Ushakumari [29] and Artalejo et al. [5]. Whereas the former provides analytical solution (for the case of constant retrial), the latter provides an algorithmic approach in a more general set up (linear retrial rate).

The first study on queueing-inventory models with unsatisfied customers is by Artalejo et al. [4]. Krishnamoorthy et al. [13] discuss a queueing-inventory system with server interruption and retrial of unsatisfied customers from the orbit with infinite size. Multiple vacation of the server with retrial of unsatisfied customers was considered by Sivakumar [28]. Padmavathi et al. [20]

extends the above work by incorporating a new feature, called *idle time* for server, in addition to vacation. At the time of stock depletion, the server idles for a random time before he goes for vacation, so that if the replenishment is received during that idle time, the server is immediately available. If the stock is replenished during vacation, he is available only at the end of vacation. A recent contribution of interest to queueing-inventory system involving registration of customers proceeding to orbit and orbital search is by Jianan Cui and Jinting Wang [10]. Anbazhagan et al. [2] investigate a continuous review base stock policy inventory system with retrial demands. Anoop and Jacob [3] discuss an inventory model with exponentially distributed service time and retrial of unsatisfied customers; wherein customer arrival is according to a Poisson process and customers are assume a multiserver queueing-inventory system with standard $(s, S)$ policy involving zero lead time. After each service completion, the customer as well the server leaves the system so that the total number of available servers is reduced by one each. As soon as the level of the inventory reaches $s$, the inventory is replenished to $S$. An arriving customer, who finds that all available servers are busy, joins an orbit of infinite size. Amirthakodi and Sivakumar [1] investigate an inventory model with feedback customers. Recently, Paul Manuel et al. [21] consider a retrial queueing-inventory model with a finite buffer and an orbit of infinite capacity; customer arrival follows Markovian arrival process ($MAP$), service time phase type distributed, inventory control policy is of the $(s, S)$ type; the stored items are subject to decay with life time exponentially distributed. They derive algorithmically several system performance measures. In the present paper we have derived a few additional characteristics of the system such as the expected waiting time of a customer in the orbit; distribution of the time until the first customer goes to orbit and also the probability of no customer going to orbit in a given interval of time. For a review of the developments on queueing-inventory models (classical and retrial cases) one may refer to Krishnamoorthy et al. [12].

In this paper we consider an $M/M/1/1$ queueing-inventory system with service time where, on arrival, if a customer encounters a busy server, proceeds to an orbit of infinite capacity. In the orbit a queue of customers is formed. The head of the queue alone retries to access the server with at least one item in the inventory, failing which it goes back to orbit and occupies the first position in the queue. The inter retrial time follows exponential distribution which is independent of the number of customers in the orbit, provided there is at least one. Arrival of external customers is according to a Poisson process, service time and lead time for replenishment of inventory are exponentially distributed.

This paper is organized as follows. Section 2 deals with the mathematical formulation of the problem. In Section 3 the condition for stability of the system is investigated, followed by the computation of the steady-state probability vector. Performance measures are provided in Section 4. In particular we compute the expected waiting time of a customer in the orbit, distribution of time until the first customer goes to orbit (during a cycle that is appropriately defined) and probability of no customer going to orbit in a given interval of time. Section 5 discusses an optimization problem. The notations used in the sequel are:

$\mathcal{N}(t)$ : Number of customers in the orbit at time $t$.

$\mathcal{I}(t)$ : Number of inventoried items at time $t$.

$\mathcal{C}(t)$ : Status of the server is idle/ busy at time $t$. That is,

$$\mathcal{C}(t) = \begin{cases} 0, \text{ if server is idle at time t.} \\ 1, \text{ if server is busy at time t.} \end{cases}$$

$I_k$ : identity matrix of order $k$.

$\mathbf{e}$ : $(1, 1, ..., 1)'$ a column vector of 1's of appropriate order.

CTMC : Continuous time Markov chain.

LIQBD: Level independent Quasi birth and death process.

PH distribution: Phase Type distribution.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

With arrival constituting a Poisson process of rate $\lambda$, service time independent identically distributed exponential random variables with parameter $\mu$, lead time for replenishment having exponential distribution with parameter $\beta$ and inter-retrial time of head of the queue in the orbit following exponential distribution with parameter $\theta$, the process $\{(\mathcal{N}(t), \mathcal{C}(t), \mathcal{I}(t)) | t \geq 0\}$ forms a

CTMC on the state space $\Omega$ given by

$$\left((\mathbb{Z}_+ \bigcup \{\mathbf{0}\}) \times \{0,1\} \times \{1,2,\ldots,S\}\right) \bigcup \left((\mathbb{Z}_+ \bigcup \{\mathbf{0}\}) \times \{\mathbf{0}\} \times \{\mathbf{0}\}\right).$$

It is to be noted that we make a strong assumption on customers getting into the system: when inventory level is zero, no customer joins the system. When the inventory level reaches a pre-specified value $s > 0$, a replenishment order is placed for $Q$ units with $Q > s$. We fix $S = Q + s$ as the maximum number of items that could be held in the system at any given time. That is the replenishment policy followed is $(s, Q)$. Further, as considered in Krishnamoorthy et.al [14] it is assumed that at the end of service a customer is provided one unit of the item from inventory with probability $\gamma$. We expected, "the assumption that no customer joins the system when inventory is zero" would enable us to arrive at, in the least, a closed form solution of the system state distribution, if not decomposition of the system. Nevertheless it turned out to be otherwise. Thus we are forced to adopt algorithmic approach for the analysis of the system described. The state space of the CTMC is partitioned in to levels $\mathcal{L}(i)$ defined as,

$$\begin{aligned}\mathcal{L}(i) = \{(0,0,j)\,|0 \le j \le S\} \cup \{(0,1,j)\,|1 \le j \le S\} \\ \cup \{(i,k,j)\,/i \ge 1; k = 0,1; 0 \le j \le S\}.\end{aligned}$$

The transitions in the Markov chain are listed below:
(a) Transitions due to arrival of customers :

$(i,0,j) \to (i,1,j)$ : the rate is $\lambda$, for $i \ge 0;\ 1 \le j \le S$.

$(i,1,j) \to (i+1,1,j)$ : the rate is $\lambda$, for $i \ge 0;\ 1 \le j \le S$.

(b) Transitions due to service completion of customers:

$(i,1,j) \to (i,0,j-1)$ : the rate is $\gamma\mu$, for $i \ge 0;\ 1 \le j \le S$.

$(i,1,j) \to (i,0,j)$ : the rate is $(1-\gamma)\,\mu$, for $i \ge 0;\ 1 \le j \le S$.

(c) Transitions due to replenishments:

$(i,0,j) \to (i,0,Q+j)$ : the rate is $\beta$, for $i \geq 0;\ 0 \leq j \leq s$.

$(i,1,j) \to (i,1,Q+j)$ : the rate is $\beta$, for $i \geq 0;\ 0 \leq j \leq s$.

(d) Transitions due to retrial of customers:

$(i,0,j) \to (i-1,1,j)$ : the rate is $\theta$, for $i \geq 1;\ 1 \leq j \leq S$.

All other transition pairs have rate zero. The infinitesimal generator $\boldsymbol{\mathcal{Q}}$ of this CTMC is given by

$$\boldsymbol{\mathcal{Q}}=\begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \dots \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $B_0$, $B_1$ and $B_2$ contain transition rates within $\mathcal{L}(0)$, transition from $\mathcal{L}(0)$ to $\mathcal{L}(1)$ and transition from $\mathcal{L}(1)$ to $\mathcal{L}(0)$ respectively; $A_0$ represents transitions from $\mathcal{L}(i)$ to $\mathcal{L}(i+1),\ i \geq 1$; $A_1$ represents transitions within $\mathcal{L}(i)$ for $i \geq 1$, and $A_2$ represents transitions from $\mathcal{L}(i)$ to $\mathcal{L}(i-1)$, $i \geq 2$. All these matrices are square matrices of order $2S+1$.

## 3. SYSTEM STABILITY AND COMPUTATION OF STEADY-STATE PROBABILITY VECTOR

The Markov chain under consideration is a LIQBD process. For this chain to be stable it is necessary and sufficient that

$$\boldsymbol{\xi} A_0 \mathbf{e} < \boldsymbol{\xi} A_2 \mathbf{e}. \tag{3.1}$$

where $\boldsymbol{\xi}$ is the unique non negative vector satisfying,

$$\boldsymbol{\xi} A = 0,\ \boldsymbol{\xi}\mathbf{e} = 1 \tag{3.2}$$

and $A = A_0 + A_1 + A_2$, is the infinitesimal generator of the finite state CTMC. Let $\boldsymbol{\xi} = (\xi_0(0), \xi_0(1), \dots, \xi_0(S), \xi_1(1), \xi_1(2), \dots \xi_1(S))$ be the steady-state vector of the generator matrix $A$. Then $\boldsymbol{\xi}A = 0$ gives the following equations

$$-\beta\xi_0(0) + \gamma\mu\xi_1(1) = 0 \tag{3.3}$$

$$-(\lambda+\theta+\beta)\xi_0(j) + (1-\gamma)\mu\xi_1(j) + \gamma\mu\xi_1(j+1) = 0,\ 1 \le j \le s \tag{3.4}$$

$$-(\lambda+\theta)\xi_0(j) + (1-\gamma)\mu\xi_1(j) + \gamma\mu\xi_1(j+1) = 0,\ s+1 \le j \le Q-1 \tag{3.5}$$

$$\beta\xi_0(j) - (\lambda+\theta)\xi_0(Q+j) + (1-\gamma)\mu\xi_1(Q+j) + \gamma\mu\xi_1(Q+j+1) = 0,\ 0 \le j \le s-1 \tag{3.6}$$

$$\beta\xi_0(s) - (\lambda+\theta)\xi_0(S) + (1-\gamma)\mu\xi_1(S) = 0 \tag{3.7}$$

$$(\lambda+\theta)\xi_0(j) - (\beta+\mu)\xi_1(j) = 0,\ 1 \le j \le s \tag{3.8}$$

$$(\lambda+\theta)\xi_0(j) - \mu\xi_1(j) = 0,\ s+1 \le j \le Q \tag{3.9}$$

$$\beta\xi_1(j) + (\lambda+\theta)\xi_0(Q+j) - \mu\xi_1(Q+j) = 0,\ 1 \le j \le s \tag{3.10}$$

The LIQBD process with infinitesimal generator $\boldsymbol{\mathcal{Q}}$ is stable if and only if $\boldsymbol{\xi}A_0\mathbf{e} < \boldsymbol{\xi}A_2\mathbf{e}$. That is,

$$\theta\left(\xi_0(1) + \xi_1(2) + \cdots + \xi_0(S)\right) > \lambda\left(\xi_1(1) + \xi_1(2) + \cdots + \xi_1(S)\right),$$

which simplifies to $\frac{\lambda}{\mu} < \frac{\theta}{\lambda+\theta}$. Thus we have the following lemma for the stability of the system:

**Lemma 3.1.** *The CTMC $\Omega$ is stable if and only if $\lambda < \frac{\mu\theta}{\lambda+\theta}$.*

Now we compute the steady-state probability vector of $\boldsymbol{\mathcal{Q}}$ under the stability condition. Let $\boldsymbol{x}$ denote the steady-state probability vector of the infinitesimal generator $\boldsymbol{\mathcal{Q}}$. Then the steady-state probability vector must satisfy the relations,

$$\boldsymbol{x}\boldsymbol{\mathcal{Q}} = 0,\ \boldsymbol{x}\mathbf{e} = 1. \tag{3.11}$$

Let us partition $\boldsymbol{x}$ by levels as

$$\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots), \tag{3.12}$$

where the sub-vectors of $\boldsymbol{x}$ are further partitioned as,

$$\boldsymbol{x_i} = \left(x_i(0,0), x_i(0,1), x_i(0,2), ..., x_i(0,S), x_i(1,1), x_i(1,2), ..., x_i(1,S)\right),\quad i \ge 0. \tag{3.13}$$

Since the state space $\Omega$ is a LIQBD process, its steady-state vector is given by

$$\boldsymbol{x_i} = \boldsymbol{x_1} R^{i-1}, \ i \geq 2. \tag{3.14}$$

(see Neuts [19]), where $R$ is the minimal non-negative solution to the matrix quadratic equation $R^2 + RA_1 + A_0 = 0$. For finding the boundary vectors $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$, we have from $\boldsymbol{x}\boldsymbol{Q}$=0,

$$\begin{aligned} \boldsymbol{x_0} B_1 + \boldsymbol{x_1} A_1 + \boldsymbol{x_2} A_2 = 0 \\ \Longleftrightarrow \boldsymbol{x_0} B_1 + \boldsymbol{x_1} \left(A_1 + RA_2\right) = 0 \\ \Longleftrightarrow \boldsymbol{x_1} = -\boldsymbol{x_0} B_1 \left(A_1 + RA_2\right)^{-1} \\ \Longleftrightarrow \boldsymbol{x_1} = \boldsymbol{x_0} D, \text{ where } D = -B_1 \left(A_1 + RA_2\right)^{-1}. \end{aligned}$$

Further,

$$\begin{aligned} \boldsymbol{x_0} B_0 + \boldsymbol{x_1} B_2 = 0 \\ \Longleftrightarrow \boldsymbol{x_0} \left(B_0 + DB_2\right) = 0. \end{aligned}$$

First we take $\boldsymbol{x_0}$ as the steady-state vector of the generator matrix $B_0 + DB_2$. Then $\boldsymbol{x_i}$, for $i \geq 1$, can be found using the formula $\boldsymbol{x_1} = \boldsymbol{x_0} D$ and $\boldsymbol{x_i} = \boldsymbol{x_1} R^{i-1}$, for $i \geq 2$. Finally, the steady-state probability distribution of the system under study is obtained by dividing each $\boldsymbol{x_i}$ with normalizing condition

$$\boldsymbol{x_0}\mathbf{e} + \left(\boldsymbol{x_1} + \boldsymbol{x_2} + \dots\right)\mathbf{e} = \boldsymbol{x_0}\left(I + D\left(I - R\right)^{-1}\right)\mathbf{e}.$$

Since $R$ cannot be computed explicitly we explore the possibility of algorithmic computation. Thus, one can use logarithmic reduction algorithm as given in Latouche and Ramaswami [18] for computing $R$. The main steps involved in the logarithmic reduction algorithm for computing $R$ are as follows:

**Step 0:** $H \leftarrow (-A_1)^{-1} A_0$, $L \leftarrow (-A_1)^{-1} A_2$, $G = L$, and $T = H$.

**Step 1:**

$$\begin{gathered} U = HL + LH \\ M = H^2 \\ H \leftarrow (I - U)^{-1} M \\ M \leftarrow L^2 \\ L \leftarrow (I - U)^{-1} M \\ G \leftarrow G + TL \\ T \leftarrow TH \end{gathered}$$

Continue Step 1 until $||\mathbf{e} - G\mathbf{e}||_\infty < \epsilon$.

**Step 2:** $R = -A_0(A_1 + A_0G)^{-1}$.

Thus we arrive at the following theorem:

**Theorem 3.2.** *When the stability condition holds, the steady state probability vector* $\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots)$ *is given by* $\boldsymbol{x_1} = \boldsymbol{x_0}D$ *and* $\boldsymbol{x_i} = \boldsymbol{x_1}R^{i-1}$, *for* $i \geq 2$, *where* $D = -B_1\left(A_1 + RA_2\right)^{-1}$, $R$ *is the minimal non-negative solution to the matrix quadratic equation* $R^2 + RA_1 + A_0 = 0$ *and the boundary vectors* $\boldsymbol{x_0}$, $\boldsymbol{x_1}$ *are obtained by solving the following relations:*

$$\boldsymbol{x_0}B_1 + \boldsymbol{x_1}\left(A_1 + RA_2\right) = 0,$$

*and*

$$\boldsymbol{x_0}\left(B_0 + DB_2\right) = 0,$$

*subject to the normalizing condition*

$$\boldsymbol{x_0}\left(I + D\left(I - R\right)^{-1}\right)\boldsymbol{e} = 1.$$

Now we compute a few characteristics of the system such as the expected waiting time of a customer in the orbit, distribution of the time until the first customer goes to orbit and the probability of no customer going to orbit in a given interval of time. These results are given in the following theorems:

**Theorem 3.3.** *Expected waiting time of an orbital customer is* $E(\mathcal{W}_L) = \sum_{r=1}^{\infty} \boldsymbol{x_r}\mathcal{W}^r$, *where* $\boldsymbol{x_r}$ *is a* $1 \times 2S$ *dimensional row vector defined by*

$$\boldsymbol{x_r} = \left(x_r(0,1), x_r(0,2), ..., x_r(0,S), x_r(1,1), x_r(1,2), ..., x_r(1,S)\right),$$

*and* $\mathcal{W}^r$ *denotes the waiting time distribution of the tagged customer who joins as the* $r^{th}$ *customer in the orbit is given by*

$$\mathcal{W}^r = \hat{I}_{2S}(-\boldsymbol{\mathcal{G}_1}^{\boldsymbol{-1}})\boldsymbol{e},$$

*where* $\hat{I}_{2S} = [0 \quad I_{2S}]_{(2S)\times\{(r-1)(2S+1)+S\}}$.

**Proof.** For computing the expected waiting time in the orbit of a tagged customer who joins as $r^{th}$ customer in the orbit, we consider the CTMC,

$$\boldsymbol{\Psi_1} = \left\{(\widehat{\mathcal{N}(t)}, \mathcal{C}(t), \mathcal{I}(t)) | t \geq 0\right\},$$

where $\widehat{\mathcal{N}(t)}$ denotes the rank, which is the position of the tagged customer in the orbit at the time he joins the system. The state space of the CTMC $\mathbf{\Psi_1}$ is given by

$$\Im_1 = \{(i,0,m), 1 \le i \le r-1; 0 \le m \le S\} \bigcup \{(i,1,m), 1 \le i \le r; \\ 1 \le m \le S\} \bigcup \{\Delta_1\},$$

where $\{\Delta_1\}$ is an absorbing state which corresponds to the tagged customer being taken for service. The infinitesimal generator of the chain $\mathbf{\Psi_1}$ is given by

$$\boldsymbol{\mathcal{H}_1} = \begin{bmatrix} \boldsymbol{\mathcal{G}_1} & \boldsymbol{\mathcal{G}_1}^{\mathbf{0}} \\ \mathbf{0} & 0 \end{bmatrix},$$

where $\boldsymbol{\mathcal{G}_1}^{\mathbf{0}}$ is an $\{(r-1)(2S+1)+S\} \times 1$ matrix such that $\boldsymbol{\mathcal{G}_1}^{\mathbf{0}}(i,1) = \theta$, for $1 \le i \le S$ and

$$\boldsymbol{\mathcal{G}_1} = \begin{bmatrix} B & 0 & 0 & \dots & \dots & 0 \\ \tilde{A}_2 & B & 0 & \dots & \dots & 0 \\ 0 & \tilde{A}_2 & B & \dots & \dots & 0 \\ & & \ddots & \ddots & & \\ & & & & \tilde{\tilde{A}}_2 & \tilde{B} \end{bmatrix},$$

where

$$B = \begin{bmatrix} B_1 & 0 & B_2 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_6 & 0 & 0 & 0 \\ B_8 & 0 & 0 & B_9 & 0 & B_{10} \\ B_3 & B_{11} & 0 & 0 & B_{12} & 0 \\ 0 & B_5 & B_{13} & 0 & 0 & B_{14} \end{bmatrix},$$

with

$$B_1 = \begin{bmatrix} -\beta & 0 \\ 0 & -(\beta+\theta)I_s \end{bmatrix}, \quad B_2 = \beta I_{s+1}, \quad B_3 = \begin{bmatrix} 0 & \gamma\mu \\ 0 & 0 \end{bmatrix}_{(S-2s-1)\times(s+1)},$$

$$B_4 = -\theta I_{S-2s-1}, \quad B_5 = \begin{bmatrix} 0 & \gamma\mu \\ 0 & 0 \end{bmatrix}_{(s+1)\times(S-2s)}, \quad B_6 = -\theta I_{s+1},$$

$$B_8 = \begin{bmatrix} \gamma\mu & (1-\gamma)\mu & & & & \\ & \gamma\mu & (1-\gamma)\mu & & & \\ & & \ddots & \ddots & & \\ & & & \gamma\mu & (1-\gamma)\mu & \end{bmatrix}_{s\times(s+1)}, \quad B_9 = -\beta I_s,$$

$$B_{11} = \begin{bmatrix} (1-\gamma)\mu & & & & \\ \gamma\mu & (1-\gamma)\mu & & & \\ & \ddots & \ddots & & \\ & & \gamma\mu & (1-\gamma)\mu & \end{bmatrix}_{(S-2s-1)\times(S-2s-1)},$$

$$B_{13} = \begin{bmatrix} (1-\gamma)\mu & & & & \\ \gamma\mu & (1-\gamma)\mu & & & \\ & \ddots & \ddots & & \\ & & \gamma\mu & (1-\gamma)\mu & \end{bmatrix}_{(s+1)\times(s+1)},$$

$$B_{10} = \begin{bmatrix} 0 & \beta I_s \end{bmatrix}_{s\times(s+1)}, \quad B_{12} = -\mu I_{S-2s-1}, \quad B_{14} = -\mu I_{s+1},$$

$$\tilde{B}(i,j) = B(i+1, j+1) \text{ for } 1 \le i,\ j \le 2S;$$

$$\tilde{\tilde{A}}(i,j) = \tilde{\tilde{A}}_2(i+1, j) \text{ for } 1 \le i \le 2S,\ 1 \le j \le 2S+1;$$

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0 & F_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with

$$F_1 = \begin{bmatrix} 0 \\ \theta I_s \end{bmatrix}_{(s+1)\times s},$$

$F_2 = \theta I_{Q-s}$ and $F_3 = \theta I_{(s+1)\times(s+1)}$.

Now the waiting time distribution $\mathcal{W}^r$ of the tagged customer who joins as the $r^{th}$ customer in the orbit, is the time until absorption in the CTMC $\mathbf{\Psi_1}$ , and given by the column vector

$$\mathcal{W}^r = \hat{I}_{2S}(-\boldsymbol{\mathcal{G}_1}^{\mathbf{-1}})\mathbf{e},$$

where $\hat{I}_{2S} = [0 \;\; I_{2S}]_{(2S)\times\{(r-1)(2S+1)+S\}}$. Hence, the expected waiting time of a general customer is given by $E(\mathcal{W}_L) = \sum_{r=1}^{\infty} \boldsymbol{x_r}\mathcal{W}^r$, where $\boldsymbol{x_r}$ is a $1 \times 2S$ dimensional row vector defined by

$$\boldsymbol{x_r} = (x_r(0,1), x_r(0,2), ..., x_r(0,S), x_r(1,1), x_r(1,2), ..., x_r(1,S)) . \qquad \square$$

In a similar manner, we can find the second moment of the waiting time of an orbital customer, which is given as the following corollary:

**Corollary 3.4.** *The second moment of the waiting time distribution of an orbital customer is*

$$E(\mathcal{W}_L^2) = \sum_{r=1}^{\infty} \boldsymbol{x_r}\mathcal{W}_2^r,$$

*where* $\mathcal{W}_2^r = 2\hat{I}_{2S}(\boldsymbol{\mathcal{G}_1}^{\boldsymbol{-2}})\boldsymbol{e}$ *(see Neuts [19]).*

**Theorem 3.5.** *The distribution of the time until the first customer goes to orbit in a cycle is* $E(\chi) = -\boldsymbol{\alpha}\left(\boldsymbol{\mathcal{G}_2^{-1}}\right)\boldsymbol{e}$.

**Proof.** We now compute the distribution of the time till the first customer arriving in a cycle goes to orbit. By a cycle we shall mean that starting with no customer in orbit, until the next epoch when all customers in the orbit are served out. We also assume that at the beginning of a cycle the inventory level is $S$ and there is no customer in service. A customer arrives and straight enters for service. During this service time, if another customer arrives, then he is the first to go to orbit.

We consider the CTMC $\boldsymbol{\Psi_2} = \{(\mathcal{C}(t), \mathcal{I}(t))| t \geq 0\}$, where $\mathcal{C}(t)$ and $\mathcal{I}(t)$ are as defined at the beginning of this paper. The state space of this CTMC $\boldsymbol{\Psi_2}$ is

$$\Im_2 = \{\mathbf{0}\} \bigcup \{(\ell, m)| \; \ell = 0, 1; \; 1 \leq m \leq S\} \bigcup \{\Delta_2\},$$

where $\{\Delta_2\}$ is the absorbing state which represents the state "first customer to go to orbit" from the state $\{(1, m)| \; 1 \leq m \leq S\}$. Clearly, $\Im_2$ is a finite state space Markov chain. The possible transitions and the corresponding rates are given in Table 1.

Thus the infinitesimal generator $\boldsymbol{\mathcal{H}_2}$ of the Markov chain $\boldsymbol{\Psi_2}$ is of the form $\boldsymbol{\mathcal{H}_2} = \begin{bmatrix} \boldsymbol{\mathcal{G}_2} & \boldsymbol{\mathcal{G}_2}^{\mathbf{0}} \\ \mathbf{0} & 0 \end{bmatrix}$ with initial probability vector $\boldsymbol{\alpha} = (0, 0, \ldots, 1, 0)$ where 1 is in the $S^{th}$ position; $\boldsymbol{\mathcal{G}_2}$ is of order $2S+1$; $\boldsymbol{\mathcal{G}_2}^{\mathbf{0}}$ is a $2S+1$ component

Table 1: The transitions in the CTMC $\mathbf{\Psi_2}$ and corresponding rates

| Form | To | Rate | |
|---|---|---|---|
| $(0,0)$ | $(0,0)$ | $-\beta$ | |
| $(0,m)$ | $(0,m)$ | $-(\beta+\lambda)$ | $m=1,2,\ldots,s.$ |
| $(0,m)$ | $(0,m)$ | $-\lambda$ | $m=s+1,s+2,\ldots,S.$ |
| $(1,m)$ | $(1,m)$ | $-(\lambda+\mu+\beta)$ | $m=1,2,\ldots,s.$ |
| $(1,m)$ | $(1,m)$ | $-(\lambda+\mu)$ | $m=s+1,s+2,\ldots,S.$ |
| $(1,m)$ | $(0,m-1)$ | $\mu$ | $m=1,2,\ldots,S.$ |
| $(\ell,m)$ | $(\ell,m+Q)$ | $\beta$ | $\ell=0,1;\ m=0,1,\ldots,s.$ |
| $(0,m)$ | $(1,m)$ | $\lambda$ | $m=1,2,\ldots,S.$ |
| $(1,m)$ | $\{\Delta_2\}$ | $\lambda$ | $m=1,2,\ldots,S.$ |

column vector such that $\boldsymbol{\mathcal{G}_2}\mathbf{e}+\boldsymbol{\mathcal{G}_2}^{\mathbf{0}}=0$. Let $\chi$ represent the random variable "time until the first customer goes to orbit in a cycle". This time duration follows PH distribution with representation $(\boldsymbol{\alpha},\boldsymbol{\mathcal{G}_2})$. Therefore the expected time until the first customer goes to the orbit is

$$E(\chi)=-\boldsymbol{\alpha}\left(\boldsymbol{\mathcal{G}_2^{-1}}\right)\mathbf{e}.$$

$\square$

**Theorem 3.6.** *Probability that all customer arrivals (demands) in a time duration of length $t$ do not go to the orbit $P_t$, is*

$$P_t=\sum_{n=1}^{\infty}\sum_{n_1=1}^{n}\sum_{i=1}^{n_1}\left(\sum_{j=1}^{Q+s}x_0(0,j)\left(1-e^{-\mu(x_i-x_{i-1})}\right)\right)^{n_1}\begin{pmatrix}n\\n_1\end{pmatrix}\mathcal{J}\left(x_0(0,0)\right)^{n-n_1},$$

*where* $\mathcal{J}=\left(\frac{n!}{t^n}\int\limits_0^{x_1}\cdots\int\limits_{x_{n-2}}^{x_{n-1}}\int\limits_{x_{n-1}}^{x_n}dx_n\cdots dx_1\right).$

**Proof.** Consider an interval of duration $t$ in the steady-state regime. The objective is to compute the probability that no customer arriving during this time period goes to orbit. This means that all customer arrivals in this interval either meet an idle server with positive inventory or during the stock out period. Thus customers do not join the system when the inventory level is zero (by model assumption). Assume that there is no customer in the orbit at the beginning of this interval. Assume if a total of $n$ arrivals take place during this interval, of which $n_1$ arrivals find positive inventory with server idle. The remaining $n - n_1$ arrivals are such that upon their arrival the server is found to be idle with no item in the inventory. Then the required probability $P_t$ is given by

$$P_t = \sum_{n=1}^{\infty} \sum_{n_1=1}^{n} \sum_{i=1}^{n_1} \left( \sum_{j=1}^{Q+s} x_0(0,j) \left(1 - e^{-\mu(x_i - x_{i-1})}\right) \right)^{n_1} \begin{pmatrix} n \\ n_1 \end{pmatrix} \mathcal{J} \left(x_0(0,0)\right)^{n-n_1},$$

where $\mathcal{J} = \left( \frac{n!}{t^n} \int\limits_0^{x_1} \cdots \int\limits_{x_{n-2}}^{x_{n-1}} \int\limits_{x_{n-1}}^{x_n} dx_n \cdots dx_1 \right)$. $\square$

## 4. PERFORMANCE MEASURES

- Mean number of customers in the orbit,
  $L_O = \left( \sum\limits_{i=1}^{\infty} \sum\limits_{j=0}^{Q+s} i x_i(0,j) + \sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{Q+s} i x_i(1,j) \right)$.
- Mean inventory level, $E_{inv} = \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{Q+s} j x_i(0,j) + \sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{Q+s} j x_i(1,j)$.
- Depletion rate of inventory, $D_{inv} = \gamma \mu_2 \left( \sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{Q+s} x_i(1,j) \right)$.
- Mean number of replenishments per unit time,
  $R_r = \beta \left( \sum\limits_{j=0}^{s} \left( \sum\limits_{i=0}^{\infty} x_i(0,j) + \sum\limits_{i=1}^{\infty} x_i(1,j) \right) \right)$.
- Expected loss rate of customers, $E_{loss} = \lambda \left( \sum\limits_{i=1}^{\infty} x_i(0,0) \right)$.
- Probability that the server is busy, $P_{busy} = \sum\limits_{i=0}^{\infty} \sum\limits_{j=1}^{Q+s} x_i(1,j)$.
- Successful rate of retrials, $E_{retrial} = \theta \left( \sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{Q+s} x_i(0,j) \right)$.

- Mean number of departures per unit time,
$D_m = \mu \left( \sum_{i=0}^{\infty} \sum_{j=1}^{Q+s} x_i(1,j) \right)$.

- Mean number of customers waiting in the orbit when inventory is available, $\widetilde{W_O} = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{Q+s} i x_i(0,j) + \sum_{i=1}^{\infty} \sum_{j=1}^{Q+s} i x_i(1,j) \right)$.

- Mean number of customers waiting in the orbit during the stock out period, $\widetilde{\widetilde{W_O}} = \left( \sum_{i=1}^{\infty} i x_i(0,0) \right)$.

## 5. OPTIMIZATION PROBLEM

In this section we provide the optimal values of the inventory level $s$ and the fixed order quantity $Q$ of the model. For checking the optimality of $s$ and $Q$, the following cost function is constructed. Define $\mathcal{F}(s,Q)$ as the expected total cost per unit time in the long run. Then

$$\mathcal{F}(s,Q) = h.E_{inv} + c_1.E_{loss} + c_2.(1 - P_{busy}) + (K + Q.c_3).R_r,$$

where $K$ is the fixed cost for placing an order, $c_1$ is the cost incurred due to loss per customer, $c_2$ is the waiting cost per unit time per customer during the stock out period, $c_3$ is the variable procurement cost per item and $h$ is the unit holding cost of inventory for one unit of time. Table 2 provides the optimal pair $(s,Q)$ and the corresponding minimum cost (in Dollars) by using MATLAB program. Here $\gamma$ is varied from 0.1 to 1, at an interval of 0.1. The values for the input parameters are given as follows $\lambda = 2,\ \mu = 5,\ \theta = 4,\ \beta = 3,\ K = \$500,\ c_1 = \$25, c_2 = \$50, c_3 = \$35, h = \$3.5$. We provide a numerical comparison based on a few performance measures in Table 3.

For numerical comparison we assign the same input values as for Table 2 with $s = 10$ and $S = 31$. For example we observe from Table 3 that the mean number of replenishments and loss rate of customer is larger for $\gamma = 1$ compared to that for $\gamma\ (= 0.5)$. Further $P_{busy}$ and $E_{inv}$ are higher for $\gamma = 0.5$ compared to that for $\gamma = 1$. These are all on expected lines.

Table 2: Optimal $(s, Q)$ pair and minimum cost

| $\gamma$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| Optimal $(s, Q)$ pair & minimum cost | (1,29) 242.353 | (1,29) 241.978 | (1,29) 241.585 | (1,29) 241.181 | (1,29) 240.767 |
| $\gamma$ | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| Optimal $(s, Q)$ pair & minimum cost | (1,29) 240.347 | (1,29) 239.922 | (1,29) 239.494 | (1,29) 239.062 | (1,29) 238.629 |

Table 3: Effect of $\gamma$ on various performance measures

| Performance measures | with $\gamma = 0.5$ | with $\gamma = 1$ (classical queueing-inventory system) |
|---|---|---|
| $P_{busy}$ | 0.39999911 | 0.39999864 |
| $E_{inv}$ | 20.6666431 | 20.3333149 |
| $D_{inv}$ | 0.14285707 | 0.14285702 |
| $R_r$ | 0.03213748 | 0.06427498 |
| $L_O$ | 1.09998846 | 1.09998834 |
| $E_{loss}$ | 0.00000070 | 0.00000286 |

## CONCLUSION

In this paper we discussed an $M/M/1/1$ queueing-inventory system with retrial of unsatisfied customers. Arrivals taking place when server busy, proceed to an orbit of infinite capacity. From the orbit, the head of the queue alone retires to access the server. Failed attempts to access an idle server with positive inventory results in the retrial customer returning to orbit. The inter-retrial times are independent identically distributed exponential random variables, irrespective the number of customers in the orbit. We computed the condition for stability and then employed algorithmic approach to obtain the system steady-state probability. The expected waiting time of a customer in the orbit, distribution of the time until the first customer goes to orbit and the probability of no customer going to orbit in a given interval of time were computed. An optimization problem is also numerically investigated.

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