Communications in Applied Analysis, **21**, No. 2 (2017), 249-268 ISSN: 1083-2564

ON L-FRACTIONAL DERIVATIVES AND L-FRACTIONAL HOMOGENEOUS EQUATIONS

A.K. LAZOPOULOS[1] AND D. KARAOULANIS[2]

[1]Mathematical Sciences Department
Hellenic Army Academy
Vari, GR16673, GREECE
[2]Strength of Materials Laboratory
School of Applied Sciences
National Technical University of Athens
Athens, GREECE

ABSTRACT: Many different fractional derivatives exist that serve different aspects of fractional calculus. Nevertheless, they have failed to correspond to a reliable fractional differential. Lazopoulos [22] has introduced the L-fractional derivative having meaningful geometrical configuration. Moreover, the L-fractional derivative has a meaningful fractional differential as well. Hence, Fractional Differential Geometry could be established. The solutions of the linear homogenous L-fractional differential equation with constant coefficients will be studied in the present work. The solutions will be obtained with the help of power series expansions. Numerical solutions are presented for various fractional differentiation orders. Finally, comparison is presented between the cases of fractional differential homogeneous equations with Riemann-Liouville derivatives and Leibnitz derivatives. The similarities and differences are spotted. Those differences support that the solutions of the equations with L-fractional derivatives lie between the conventional (where $\alpha = 1$) and the corresponding ones with Riemann-Liouville derivatives.

AMS Subject Classification: 26A33, 34A08, 34K37

Key Words: L-fractional derivatives, fractional differential geometry, fractional differential equations

Received: December 19, 2016 ; **Accepted:** January 28, 2017 ;
Published: March 26, 2017. **doi:** 10.12732/caa.v21i2.6
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

Fractional Calculus has lately been an indispensable tool for studying modern problems in physics, engineering and finance. Leibniz [1], Liouville [2] and Riemann [3] originated Fractional Calculus, but its importance has lately been recognized in working with possibly non-differentiable functions. Various material models based upon Fractional time derivatives have been presented, for describing their viscoelastic interaction [4,5]. Fractional derivative models account for long-range (non-local) dependence of phenomena and result in better description of their behavior. Lazopoulos [6] has proposed a model for lifting Noll's axiom of local-action. Applications in various physical areas may be also found in various books [9,10,11,12]. Carpinteri et al. [8] have also proposed a fractional approach to non-local mechanics. Fractional Geometry of Manifolds [13, 14, 15, 25] with applications in fields of mechanics, quantum mechanics, relativity, finance, probabilities etc. has recently been proposed. Nevertheless, researchers are raising doubts about the existence of Fractional Differential Geometry and their argument is not easily rejected.

Adda [7, 17] has proposed the fractional differential in the form:

$$d^a f = g(x)(dx)^a. \tag{1}$$

However, that form of the differential is not valid for negative increments dx. That is the reason many researchers reject the existence of fractional differential geometry along with its applications in physics, engineering, finance and probabilities, chaos and control.

Nevertheless, the variable x accepts its own fractional differential:

$$d^\alpha x = \sigma(x)(dx)^\alpha, \tag{2}$$

with $\sigma(x) \neq 1$, differently of the conventional case when $a = 1$, where $\sigma(x)$ is always one. Relating both equations, it appears that:

$$d^a f = \frac{g(x)}{\sigma(x)} d^a x. \tag{3}$$

In this case $d^\alpha x$ is always a real quantity accepting positive or negative incremental real values alike. On these bases the development of Fractional Differential Geometry may be established.

Furthermore, fractal functions exhibiting self-similarity are non-differentiable functions but they exhibit fractional differentiability of order $0 < \alpha < 1$ (see [25, 27, 28, 29, 30]).

Fractional Calculus in mechanics has been suggested by many researchers, Tarasov [13,16], Drapaca & Sivaloganathan [20], Lazopoulos [15], in problems of hydrodynamics [13, 21] and many other applications Yang [32]. Recently Fractional, rigid body dynamics, in holonomic and non-holonomic systems [25, 18, 31] and viscoelastic responses [4,5]. Continuum mechanics with microstructure demands non-local constitutive relations. Hence, Fractional Continuum Mechanics has been applied to various problems in [23, 26], just to express the nonlocality of the constitutive laws, due to various cracks, inhomogeneities etc. Recent applications in Quantum Mechanics, Physics and relativity demand differential geometry revisited by Fractional Calculus.

In the present work, the fractional differential, established in Lazopoulos and Lazopoulos [15] will be recalled along with the introduced L-fractional derivatives for revisiting the solutions oh the L-fractional homogeneous equations. It is again pointed out here that L-derivative is the only fractional derivative with physical meaning. Implementing the theory, graphs of numerical solutions are also included. Finally, the solutions of the L-Fractional derivatives are compared to the ones corresponding to the same fractional differential equations expressed in Riemann-Liouville fractional derivatives.

2. BASIC PROPERTIES OF FRACTIONAL CALCULUS

Fractional Calculus has become lately a branch of pure mathematics with many applications in Physics and Engineering, Tarasov [13,16]. The various types of the fractional derivatives exhibit some advantages over the others, but they exhibit non-local characteristics, contrary to the conventional ones.

The detailed properties of fractional derivatives may be found in Samko et al. [11], Podlubny [10], Kilbas et al. [9]. Starting from Cauchy formula for the n-fold integral of a primitive function $f(x)$ we define the left and right fractional integral of f as:

$$ {}_aI_x^a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-a}} ds, \tag{4}$$

$$ {}_xI_b^a f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-a}} ds, \tag{4}$$

where α is the order of fractional integrals with $0 < a \leq 1$, considering $\Gamma(x) =$*(x-1)!* with $\Gamma(\alpha)$ Euler's Gamma function.

Thus the left and right Riemann-Liouville (R-L) derivatives are defined by:

$$ {}_aD_x^a f(x) = \frac{d}{dx} ({}_aI_x^{1-a}\ f(x)), \tag{6}$$

and

$$ {}_xD_b^a f(x) = -\frac{d}{dx} ({}_bI_x^{1-a}\ f(x)). \tag{7}$$

Nevertheless the fact that the R-L derivatives of a constant c are not zero, imposed the need for Caputo's derivative that is more friendly in the description of physical systems, although it is more restrictive. In fact Caputo's derivative is defined by:

$$ {}_a^cD_x^a f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(s)}{(x-s)^a} ds, \tag{8}$$

and

$$ {}_x^cD_b^a f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{f'(s)}{(s-x)^a} ds. \tag{9}$$

Evaluating Caputo's derivatives for functions of the type $f(x) = (x-a)^n$ or $f(x) = (b-x)^n$, we get:

$$ {}_a^cD_x^a (x-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (x-a)^{\nu-\alpha}, \tag{10}$$

and for the corresponding right Caputo's derivative:

$$ {}_{x}^{c}D_{b}^{a}(b-x)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(b-x)^{\nu-\alpha}. \tag{11} $$

Likewise, Caputo's derivatives are zero for constant functions:

$$ {}_{x}^{c}D_{b}^{a}c = 0. \tag{12} $$

3. THE GEOMETRY OF FRACTIONAL DIFFERENTIAL, THE L-FRACTIONAL DERIVATIVE

The a-Fractional differential of a function f(x) is defined by, [17]:

$$ d^{a}f(x) = {}_{0}^{c}D_{x}^{\alpha}(dx)^{a}. \tag{13} $$

The problem here is that the fractional differential defined by equation (13) is valid for positive incrementals *dx,* whereas for negative ones that differential might be complex. Thus considering for the moment that the increment dx is positive, and recalling that ${}_{0}D_{x}^{a}x \neq 1$, the a-fractional differential of the variable x is:

$$ d^{a}x = {}_{0}^{c}D_{x}^{a}x(dx)^{a}. \tag{14} $$

Hence

$$ d^{a}f(x) = \frac{{}_{0}^{c}D_{x}^{a}f(x)}{{}_{0}^{c}D_{x}^{a}x}d^{a}x. \tag{15} $$

Therefore $d^{a}f(x)$ is a non-linear function of dx, although it is a linear function of $d^{a}x$. That fact makes the consideration of the virtual fractional tangent space that we propose possible. Now the definition of fractional differential, equation (15), is imposed either for positive or negative variable differentials *dx.* In addition the L-fractional derivative ${}_{0}^{L}D_{x}^{a}f(x)$ is defined by,

$$ d^{a}f(x) = {}_{0}^{L}D_{x}^{a}f(x)d^{a}x, \tag{16} $$

with the L-fractional derivative,

$$ {}_{0}^{L}D_{x}^{a}f(x) = \frac{{}_{0}^{c}D_{x}^{a}f(x)}{{}_{0}^{c}D_{x}^{a}x}. \tag{17} $$

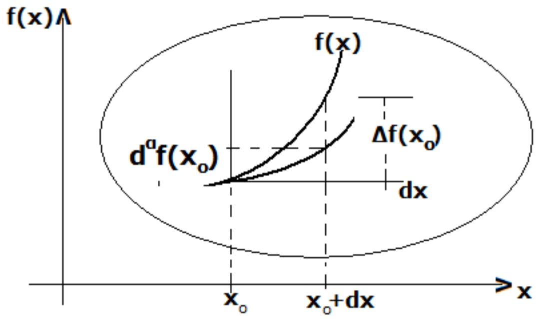

Figure 1: The non-linear differential of $f(x)$

In addition, recalling equation (4),

$$f(x) - f(a) = {}_a^L I_x^a \left({}_a^L D_x^\alpha f(x) \right)$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(2-a)} \int_a^x \frac{(s-a)^{1-\alpha}}{(x-s)^{1-a}} {}_a^L D_x^\alpha f(s) ds. \quad (18)$$

In fact the function y=f(x) has been drawn in Figure 1, with the corresponding first differential space at a point x according to Adda's definition, equation (13).

It is well known that the differential space is not tangent to the function at x_0, but intersects the figure y=f(x) at least at one point x_0.Furthermore, considering a point x_0+dx in the neighborhood of x_0, we have

$$\Delta f = f(x_0 + \Delta x) - f(x_0). \quad (19)$$

Now magnifying the neighborhood of x_0 we may define a space (d^αx, Δf), see Figure 2.

According to equation (16), the first differential $d^a f(x)$ is a linear function of $d^a x$ and tangent space of Δf at the point x_0. This space, we introduce, is the tangent space, that is updated at any point x. Likewise, we may consider in addition to the Fractional tangent space, the normal at any point.

The innovation of this article is the introduction of L-Derivative in Homogeneous Fractional Differential Equations. As it will be seen later, this derivative gives new meaning to these equations.

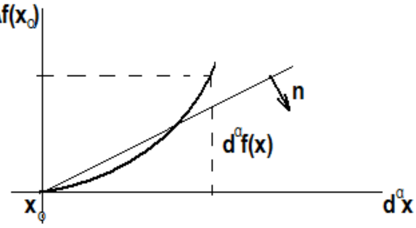

Figure 2: The virtual tangent space of the $f(x)$

4. SOLUTION OF LINEAR HOMOGENEOUS FRACTIONAL DIFFERENTIAL EQUATION WITH MITTAG-LEFFLER TYPE FUNCTIONS

Let us consider the linear homogenous fractional differential equation:

$$b_n \cdot ({}_aD_x^a)^n y(x) + b_{n-1} \cdot ({}_aD_x^a)^{n-1} y(x) + b_{n-2} \cdot ({}_aD_x^a)^{n-2} y(x) + ... \\ + b_1 \cdot ({}_aD_x^a) y(x) + b_0 \cdot y(x) = 0, \quad (20)$$

where $({}_aD_x^a)^m$ (m=1,2,3,...) is the m-fold Riemann-Liouville operator as defined in equation (6), and b_i (i=0,1,2,...) are constants.

According to Sabatier et.al [34], the solution of the above homogeneous equation has the form of a Mittag-Leffler type function, the α-exponential type function:

$$y(x) = e_\alpha^{\lambda(x-a)} = (x-a)^{a-1} \sum_{k=0}^{\infty} \frac{\lambda^k (x-a)^{k\alpha}}{\Gamma[(k+1)\alpha]} \\ (x > a, \quad \lambda \in \mathrm{C}, \quad \alpha \in \mathrm{R}^+, \quad a \in \mathrm{R}). \quad (21)$$

The α-exponential type function has the interesting property:

$${}_0D_x^a e_a^{\lambda(x-a)} = \lambda e_a^{\lambda(x-a)}. \quad (22)$$

Substituting $e_a^{\lambda(x-a)}$ in (20) we get:

$$b_n \cdot \lambda^n e_a^{\lambda(x-a)} + b_{n-1} \cdot \lambda^{n-1} e_a^{\lambda(x-a)} + b_{n-2} \cdot \lambda^{n-2} e_a^{\lambda(x-a)} \\ + ... + b_1 \cdot \lambda e_a^{\lambda(x-a)} + b_0 \cdot e_a^{\lambda(x-a)} = 0. \quad (23)$$

From where the following characteristic equation occurs:

$$b_n \cdot \lambda^n + b_{n-1} \cdot \lambda^{n-1} + b_{n-2} \cdot \lambda^{n-2} + ... + b_1 \cdot \lambda + b_0 = 0. \tag{24}$$

The roots of equation (26) compose the solution of the homogeneous equation:

$$c_n \cdot e_a^{\lambda_n (x-a)} + c_{n-1} \cdot e_a^{\lambda_{n-1}(x-a)} + c_{n-2} \cdot e_a^{\lambda_{n-2}(x-a)} + ... + c_1 \cdot e_a^{\lambda_1 (x-a)} = 0, \tag{25}$$

where $c_1, c_2, c_3, ..., c_{\mathrm{n}}$ are determined by the boundary conditions.

5. SOLUTION OF LINEAR HOMOGENOUS L-FRACTIONAL DIFFERENTIAL EQUATION

Let us consider the linear homogenous fractional differential equation:

$$\begin{aligned} b_n \cdot ({}_0^L D_x^a)^n y(x) + b_{n-1} \cdot ({}_0^L D_x^a)^{n-1} y(x) + b_{n-2} \cdot ({}_0^L D_x^a)^{n-2} y(x) + ... \\ + b_1 \cdot ({}_0^L D_x^a) y(x) + b_0 \cdot y(x) = 0, \end{aligned} \tag{26}$$

with initial value conditions: $({}_0^L D_x^a)^m y(x)|_{x=0} = c_m,\ m = 0, 1, 2, ..., n-1$, where $({}_0^L D_x^a)^m$ (m=1,2,3,...) is the m-fold L-fractional operator as defined in equation (17), and $\mathrm{b_i}$ (i=0,1,2,...) are constants.

We assume a solution for the equation (26) in a power series form: $y(x) = \sum\limits_{k=0}^{\infty} y_k x^k$.The application of L-fractional differential operator $({}_0^L D_x^a)$on y(x) repeatedly once and twice results in:

$$({}_0^L D_x^a) y(x) = \sum_{k=1}^{\infty} y_k \frac{\Gamma(2-\alpha) \cdot \Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha)} x^{\kappa-1} \tag{2}$$

$$= \sum_{k=0}^{\infty} y_{k+1} \frac{\Gamma(2-\alpha) \cdot \Gamma(\kappa+2)}{\Gamma(\kappa+2-\alpha)} x^{\kappa}, \tag{27}$$

$$({}_0^L D_x^a)^2 y(x) = \sum_{k=0}^{\infty} y_{k+2} \frac{(\Gamma(2-\alpha))^2 \cdot \Gamma(\kappa+3) \cdot \Gamma(\kappa+2)}{\Gamma(\kappa+3-\alpha) \cdot \Gamma(\kappa+2-\alpha)} x^{\kappa}. \tag{28}$$

Performing this operation n time, we can inductively produce the following result:

$$({}_0^L D_x^a)^n y(x) = \sum_{k=0}^{\infty} y_{k+n} \cdot [(\Gamma(2-\alpha))^n \prod_{l=0}^{n-1} (\frac{\Gamma(k+n+1-l)}{\Gamma(k+n+1-\alpha-l)})] \cdot x^{\kappa}. \tag{29}$$

By defining:

$$P[n,k] = [(\Gamma(2-\alpha))^n \prod_{l=0}^{n-1} (\frac{\Gamma(k+n+1-l)}{\Gamma(k+n+1-\alpha-l)})],\ \forall n \geq 1,$$

with $P[0,k] = 1$ $(k = 0,1,2,\dots)$, equation (29) can be written as:

$$({}_0^L D_x^a)^n y(x) = \sum_{k=0}^{\infty} y_{k+n} \cdot P[n,k] \cdot x^\kappa,\ \forall n \geq 0. \tag{30}$$

The substitution of the latter result in equation (26), gives:

$$\sum_{k=0}^{\infty} b_n \cdot y_{k+n} \cdot P[n,k] \cdot x^\kappa + \sum_{k=0}^{\infty} b_{n-1} \cdot y_{k+n-1} \cdot P[n-1,k] \cdot x^\kappa + ... \\ + \sum_{k=0}^{\infty} b_1 y_{k+1} \cdot P[1,k] \cdot x^\kappa + \sum_{k=0}^{\infty} b_0 \cdot y_k \cdot P[0,k] \cdot x^\kappa = 0. \tag{31}$$

From equation (31) we end up in the following recurrence formula for the coefficients $\mathrm{y_k}$ of the power series:

$$b_n \cdot y_{k+n} \cdot P[n,k] + b_{n-1} \cdot y_{k+n-1} \cdot P[n-1,k] + b_{n-2} \cdot y_{k+n-2} \cdot P[n-2,k] + ... \\ + b_1 \cdot y_{k+1} \cdot P[1,k] + b_0 \cdot y_k = 0, \tag{32}$$

for every k$\geq$0, where the first n-1,coefficients $\mathrm{y_k}$ of the power series are given by the relation:

$$y_m = \frac{c_m}{P[m,0]}, \quad m = 0,1,2,...,n-1. \tag{33}$$

6. NUMERICAL RESULTS AND REMARKS FOR THE L-FRACTIONAL LINEAR HOMOGENOUS DIFFERENTIAL EQUATION

We examine the behavior of the power series solution y(x) for certain values of n and fractional derivative order α. We investigate the dependence of the solution upon different values of α and different initial values.

In the above diagrams, we present numerical results for second order L-fractional differential equation. In Figure 3 and 4, it is shown the graph of the series solution for n=2, when α varies from 0.3 to 0.9, for randomly

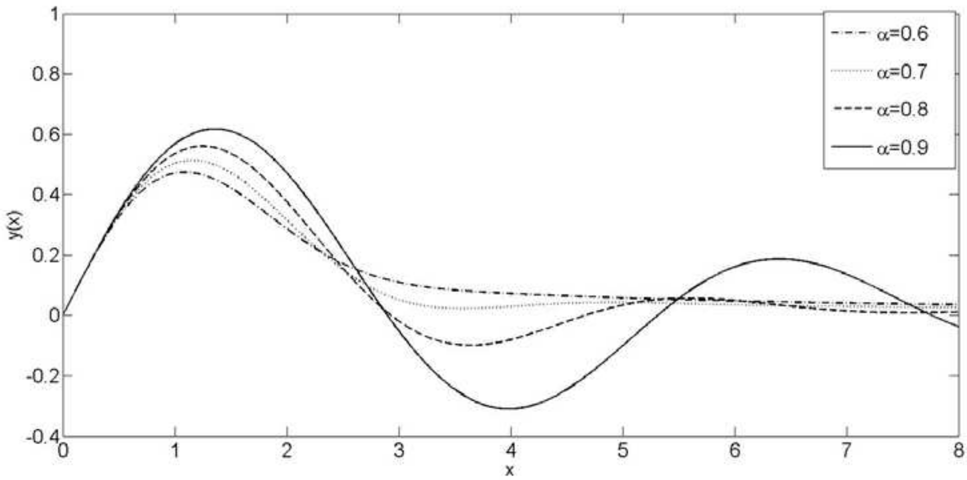

Figure 3: Power series solution $y(x)$ versus x for $n = 2$, $b_0 = 1$, $b_1 = 0.0292$, $b_2 = 0.9289$, $c_0 = 0$, $c_1 = 0.7303$, number of series terms k=163 and α from 0.6 to 0.9.

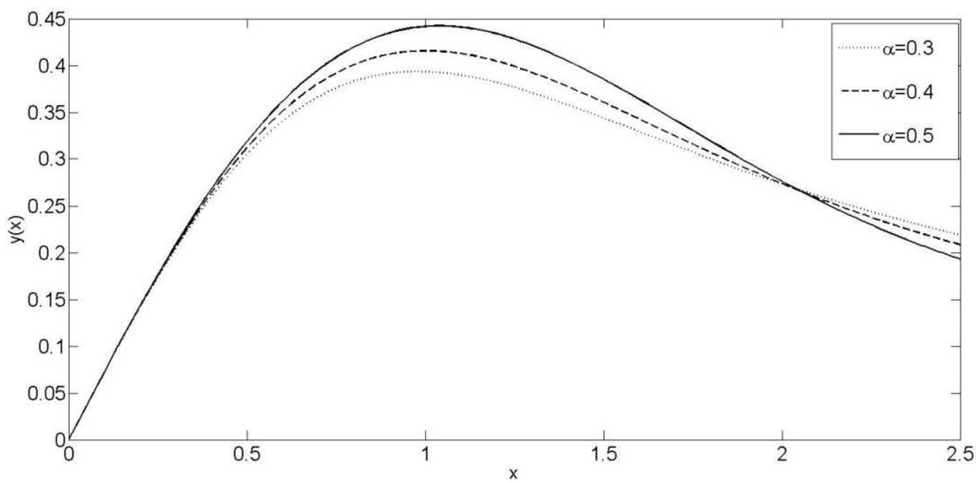

Figure 4: Power series solution y(x) versus x for n=2, $b_0 = 1$, $b_1 = 0.0292$, $b_2 = 0.9289$, $c_0 = 0$, $c_1 = 0.7303$, number of series terms k=163 and $\alpha = 0.3$ to 0.5.

selected values of coefficients b_i and initial conditions and number of terms in series k=163. Similarly, below in Figure 5 it is shown the graph of the series solution for n=2 and number of series terms k=163, but for different values of

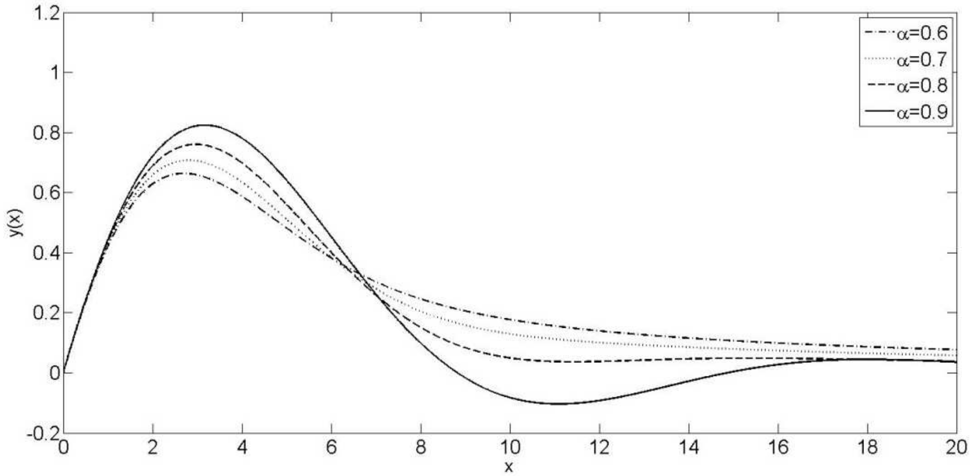

Figure 5: Power series solution y(x) versus x for n=2, b_0 =1, b_1 =1.4987, b_2 =6.5961, c_0 =0, c_1 =0.5186, number of series terms k=163 and α =0.6 to 0.9.

coefficients b_i and initial conditions.

In Figure 6, we present the solution for third order differential equation (n=3), when α varies from 0.5 to 0.9, for randomly selected values of coefficients b_i and initial conditions and number of terms in series k=164.

It is worth mentioned that for both n=2 and n=3 and α =0.6to 0.9, the shape of the solution given is similar. Furthermore, the change in α, in each case, does not affect the shape but only the values of the resulting y(x). In each graph, the greater the value of α the greater is the fluctuation of y(x) with a slight translation to the right.

Finally, in order to examine the dependency of the solution y(x) on initial conditions, we examine the latter case (n=3, b_0 =1, b_1 =0.9754, b_2 =2.785, b_3 =5.4688, k=164 and α =0.9) for different initial conditions c_1, c_2. In Figure 7 and 8 we present our results. It is obvious that in both cases, the greater the value of initial conditions the greater is the fluctuation of y(x).

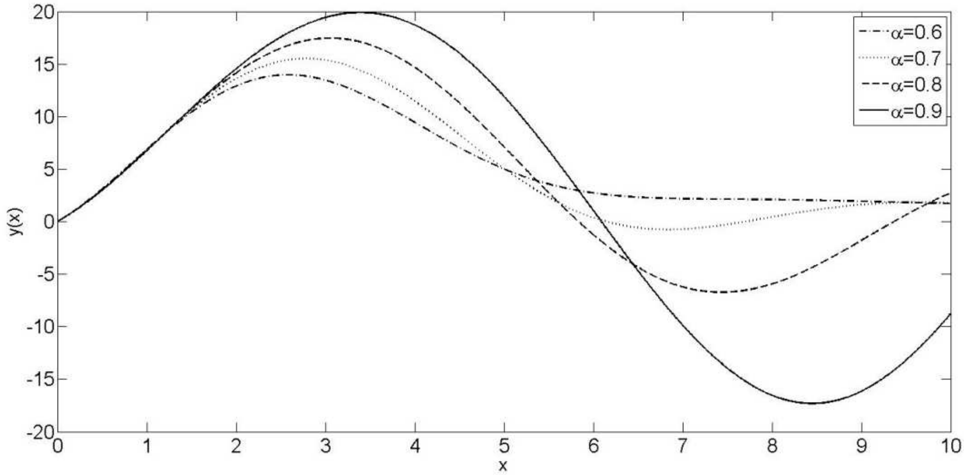

Figure 6: Power series solution y(x) versus x for n=3, b_0 =1, b_1 =0.9754, b_2 =2.785, b_3 =5.4688, c_0 =0, c_1 =4.7875, c_2 =4.8244, number of series terms k=164 and α =0.6 to 0.9.

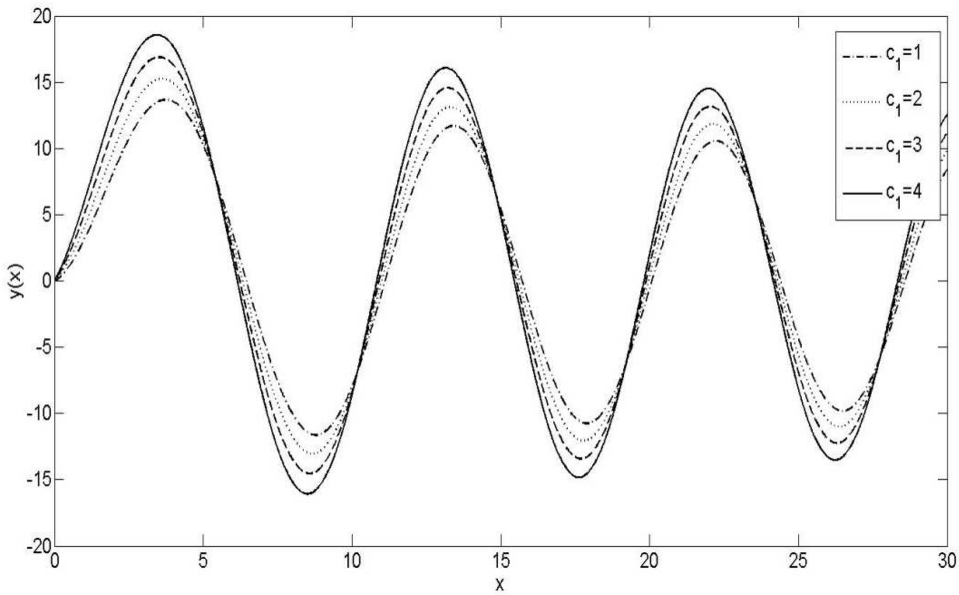

Figure 7: Power series solution y(x) versus x for n=3, b_0 =1, b_1 =0.9754, b_2 =2.785, b_3 =5.4688, number of series terms k=164, c_0 =0, c_1 =1 to 4, c_2 =4.8244 and α =0.9.

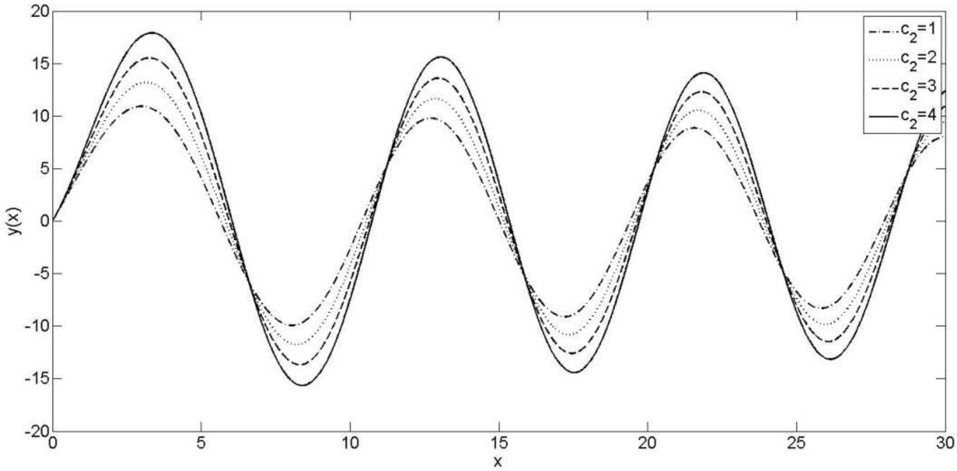

Figure 8: Power series solution y(x) versus x for n=3, b_0 =1, b_1 =0.9754, b_2 =2.785, b_3 =5.4688, number of series terms k=164, c_0 =0, c_1 =4.7875, c_2 =1 to 4 and α =0.9.

7. COMPARISON BETWEEN THE L-DERIVATIVE SOLUTION AND THE MITTAG-LEFFLER TYPE FUNCTION SOLUTION

The classical fractional case for solving homogeneous fractional differential equations is expressed by the help of the Mittag-Leffler type function solution (See Sabatier et.al. [34]). It would be interesting to compare that solution with the L-Fractional Derivative solution. Therefore, a test case has been prepared just for comparison reasons. More specifically, we examine the fractional differential equation:

$$b_2 \cdot \left(\frac{d^a}{dx^a}\right)^2 y(x) + b_1 \cdot \frac{d^a}{dx^a} y(x) + b_0 \cdot y(x) = 0,$$
$$y(0) = 0,$$
$$\frac{d^a}{dx^a} y(0) = 0.7303, \tag{34}$$

where b_0 =1, b_1 = 2.92 and b_2 =0.9289. The derivatives of the homogeneous differential equation($\frac{d^a}{dx^a} y(x)$) might be Riemann-Liouville (${}_0D^a_x$) or Leibniz (L-) derivatives (${}^L_0D^a_x y(x)$) accordingly. Due to the results (Figs 9-12) the difference between these two solutions is stupendous: Although they both agree

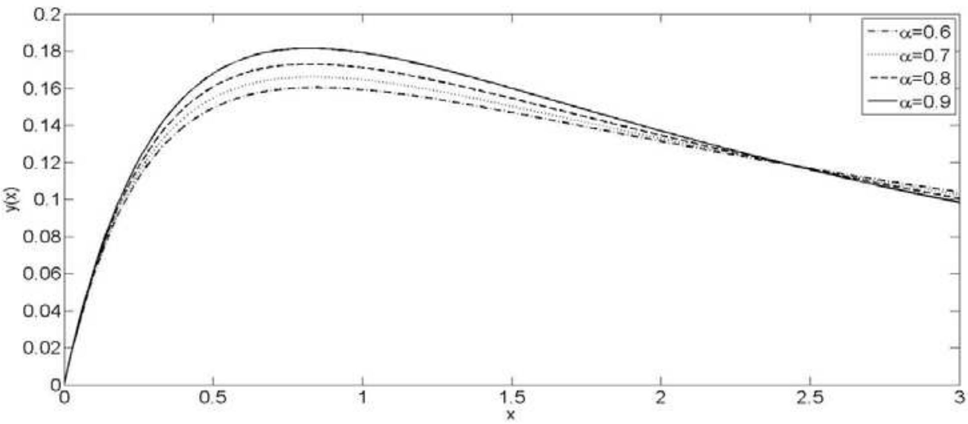

Figure 9: Power series solution y(x) versus x for n=2, b_0 =1, b_1 =2.92, b_2 =0.9289, c_0 =0, c_1 =0.7303, number of series terms k=163 and α from 0.6 to 0.9.

for α =1, to the same y(x) of the conventional case, they exhibit large differences for the rest of α. Leibnitz derivative shows a much smoother behavior (closer to the conventional case with α =1) than Riemann-Liouville Derivative. Since fractional differential equations with L-Derivatives are derived through fractional differentials, they are physically meaningful. On the contrary, Riemann-Liouville fractional differential derivatives do not correspond to any differential. Hence the differentials equations which are formed by substitution of the conventional derivatives with the Riemann-Liouville ones are meaningless. In our opinion the big differences between the conventional equations (for α =1) and the equations expressed by the Riemann-Liouville derivatives are due to the fact that the latter are meaningless. In addition the ones expressed by the proposed Leibnitz Derivatives are closer to the conventional ones (cases for α =1) and their results are meaningful since L-Fractional Derivatives correspond to differentials. On the other hand, the Mittag-Leffler approach seems much steeper and unreasonable.

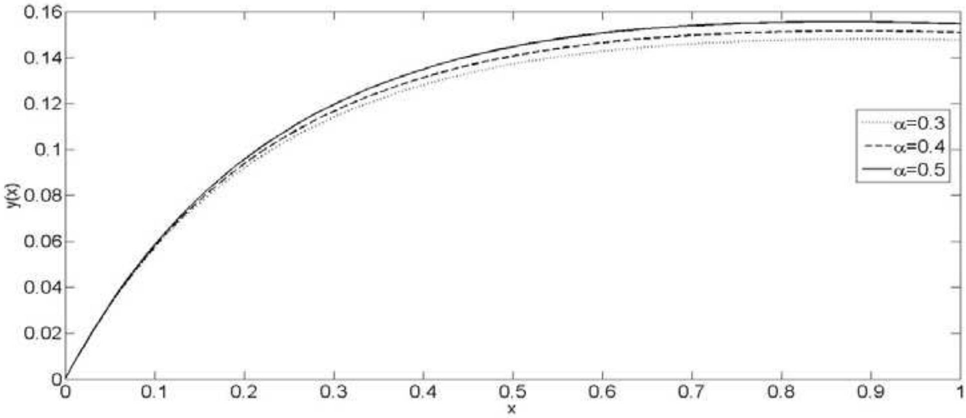

Figure 10: Power series solution y(x) versus x for n=2, b_0 =1, b_1 =2.92, b_2 =0.9289, c_0 =0, c_1 =0.7303, number of series terms k=163 and α =0.3 to 0.5.

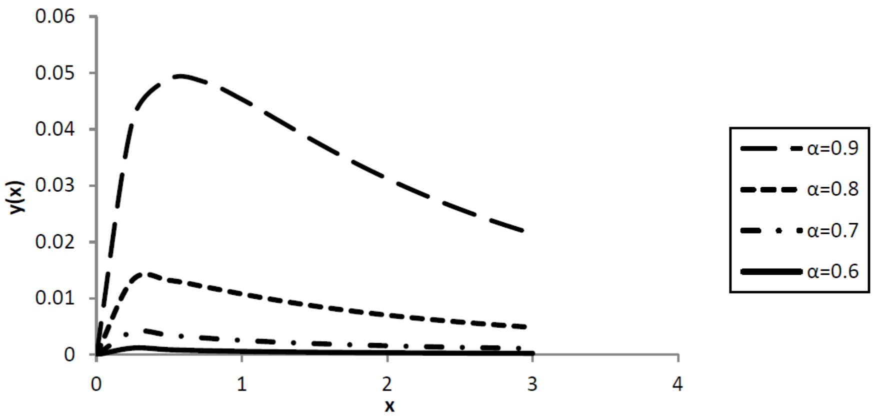

Figure 11: Mittag-Leffler type function solution y(x) versus x for n=2, b_0 =1, b_1 =2.92, b_2 =0.9289, c_0 =0, c_1 =0.7303, number of series terms k=163 and α from 0.6 to 0.9.

8. CONCLUSION

L-fractional derivatives, proposed by Lazopoulos et al. [15], define fractional differentials with either positive or negative variable increments, along with the tangents and the normals. The application of L-fractional operator in the case of linear homogenous fractional differential equations with constant

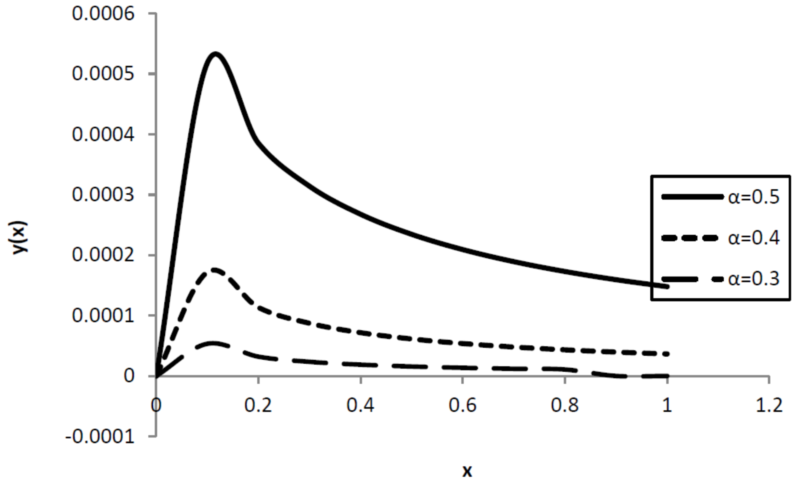

Figure 12: Mittag-Leffler type function solution y(x) versus x for n=2, b_0 =1, b_1 =2.92, b_2 =0.9289, c_0 =0, c_1 =0.7303, number of series terms k=163 and α =0.3 to 0.5.

coefficients. The solution is defined through power expansions of the variable. The power series expansion yields a recursive formula for the coefficients of the power series. The order α of fractional differentiation seems to affect the fluctuation of the resulting solution y(x) and not the shape of it. Finally, the same homogeneous differential equation is solved with the help of Mittag-Leffler type functions and results are extracted. While comparing these two methods conclusions are made, showing the properness and usefulness of the Leibnitz L-Derivative. Finally the solutions expressed by the proposed Leibnitz Derivatives are closer to the conventional ones (cases for α =1) and their results are meaningful since L-Fractional Derivatives correspond to differentials. In contrast, the Mittag-Leffler approach seems much steeper and unreasonable.

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