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A DIFFERENCE EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT: A recent Avery et al. fixed point theorem is applied to show the existence of a positive solution of the second order difference equation

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, ..., N\},$$

with boundary conditions

$$u(0) = u(N+2) = 0.$$

An example is also given.

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1. INTRODUCTION

In this paper, we give an application of an Avery *et al.* fixed point theorem to obtain at least one positive solution of the difference equation

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, ..., N\}, \tag{1.1}$$

with boundary conditions

$$u(0) = u(N+2) = 0, \tag{1.2}$$

where $f : [0, \infty) \to [0, \infty)$ and Δ^2 is the second forward difference operator which acts on u by $\Delta^2 u(k) = u(k+2) - 2u(k+1) + u(k)$. We show that under certain conditions imposed on f, (1.1), (1.2) has a positive symmetric solution in the sense that $u(N+2-k) = u(k)$ for all $k \in \{0, 1, ..., N\}$.

Recently, Avery *et al.* have been developing extensions of the Leggett-Williams fixed point theorem [15] to allow for more flexibility in the conditions required for the existence of a fixed point of an operator. For example, in [11], an extension was given that does not require either of the functional boundaries to be invariant with respect to the functional wedge. The fixed point theorem that will be applied here, which was proven in [4], only requires subsets of both boundaries to be mapped inward and outward. Also, conditions involving the norm in the Leggett-Williams fixed point theorem are replaced by conditions on a convex functional. For more Avery type fixed point theorems, see [3, 4, 5, 6, 8, 11, 13].

These Avery type fixed point theorems have been applied to continuous boundary value problems [1, 2, 9, 10, 12, 19], difference equations [7, 14, 18] and dynamic boundary value problems on time scales [16, 17]. In these papers, an operator is defined whose kernel is the Green's function corresponding to the respective boundary value problem. The fixed points of this operator are solutions to the boundary value problem. A cone is defined using the properties of the Green's function and under certain hypotheses, it is shown that the operator satisfies the fixed point theorem. When working with Dirichlet boundary conditions, the functions in the cone are defined to be nonnegative, nondecreasing, concave and symmetric about the midpoint of the interval, implying the maximum value of functions in the cone occur at the midpoint. This approach is taken here.

2. THE FIXED POINT THEOREM

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

(i) $u \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda u \in \mathcal{P}$;

(ii) $u \in \mathcal{P}$, $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\alpha : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\alpha (tu + (1-t) v) \geq t\alpha (u) + (1-t) \alpha (v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0,1]$.

Similarly, the map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\beta : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\beta (tu + (1-t) v) \leq t\beta (u) + (1-t) \beta (v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0,1]$.

Definition 2.3. Let ψ and δ be nonnegative continuous functionals on a cone $\mathcal{P}$; then, for nonnegative real numbers a, and b we define the sets

$$\mathcal{P} (\psi, b) := \{x \in \mathcal{P} : \psi (x) \leq b\},$$

and

$$\mathcal{P} (\psi, \delta, a, b) := \{x \in \mathcal{P} : a \leq \psi (x) \text{ and } \delta (x) \leq b\}.$$

Theorem 2.4 (Anderson, Avery, Henderson [4])**.** *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α is a nonnegative continuous concave functional on $\mathcal{P}$, β is a nonnegative continuous convex functional on $\mathcal{P}$, and $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. Assume there exist nonnegative numbers a, b, c, and d such that:*

(A1) $\{x \in \mathcal{P} : a < \alpha (x) \text{ and } \beta (x) < b\} \neq \emptyset$;

(A2) if $x \in \mathcal{P}$ with $\beta (x) = b$ and $\alpha (x) \geq a$, then $\beta (Tx) < b$;

(A3) if $x \in \mathcal{P}$ with $\beta (x) = b$ and $\alpha (Tx) < a$, then $\beta (Tx) < b$;

(A4) $\{x \in \mathcal{P} : c < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset$;

(A5) if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx) > c$;

(A6) if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(Tx) > d$, then $\alpha(Tx) > c$.

If

(H1) $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x) \text{ and } \alpha(x) < c\} \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, and $\mathcal{P}(\alpha, c)$ is bounded,

then T *has a fixed point* x^* *in* $\mathcal{P}(\beta, \alpha, b, c)$.
If

(H2) $c < a$, $d < b$, $\{x \in \mathcal{P} : a < \alpha(x) \text{ and } \beta(x) < d\} \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$, and $\mathcal{P}(\beta, d)$ is bounded,

then T *has a fixed point* x^{**} *in* $\mathcal{P}(\alpha, \beta, a, d)$.

3. PRELIMINARIES

Define the Banach space E to be

$$E = \{u : \{0, \ldots, N+2\} \to \mathbb{R}\}$$

with the norm

$$\|u\| = \max_{k \in \{0,1,\ldots,N+2\}} |u(k)|.$$

The Green's function for $-\Delta^2 u = 0$ satisfying the boundary conditions (1.2) is given by

$$H(k, l) = \frac{1}{N+2} \begin{cases} k(N+2-l), & k \in \{0, \ldots, l\}, \\ \\ l(N+2-k), & k \in \{l+1, \ldots N+2\}. \end{cases}$$

Thus if u is a fixed point of the operator $T : E \to E$ defined by

$$Tu(k) := \sum_{l=1}^{N+1} H(k, l) f(u(l)),$$

then u is a solution of the boundary value problem (1.1), (1.2). Notice the Green's function has the property that for any $l \in \{0, \ldots, N+2\}$,

$$wH(y,l) \geq yH(w,l) \tag{3.1}$$

for all $w, y \in \{0, \ldots, N+2\}$ with $w \geq y$. For notational purposes, define

$$\underline{N} = \left\lfloor \frac{N+2}{2} \right\rfloor$$

to be the greatest integer less than or equal to $\frac{N+2}{2}$ and

$$\overline{N} = \left\lceil \frac{N+2}{2} \right\rceil$$

to be the smallest integer greater than or equal to $\frac{N+2}{2}$. Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} := \{u \in E \ : \ u(N+2-k) = u(k), u \text{ is nonnegative and nondecreasing on } \{0,1,\ldots,\underline{N}\}, \text{ and } wu(y) \geq yu(w) \text{ for } w \geq y \text{ with } y, w \in \{0,1,\ldots,\underline{N}\}\}.$$

Due to the fact that $H(N+2-k, N+2-l) = H(k,l)$ and (3.1), $T : \mathcal{P} \to \mathcal{P}$.

For $u \in \mathcal{P}$ and $\tau \in \{1, \ldots, \underline{N}\}$, define the concave functional α on $\mathcal{P}$ by

$$\alpha(u) := \min_{k \in \{\tau, \ldots, \underline{N}\}} u(k) = u(\tau),$$

and the convex functional β on $\mathcal{P}$ by

$$\beta(u) := \max_{k \in \{0, \ldots, \underline{N}\}} u(k) = u(\underline{N}).$$

4. POSITIVE SOLUTIONS OF (1.1), (1.2)

Theorem 4.1. *If $\tau \in \{1, ..., \underline{N}\}$ is fixed, b and c are positive real numbers with $3b < c$, and $f : [0, \infty) \to [0, \infty)$ is a continuous function such that:*

(i) $f(w) > \frac{c(N+2)}{\tau(N+1-\tau)(\underline{N}-\tau)}$ *for* $w \in \left[c, \frac{c\underline{N}}{\tau}\right]$,

(ii) $f(w)$ *is decreasing for* $w \in \left[\frac{b}{\underline{N}}, \frac{b\tau}{\underline{N}}\right]$ *with* $f\left(\frac{b\tau}{\underline{N}}\right) \geq f(w)$ *for* $w \in \left[\frac{b\tau}{\underline{N}}, b\right]$,

(iii) and

$$2\sum_{l=1}^{\tau}\frac{l\left(\overline{N}\right)}{N+2}f\left(\frac{bl}{\underline{N}}\right)\leq b-f\left(\frac{b\tau}{\underline{N}}\right)\frac{1}{N+2}\left(\overline{N}\right)\left(\underline{N}-\tau\right)\left(\tau+1+\underline{N}\right),$$

then T *has a fixed point* u^* *in* $\mathcal{P}(\beta,\alpha,b,c)$. *Thus the discrete conjugate problem* (1.1), (1.2) *has at least one positive solution* $u^*\in\mathcal{P}(\beta,\alpha,b,c)$.

Proof. We show the operator T satisfies Theorem 2.4. As noted before, $T:\mathcal{P}\to\mathcal{P}$. A standard application of the Arzelà-Ascoli theorem shows T is completely continuous.

First, let $a=\frac{b\tau}{\underline{N}}$ and $d=\frac{c\underline{N}}{\tau}$. Then, we have $a=\frac{b\tau}{\underline{N}}<\frac{c\tau}{3\underline{N}}<c$ and $b<\frac{c}{3}=\frac{d\tau}{3\underline{N}}<d$.

We verify (A1) and (A4). Choose $L\in\left(\frac{2b}{(N+2-\tau)\underline{N}},\frac{2b}{(N+2-\underline{N})\underline{N}}\right)$ and define the function u_L by

$$u_L(k):=\sum_{l=1}^{N+1}LH(k,l)=\frac{Lk}{2}(N+2-k).$$

Since

$$\alpha(u_L)=u_L(\tau)=\frac{L\tau}{2}(N+2-\tau)>\frac{b\tau}{\underline{N}}=a,$$

and

$$\beta(u_L)=u_L(\underline{N})=\frac{L\underline{N}}{2}(N+2-\underline{N})<b,$$

$u_L\in\{u\in\mathcal{P}:a<\alpha(u)\text{ and }\beta(u)<b\}$.

Similarly, choose $J\in\left(\frac{2c}{\tau(N+2-\tau)},\frac{2c}{\tau(N+2-\underline{N})}\right)$ and define the function u_J by

$$u_J(k):=\sum_{l=1}^{N+1}JH(k,l)=\frac{Jk}{2}(N+2-k).$$

Since

$$\alpha(u_J)=u_J(\tau)=\frac{J\tau}{2}(N+2-\tau)>c,$$

and

$$\beta(u_J)=u_J(\underline{N})=\frac{J\underline{N}}{2}(N+2-\underline{N})<\frac{c\underline{N}}{\tau}=d,$$

$u_J\in\{u\in\mathcal{P}:c<\alpha(u)\text{ and }\beta(u)<d\}$. Hence we have

$$\{u\in\mathcal{P}:a<\alpha(u)\text{ and }\beta(u)<b\}\neq\emptyset$$

and

$$\{u \in \mathcal{P} : c < \alpha(u) \text{ and } \beta(u) < d\} \neq \emptyset.$$

Therefore conditions (A1) and (A4) hold.

Turning to (A2), let $u \in \mathcal{P}$ with $\beta(u) = b$ and $\alpha(u) \geq a$. Since $u \in \mathcal{P}$, for $l \in \{0, ..., \tau\}$,

$$u(l) \geq \left(\frac{u(\tau)}{\tau}\right) l \geq \frac{bl}{\underline{N}}.$$

and for $l \in \{\tau, ..., \underline{N}\}$,

$$\frac{b\tau}{\underline{N}} \leq u(l) \leq b.$$

Hence by (ii) and (iii), it follows that

$$\begin{aligned}
\beta(Tv) &= \sum_{l=1}^{N+1} H(\underline{N}, l) f(u(l)) \\
&\leq 2\sum_{l=1}^{N} \frac{l(\overline{N})}{N+2} f(u(l)) \\
&= 2\sum_{l=1}^{\tau} \frac{l(\overline{N})}{N+2} f(u(l)) + 2\sum_{l=\tau+1}^{N} \frac{l(\overline{N})}{N+2} f(u(l)) \\
&\leq 2\sum_{l=1}^{\tau} \frac{l(\overline{N})}{N+2} f\left(\frac{bl}{\underline{N}}\right) + 2\sum_{l=\tau+1}^{N} \frac{l(\overline{N})}{N+2} f\left(\frac{b\tau}{\underline{N}}\right) \\
&\leq b - f\left(\frac{b\tau}{\underline{N}}\right) \frac{1}{N+2} (\overline{N}) (\underline{N} - \tau)(\tau + 1 + \underline{N}) \\
&\quad + f\left(\frac{b\tau}{\underline{N}}\right) \frac{1}{N+2} (\overline{N}) (\underline{N} - \tau)(\tau + 1 + \underline{N}) \\
&= b.
\end{aligned}$$

So (A2) is satisfied.

Next, we establish (A3), and so we let $u \in \mathcal{P}$ with $\beta(u) = b$ and $\alpha(Tu) < a$. Now, by (3.1),

$$\begin{aligned}
\beta(Tu) &= \sum_{l=1}^{N+1} H(\underline{N}, l) f(u(l)) \\
&\leq \frac{\underline{N}}{\tau} \sum_{l=1}^{N+1} H(\tau, l) f(u(l))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\underline{N}}{\tau}\alpha(Tu) \\
&< \frac{a\underline{N}}{\tau} = b,
\end{aligned}
$$

so (A3) holds.

In dealing with (A5), let $u \in \mathcal{P}$ with $\alpha(u) = c$ and $\beta(u) \leq d$. Then for $l \in \{\tau, ..., N+2\}$, we have

$$c \leq u(l) \leq d = \frac{c\underline{N}}{\tau}.$$

Hence by Property (i),

$$
\begin{aligned}
\alpha(Tu) &= \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \\
&\geq \sum_{l=\tau+1}^{N+1} H(\tau, l) f(u(l)) = \\
&= \sum_{l=\tau+1}^{N+1} \frac{\tau(\underline{N}-\tau)}{N+2} f(u(l)) \\
&> \sum_{l=\tau+1}^{N+1} \frac{c}{N+1-\tau} \\
&= c,
\end{aligned}
$$

and so (A5) is valid.

Now we address (A6). So, let $u \in \mathcal{P}$ with $\alpha(u) = c$ and $\beta(Tu) > d$. Again, by (3.1),

$$
\begin{aligned}
\alpha(Tu) &= \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \geq \\
&\geq \frac{\tau}{\underline{N}} \sum_{l=1}^{N+1} H(\underline{N}, l) f(u(l)) \\
&= \frac{\tau}{\underline{N}}\beta(Tu) \\
&> \frac{\tau d}{\underline{N}} \\
&= c
\end{aligned}
$$

and so (A6) of Theorem 2.4 also holds.

Last, we show (H1) holds. Let $K \in \left(\frac{2b}{\underline{N}(N+2-\underline{N})}, \frac{2c}{3\underline{N}(N+2-\underline{N})}\right)$. Then define

$$u_K(k) = K \sum_{l=1}^{N+1} H(k,l) = \frac{Kk}{2}(N+2-k).$$

So

$$\beta(u_K) = \frac{K\underline{N}}{2}(N+2-\underline{N}) > b$$

and

$$\begin{aligned}\alpha(u_K) &= \frac{K\tau}{2}(N+2-\tau)\\ &< \frac{c\tau}{3(N+2-\underline{N})\underline{N}}(N+2-\tau)\\ &\le c.\end{aligned}$$

Thus $\{u \in \mathcal{P} : b < \beta(u) \text{ and } \alpha(u) < c\} \neq \emptyset$.

If $u \in \mathcal{P}(\beta, b)$, then

$$\alpha(u) \le \beta(u) \le b < c,$$

and hence $u \in \mathcal{P}(\alpha, c)$. Thus $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$.

Lastly, if $u \in \mathcal{P}(\alpha, c)$, then

$$\frac{\tau}{\underline{N}}\beta(u) \le \alpha(u) \le c,$$

and so

$$\|u\| = \beta(u) \le \frac{c\underline{N}}{\tau}.$$

Therefore $\mathcal{P}(\alpha, c)$ is bounded. So (H1) holds. Thus T satisfies the conditions of Theorem 2.4 and has a fixed point $u^* \in \mathcal{P}(\beta, \alpha, b, c)$. □

5. AN EXAMPLE

Example 5.1. Let $N = 10$, $\tau = 2$, $b = 2$, and $c = 7$. Notice that $3b < c$. Define a continuous function $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} \frac{1-w}{6} & 0 \le w < 1 \\ 0 & 1 \le w < 2 \\ w-2 & 2 \le w. \end{cases}$$

Then,

(i) $f(w) \geq 5 > \frac{7 \cdot 12}{2 \cdot 9 \cdot 4} = 7/6$ for $w \in [7, 21]$,

(ii) $f(w)$ is decreasing on $\left[\frac{1}{3}, \frac{2}{3}\right]$, and $f\left(\frac{2}{3}\right) \geq f(w)$ for $w \in \left[\frac{2}{3}, 2\right]$, and

(iii) $\sum_{l=1}^{2} l f\left(\frac{1}{3}l\right) = \frac{2}{9} \leq 1 = 2 - f\left(\frac{2}{3}\right) \cdot \frac{1}{2} \cdot 4 \cdot 9$.

Therefore by Theorem 4.1, the conjugate boundary value problem,

$$\Delta^2 u(k) + f(u(k)), \ k \in \{0, 1, ..., 10\},$$

with boundary conditions

$$u(0) = u(12) = 0,$$

has a positive solution u^* with $u^*(6) \geq 2$ and $u^*(2) \leq 7$. The novelty of this result is not just the existence of a solution, but the existence of a symmetric solution with the given location data.

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