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CLASSIFICATIONS OF POSITIVE SOLUTIONS OF CERTAIN SECOND ORDER FUNCTIONAL DYNAMIC EQUATIONS

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ABSTRACT: In this paper, we study the asymptotic behavior of the positive solutions of the following second order functional dynamic equation

$$\left(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\right)^{\Delta}+f(t,x^{\sigma}(t),x^{\tau}(t))=0,$$

on a time scale $\mathbb{T}$, where $\alpha>0$ is a constant, $\sigma(t)$ is the forward jump operator, and $x^{\tau}(t)=x(\tau(t))$. According to the cases, where

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s=\infty \text{ or } \int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s<\infty$$

holds, all positive solutions of the above equation are classified into three types by means of their asymptotic behavior, respectively. Our goal is to establish necessary and sufficient conditions for the existence of certain types of solutions of the dynamic equation. Furthermore, we apply the results obtained this paper to certain second order difference equations and obtain new corresponding results for the difference equations. Finally, we give an example to illustrate the main results.

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1. INTRODUCTION

In this paper, we consider the following second order functional dynamic equation

$$\left(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\right)^{\Delta}+f(t,x^{\sigma}(t),x^{\tau}(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

on a time scale $\mathbb{T}$, where $\alpha>0$ is a constant, $\sigma(t)$ is the forward jump operator, and $x^{\tau}(t)=x(\tau(t))$. Throughout this paper, we always assume that

(A1) $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^{+})$, and $\tau\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $\lim\limits_{t\to\infty}\tau(t)=\infty$;

(A2) $f\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R}^2,\mathbb{R})$ with $uf(t,u,v)>0$ for $t\in[T,\infty)_{\mathbb{T}}\subseteq[t_0,\infty)_{\mathbb{T}}$ and $uv>0$, $f(t,u,v)$ is non-decreasing in u and v for each fixed $t\in[T,\infty)_{\mathbb{T}}$.

Equation (1.1) is called a retarded dynamic equation if $\tau(t)<t$ and is called an advanced dynamic equation if $\tau(t)>t$ and ordinary if $\tau(t)=t$. Since the asymptotic behavior of solutions near infinity is our primary concern, here we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. For the related theory and concepts of the time scales, we refer the reader to [2, 3, 7]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus [11], i.e., $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}:=\{t:t=q^n,n\in\mathbb{N}_0,q>1\}$. Dynamic equations on time scale have an enormous potential for applications such as in population dynamic, quantum mechanics, electrical engineering, neutral networks, heat transfer, and combinatorics [2, 7, 11].

By a solution of Eq.(1.1), we mean a nontrivial real-valued function $x\in C^1_{rd}\ ([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ which has the property $r|x^{\Delta}|^{\alpha-1}x^{\Delta}\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies Eq.(1.1) for all $t\in[t_0,\infty)_{\mathbb{T}}$, where C_{rd} is the space of rd-continuous functions. Our attention is restricted to those solutions $x(t)$ of Eq.(1.1) which exist on some half-linear $[t_y,\infty)_{\mathbb{T}}\subseteq[t_0,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|:t\in[T,\infty)_{\mathbb{T}}\}>0$ for any $T\in[t_y,\infty)_{\mathbb{T}}$.

We note that Eq.(1.1) in its general form includes several types of differential and difference equations with delay or advanced arguments or both. In recent years, there has been an interest in the studying the asymptotic behavior solutions of dynamic equations on a time scale. This has lead to many attempts to harmonize the asymptotic theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to papers [1, 4, 5, 7, 9, 10], and the reference cites therein. In this paper, motivated by the works of [6, 12], we study the asymptotic behavior of Eq.(1.1) and give a classification scheme for eventually positive solutions of Eq.(1.1) in term of their asymptotic magnitude, i.e., all positive solutions are classified into three types by means of their asymptotic behavior under the case where

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} = \infty, \tag{1.2}$$

or

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)} < \infty. \tag{1.3}$$

holds, respectively. We will present necessary and sufficient conditions for the existence of certain types of solutions of Eq.(1.1). Furthermore, we apply the results obtained this paper to certain second order difference equations and obtain new corresponding results for the difference equations. Finally, we give an example to illustrate the main results.

2. THE CASE $R(T_0) = \infty$

In this section, we will restrict our attention to the case where (1.2) holds. For simplicity, let

$$R(t,T) = \int_{T}^{t} \frac{\Delta s}{r^{1/\alpha}(s)}, \quad t > T \geq t_0.$$

Lemma 2.1. *Let* (1.2) *hold, and let* $x(t)$ *be an eventually positive solution of Eq.(1.1). Then*

(1) $x^{\Delta}(t)$ *is eventually positive.*

(2) There exist $T_0 \in [t_0,\infty)_{\mathbb{T}}$ *and* $l_1,\ l_2 > 0$ *such that* $l_1 R(t,T_0) \leq x(t) \leq l_2 R(t,T_0)$ *for* $t \in [T_0,\infty)_{\mathbb{T}}$.

Proof. (1) Without loss of generality, we assume that there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x^{\sigma}(t) > 0$ and $x^{\tau}(t) > 0$ for $t \in [T_0,\infty)_{\mathbb{T}}$. From

(1.1), we get

$$\big(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\big)^{\Delta} = -f(t, x^{\sigma}(t), x^{\tau}(t)) < 0,$$

so, $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is decreasing for $t \in [T_0, \infty)_{\mathbb{T}}$. Thus, $x^{\Delta}(t)$ is eventually one sign, and

$$r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) \le r(T_0)|x^{\Delta}(T_0)|^{\alpha-1}x^{\Delta}(T_0),\ t \in [T_0, \infty)_{\mathbb{T}}. \tag{2.1}$$

If $x^{\Delta}(t) < 0$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, then, by (2.1),

$$r(t)(-x^{\Delta}(t))^{\alpha} \ge r(T_0)(-x^{\Delta}(T_0))^{\alpha}, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

i.e.,

$$x^{\Delta}(t) \le r^{1/\alpha}(T_0)x^{\Delta}(T_0)\frac{1}{r^{1/\alpha}(t)}, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

which implies

$$x(t) \le x(T_0) + r^{1/\alpha}(T_0)x^{\Delta}(T_0)\int_{T_0}^{t}\frac{\Delta s}{r^{1/\alpha}(s)} \to -\infty \text{ as } \mathrm{t} \to \infty,$$

which contradicts $x(t) > 0$. Hence, $x^{\Delta}(t) > 0$ on $[T_0, \infty)_{\mathbb{T}}$, i.e., (1) holds.

(2) By (1) and (2.1),

$$x(t) \le x(T_0) + r^{1/\alpha}(T_0)x^{\Delta}(T_0)\int_{T_0}^{t}\frac{\Delta s}{r^{1/\alpha}(s)}, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

consequently, there exists $c_2 > 0$ such that $x(t) \le c_2R(t, T_0)$ for large t. Note that $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is eventually decreasing, there exists $l > 0$ such that $r(t)(x^{\Delta}(t))^{\alpha} \ge l$, i.e.,

$$x^{\Delta}(t) \ge \Big(\frac{l}{r(t)}\Big)^{1/\alpha}, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

which follows, for $t \in [T_0, \infty)_{\mathbb{T}}$,

$$x(t) \ge x(T_0) + l^{1/\alpha}\int_{T_0}^{t}\frac{\Delta s}{r^{1/\alpha}(s)} \ge l^{1/\alpha}R(t, T_0).$$

Hence, (2) holds. □

Let P denote the set of all eventually positive solutions of Eq.(1.1). Based on Lemma 2.1, we get the following classification.

Theorem 2.1. *Let (1.2) hold. Then any eventually positive solution $x(t)$ of Eq.(1.1) must belong to one of the following classes:*

$$\begin{aligned}
P_c^0 &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = c > 0,\ \lim_{t\to\infty} r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = 0\big\};\\
P_\infty^0 &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = +\infty,\ \lim_{t\to\infty} r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = 0\big\};\\
P_\infty^\gamma &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = +\infty,\ \lim_{t\to\infty} r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = \gamma > 0\big\}.
\end{aligned}$$

Proof. Recall that $x(t) \in P$. Using Lemma 2.1, we see $x^\Delta(t)$ and $r(t)$ $|x^\Delta(t)|^{\alpha-1}$ $x^\Delta(t)$ are eventually positive. Hence $x(t)$ either tends to a positive constant or to positive infinity, and $r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)$ tends to a nonnegative constant as $t \to \infty$.

Note that if $x(t)$ tends to a positive constant, then $r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)$ must tend to zero. Otherwise, $r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) \geq d > 0$ for $t \in [T_0,\infty)_{\mathbb{T}}$, so that $x^\Delta(t) \geq \big(\frac{d}{r(t)}\big)^{1/\alpha}$. Consequently,

$$x(t) \geq x(T_0) + d^{1/\alpha}\int_{T_0}^t \frac{\Delta s}{r^{1/\alpha}(s)} \to \infty \quad \text{as } t \to \infty,$$

which is a contradiction. □

In order to justify our classification of the positive solutions of Eq.(1.1), we present the following results.

Theorem 2.2. *Let (1.2) hold. A necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution $x(t) \in P_c^0$ is that for some $l > 0$, $T_0 \in [t_0,\infty)_{\mathbb{T}}$,*

$$\int_{T_0}^\infty \Big(\frac{1}{r(t)}\int_t^\infty f(s,l,l)\Delta s\Big)^{1/\alpha}\Delta t < \infty. \tag{2.2}$$

Proof. (Sufficiency) Let $x(t)$ be an eventually positive solution of (1.1) with $x(t) \in P_c^0$, i.e., $\lim_{t\to\infty} x(t) = c > 0$ and $\lim_{t\to\infty} r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = 0$. Then there exist $l > 0$ and $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that $x^\sigma(t) \geq l$ and $x^\tau(t) \geq l$ for $t \in [T_0,\infty)_{\mathbb{T}}$. From (1.1),

$$r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = \int_t^\infty f(s,x^\sigma(s),x^\tau(s))\Delta s, \quad t \in [T_0,\infty)_{\mathbb{T}},$$

i.e.,

$$x^\Delta(t) = \Big(\frac{1}{r(t)}\int_t^\infty f(s,x^\sigma(s),x^\tau(s))\Delta s\Big)^{1/\alpha}, \quad t \in [T_0,\infty)_{\mathbb{T}},$$

since $x^{\Delta}(t) > 0$ by Lemma 2.1. Integrating the above equation from T_0 to ∞, and noting that (A2), it follows

$$\begin{aligned}&\int_{T_0}^{\infty}\Big(\frac{1}{r(t)}\int_t^{\infty} f(s,l,l)\Delta s\Big)^{1/\alpha}\Delta t\\ &\quad\le \int_{T_0}^{\infty}\Big(\frac{1}{r(t)}\int_t^{\infty} f(s,x^{\sigma}(s),x^{\tau}(s))\Delta s\Big)^{1/\alpha}\Delta t \le c - x(T_0) < \infty,\end{aligned}$$

i.e., (2.2) holds.

(Necessity) Let (2.2) hold. Then for the given $l > 0$, there exists $T \in [T_0,\infty)_{\mathbb{T}}$, sufficiently large, such that

$$\int_{T}^{\infty}\Big(\frac{1}{r(t)}\int_t^{\infty} f(s,l,l)\Delta s\Big)^{1/\alpha}\Delta t \le \frac{l}{2}. \tag{2.3}$$

We define a bounded, convex and closed subset Ω of $C([T,\infty)_{\mathbb{T}},\mathbb{R})$ as

$$\Omega = \Big\{x(t) \in C([T,\infty)_{\mathbb{T}},\mathbb{R}^+) : \frac{l}{2} \le x(t) \le l,\ t \in [T,\infty)_{\mathbb{T}}\Big\},$$

and an operator $\Phi : \Omega \to C([T,\infty)_{\mathbb{T}},\mathbb{R}^+)$ by

$$(\Phi x)(t) = l - \int_t^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty} f(u,x^{\sigma}(u),x^{\tau}(u))\Delta u\Big)^{1/\alpha}\Delta s, \quad t \in [T,\infty)_{\mathbb{T}}.$$

Then the mapping Φ has the following properties.

(i) Φ maps Ω into itself. Indeed, for any $x \in C([T,\infty)_{\mathbb{T}},\mathbb{R})$ and for $t \in [T,\infty)_{\mathbb{T}}$, by (2.3),

$$\begin{aligned} l \ge &(\Phi x)(t) = l - \int_t^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty} f(u,x^{\sigma}(u),x^{\tau}(u))\Delta u\Big)^{1/\alpha}\Delta s,\\ \ge & l - \int_T^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty} f(u,l,l)\Delta u\Big)^{1/\alpha}\Delta s \ge \frac{l}{2}.\end{aligned}$$

So that $(\Phi x)(t) \in \Omega$.

(ii) The operator Φ is continuous on Ω. To see this, let $\epsilon > 0$, choose $T^* \in [T_0,\infty)_{\mathbb{T}}$ so large such that for $t \in [T^*,\infty)_{\mathbb{T}}$,

$$\int_t^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty} f(u,l,l)\Delta u\Big)^{1/\alpha}\Delta s \le \frac{\epsilon}{2}. \tag{2.4}$$

Let $\{x_n\}$ be a sequence in Ω converging to an element x of Ω, i.e., $\lim_{n\to\infty}\|x_n - x\| = 0$, since Ω is closed. It is easy to verify that for $t \in [T^*,\infty)_{\mathbb{T}}$, by (2.4),

$$|(\Phi x_n)(t) - (\Phi x)(t)| \le \int_t^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty} f(u,x_n^{\sigma}(u),x_n^{\tau}(u))\Delta u\Big)^{1/\alpha}\Delta s$$

$$
\begin{aligned}
&+\int_t^\infty \Big(\frac{1}{r(s)}\int_s^\infty f(u,x^\sigma(u),x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s\\
\le& 2\int_t^\infty \Big(\frac{1}{r(s)}\int_s^\infty f(u,l,l)\Delta u\Big)^{1/\alpha}\Delta s\\
\le& \epsilon.
\end{aligned}
$$

It follows that $\lim_{n\to\infty}\|(\Phi x_n)(t)-(\Phi x)(t)\|=0$, i.e., Φ is continuous on Ω.

(iii) $\Phi(\Omega)$ is precompact. The uniform bounded easily follows taking into account that $\Phi(\Omega)\subset\Omega$. Let us prove the equi-continuity. Indeed, for t_2, $t_1\in[T^*,\infty)_{\mathbb{T}}$, $x\in C([T^*,\infty)_{\mathbb{T}},\mathbb{R})$, by (2.4),

$$
\begin{aligned}
&|(\Phi x)(t_2)-(\Phi x)(t_1)|\\
&\quad\le \int_{t_2}^\infty \Big(\frac{1}{r(s)}\int_s^\infty f(u,l,l)\Delta u\Big)^{1/\alpha}\Delta s+\int_{t_1}^\infty \Big(\frac{1}{r(s)}\int_s^\infty f(u,l,l)\Delta u\Big)^{1/\alpha}\Delta s\\
&\quad\le \epsilon,
\end{aligned}
$$

which follows $\Phi(\Omega)$ is equi-continuous. Then $\Phi(\Omega)$ is precompact. Hence, by Schauder's fixed point theorem, we know that Φ has a fixed point $x\in\Omega$ such that $\Phi x=x$. It is easily checked that x is an eventually positive solution of (1.1), and

$$
x(t)=l-\int_t^\infty \Big(\frac{1}{r(s)}\int_s^\infty f(u,x^\sigma(u),x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s\to l \ \text{ as } \ t\to\infty,
$$

and

$$
r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)=\int_t^\infty f(u,x^\sigma(u),x^\tau(u))\Delta u\to 0 \quad \text{as} \ \ t\to\infty,
$$

i.e., $x(t)\in P_l^0$. □

Theorem 2.3. *Let* (1.2)*) hold. A necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution $x(t)\in P_\infty^\gamma$ is that for some $l>0$, $T_0\in[t_0,\infty)_{\mathbb{T}}$,*

$$
\int_{T_0}^\infty f\big(t,lR(\sigma(t),T_0),lR(\tau(t),T_0)\big)\Delta t<\infty. \tag{2.5}
$$

Proof. (Sufficiency) Let $x(t)$ be an eventually positive solution of (1.1) with $x(t)\in P_\infty^\gamma$, i.e., $\lim_{t\to\infty}x(t)=+\infty$ and $\lim_{t\to\infty}r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)=\gamma>0$. Using Lemma 2.1(2), we know that there exist $l>0$ and $T_0\in[t_0,\infty)_{\mathbb{T}}$ such that

$x(t) \geq lR(t, T_0)$, $x^\sigma(t) \geq lR(\sigma(t), T_0)$ and $x^\tau(t) \geq lR(\tau(t), T_0)$ for $t \in [T_0, \infty)_{\mathbb{T}}$. Integrating (1.1) from T_0 to ∞, and by (A2), we get

$$\begin{aligned}\int_{T_0}^{\infty} & f(t, lR(\sigma(t), T_0), lR(\tau(t), T_0))\Delta t \\ & \leq \int_{T_0}^{\infty} f(t, x^\sigma(t), x^\tau(t))\Delta t \leq -\gamma + r(T_0)(x^\Delta(T_0))^\alpha < \infty,\end{aligned}$$

which follows (2.5) holds.

(Necessity) Let (2.5) hold with $l > 0$. Choose $T \in [T_0, \infty)_{\mathbb{T}}$ so large such that

$$\int_t^{\infty} f(t, lR(\sigma(t), T), lR(\tau(t), T))\Delta t \leq (2^\alpha - 1)m^\alpha,\ t \in [T, \infty)_{\mathbb{T}},$$

in which $m > 0$ and $2m \leq l$. We define a bounded, convex and closed subset Ω of $C([T, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ as

$$\Omega = \Big\{x(t) \in C([T, \infty)_{\mathbb{T}}, \mathbb{R}^+) : mR(t, T) \leq x(t) \leq 2mR(t, T),\ t \in [T, \infty)_{\mathbb{T}}\Big\},$$

and an operator $\Phi : \Omega \to C([T, \infty)_{\mathbb{T}}, \mathbb{R})$ by

$$(\Phi x)(t) = \int_T^t \frac{1}{r^{1/\alpha}(s)}\Big(m^\alpha + \int_s^{\infty} f(u, x^\sigma(u), x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Arguing as in the proof of Theorem 2.2, it can be shown that Φ is a continuous operator which maps Ω into a compact subset of $C([T, \infty)_{\mathbb{T}}, \mathbb{R})$. Using Schauder's fixed point theorem, we see that Φ has a fixed point $x \in \Omega$ such that $\Phi x = x$. It is easily checked that x is an eventually positive solution of (1.1), and

$$x(t) = \int_T^t \frac{1}{r^{1/\alpha}(s)}\Big(m^\alpha + \int_s^{\infty} f(u, x^\sigma(u), x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s \to \infty \quad \text{as } t \to \infty,$$

and

$$r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t) = m^\alpha + \int_t^{\infty} f(u, x^\sigma(u), x^\tau(u))\Delta u \to m^\alpha \text{ as } t \to \infty.$$

Consequently, $x(t) \in P_\infty^{m^\alpha}$. The proof is complete. □

Theorem 2.4. *Let (1.2) hold. If for any l, $d > 0$ such that, for some $T_0 \in [t_0, \infty)_{\mathbb{T}}$,*

$$\int_{T_0}^{\infty}\Big(\frac{1}{r(t)}\int_t^{\infty} f(s, l, l)\Delta s\Big)^{1/\alpha}\Delta t = \infty, \tag{2.6}$$

and

$$\int_{T_0}^{\infty} f\big(t, dR(\sigma(t), T_0), dR(\tau(t), T_0)\big)\Delta t = \infty. \tag{2.7}$$

Then Eq.(1.1) has an eventually positive solution $x(t) \in P_{\infty}^{0}$.

Proof. By Theorem 2.1, we know that the eventually positive solution $x(t)$ of Eq.(1.1) satisfies $x(t) \in P_c^0 \cup P_{\infty}^0 \cup P_{\infty}^{\gamma}$. Note that condition (2.6) (by Theorem 2.2) implies that $x(t) \in P_{\infty}^0 \cup P_{\infty}^{\gamma}$. Also, condition (2.7) (by Theorem 2.3) implies that $x(t) \in P_c^0 \cup P_{\infty}^0$. Therefore, the conditions (2.6) and (2.7) follow that $x(t) \in P_{\infty}^0$. □

3. THE CASE $R(T_0) < \infty$

In this section, we will restrict our attention to the case where (1.3) holds. For simplicity, let

$$R(t) = \int_t^{\infty} \frac{\Delta s}{r^{1/\alpha}(s)}, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Lemma 3.1. *Let* (1.3) *hold, and let* $x(t)$ *be an eventually positive solution of Eq.(1.1) on* $[t_0, \infty)_{\mathbb{T}}$. *Then*

(1) $x^{\Delta}(t)$ is of constant sign eventually.

(2) $\lim_{t\to\infty} x(t)$ exists.

(3) There exist l_1, $l_2 > 0$ and $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $l_1 R(t) \le x(t) \le l_2$ for $t \in [T_0, \infty)_{\mathbb{T}}$.

(4) It holds,

$$\int_{t_0}^{\infty} \Big(\frac{1}{r(t)} \int_{t_0}^{t} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s\Big)^{1/\alpha} \Delta t < \infty. \tag{3.1}$$

Proof. (1) By the proof of Lemma 2.1(1), we know $x^{\Delta}(t)$ is of constant sign eventually.

(2) Note that $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is decreasing eventually. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) \le r(t_1)|x^{\Delta}(t_1)|^{\alpha-1}x^{\Delta}(t_1),\ t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.2}$$

By (1), we have that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for all t sufficiently large.

If $x^{\Delta}(t) > 0$ for all large t, then, by (3.2)

$$x^{\Delta}(t) \le r^{1/\alpha}(t_1)x^{\Delta}(t_1)\frac{1}{r^{1/\alpha}(t)},\ t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.3}$$

consequently,

$$x(t) \le x(t_1) + r^{1/\alpha}(t_1)x^{\Delta}(t_1)R(t, t_1) < \infty,$$

which follows that $\lim_{t\to\infty} x(t)$ exists.

If $x^{\Delta}(t) < 0$ for all large t, then, note that $x(t) > 0$, we get $\lim_{t\to\infty} x(t)$ exists.

(3) It follows from (1) and (2) that there exist $T_0 \in [t_0, \infty)_{\mathbb{T}}$ and $l_2 > 0$ such that $x(t) \le l_2$, and $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for $t \in [T_0, \infty)_{\mathbb{T}}$.

If $x^{\Delta}(t) > 0$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, then $R(t) \le x(t)$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, since $\lim_{t\to\infty} R(t) = 0$.

If $x^{\Delta}(t) < 0$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, then, by (3.2),

$$r(t)(-x^{\Delta}(t))^{\alpha} \ge r(t_1)(-x^{\Delta}(t_1))^{\alpha},\ t \in [t_1, \infty)_{\mathbb{T}},$$

i.e.,

$$x^{\Delta}(t) \le r^{1/\alpha}(t_1)x^{\Delta}(t_1)\frac{1}{r^{1/\alpha}(t)},\ t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

By (3.4), we get

$$x(s) - x(t) \le r^{1/\alpha}(T_0)x^{\Delta}(T_0)R(s, t),\ s \ge t \ge T_0.$$

Taking the limit as $s \to \infty$ on both sides of the above inequality, we get

$$x(t) \ge -r^{1/\alpha}(T_0)x^{\Delta}(T_0)R(t) = l_1R(t),\ t \in [T_0, \infty)_{\mathbb{T}}.$$

Hence, (3) holds.

(4) Integrating (1.1) from T_0 to t, which follows

$$\int_{T_0}^{t} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s = r(T_0)|x^{\Delta}(T_0)|^{\alpha-1}x^{\Delta}(T_0) - r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t).$$

By (1), recall that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for all large t.

If $x^{\Delta}(t) > 0$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, we have, by (3.4).

$$\int_{T_0}^{u} \Big(\frac{1}{r(t)}\int_{T_0}^{t} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s\Big)^{1/\alpha}\Delta t$$

$$\leq r^{1/\alpha}(T_0)x^{\Delta}(T_0)\int_{T_0}^{u}\frac{1}{r^{1/\alpha}(t)}\Delta t \leq r^{1/\alpha}(T_0)x^{\Delta}(T_0)R(T_0) < \infty.$$

i.e., (3.1) holds.

If $x^{\Delta}(t) < 0$ eventually for $t \in [T_0, \infty)_{\mathbb{T}}$, we have, by (3.4) again,

$$\int_{T_0}^{u}\frac{1}{r^{1/\alpha}(t)}\Big(\int_{T_0}^{t} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s\Big)^{1/\alpha}\Delta t \leq -\int_{T_0}^{u} x^{\Delta}(t)\Delta t \leq x(T_0) < \infty.$$

i.e., (3.1) also holds. □

In view of Lemma 3.1, we get the following classification.

Theorem 3.1. *Let* (1.3) *hold. Then any eventually positive solution of Eq.(1.1) must belong to one of the following classes:*

$$\begin{aligned} P_0 &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = 0\big\}; \\ P_c^{\gamma} &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = c > 0,\ \lim_{t\to\infty} r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) = \gamma\big\}; \\ P_c^{-\infty} &= \big\{x(t) \in P \mid \lim_{t\to\infty} x(t) = c > 0,\ \lim_{t\to\infty} r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) = -\infty\big\}. \end{aligned}$$

The following lemma and theorems will be derived to justify the above classification scheme.

Lemma 3.2. *Let* (1.3) *hold. Then a necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution* $x(t)$ *satisfying* $\lim_{t\to\infty} x(t) = c > 0$ *is that for some* $l > 0$,

$$\int_{t_0}^{\infty}\Big(\frac{1}{r(t)}\int_{t_0}^{t} f(s, l, l)\Delta s\Big)^{1/\alpha}\Delta t < \infty. \tag{3.5}$$

Proof. (Sufficiency) Let $x(t)$ be any eventually positive solution of (1.1) with $\lim_{t\to\infty} x(t)$ $= c > 0$. Then there exist $l > 0$ and $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $x^{\sigma}(t) \geq l$ and $x^{\tau}(t) \geq l$ for $t \in [T_0, \infty)_{\mathbb{T}}$. Besides, by Lemma 3.1(4), and (A2), we have

$$\begin{aligned} &\int_{T_0}^{\infty}\Big(\frac{1}{r(t)}\int_{t_0}^{t} f(s, l, l)\Delta s\Big)^{1/\alpha}\Delta t \\ &\qquad \leq \int_{T_0}^{\infty}\Big(\frac{1}{r(t)}\int_{t_0}^{t} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s\Big)^{1/\alpha}\Delta t < \infty. \end{aligned}$$

So, (3.5) holds.

(Necessity) Let (3.5) hold with $l>0$, we choose $T\in[t_0,\infty)_{\mathbb{T}}$ so large such that

$$\int_T^\infty\Big(\frac{1}{r(t)}\int_{t_0}^t f(s,l,l)\Delta s\Big)^{1/\alpha}\Delta t\le m,$$

where $m>0$ and $2m\le l$. We define a bounded, convex and closed subset Ω of $C([T,\infty)_{\mathbb{T}},\mathbb{R}^+)$ as

$$\Omega=\Big\{x\in C([T,\infty)_{\mathbb{T}},\mathbb{R}^+): m\le x(t)\le 2m,\ t\in[T,\infty)_{\mathbb{T}}\Big\},$$

and an operator $\Phi:\Omega\to C([T,\infty)_{\mathbb{T}},\mathbb{R}^+)$ by

$$(\Phi x)(t)=m+\int_t^\infty\Big(\frac{1}{r(s)}\int_{t_0}^s f(u,x^\sigma(u),x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s,\quad t\in[T,\infty)_{\mathbb{T}}.$$

As the proof of Theorem 2.2, it is routinely verified that

(i) Φ maps Ω into itself.

(ii) The operator Φ is continuous on Ω.

(iii) $\Phi(\Omega)$ is precompact.

Hence, It follows from Schauder's fixed point theorem that Φ has a fixed point $x\in\Omega$. It is easily checked that x is an eventually positive solution of (1.1), and

$$x(t)=m+\int_t^\infty\Big(\frac{1}{r(s)}\int_{t_0}^s f(u,x^\sigma(u),x^\tau(u))\Delta u\Big)^{1/\alpha}\Delta s\to m\ \text{ as }\ t\to\infty.$$

The proof is complete. □

Theorem 3.2. *Let* (1.3) *hold. If for some* $l>0$,

$$\int_{t_0}^\infty f(t,lR^\sigma(t),lR^\tau(t))\Delta t<\infty, \tag{3.6}$$

then Eq.(1.1) with $\alpha=1$ *has an eventually positive solution* $x(t)\in P_0$.

Proof. By (3.6), we choose $T\in[t_0,\infty)_{\mathbb{T}}$ so large such that

$$\int_T^\infty f(t,lR^\sigma(t),lR^\tau(t))\Delta t\le\frac{l}{2}.$$

Consider the equation

$$x(t)=R(t)\Big(\frac{l}{2}+\int_T^t f(s,x^\sigma(s),x^\tau(s))\Delta s\Big)$$

$$+\int_t^\infty R^\sigma(s)f(s,x^\sigma(s),x^\tau(s))\Delta s, \quad (3.7)$$

for $t\in[T,\infty)_{\mathbb{T}}$. It is easy to check that a solution of (3.7) is also a solution of (1.1) with $\alpha=1$. Now, we show that (3.7) has a positive solution $x(t)\in P_0$. At fact, consider the sequence $\{x_k(t)\}_{k=1}^\infty$ defined as follows

$$x_1(t)=0,\ \cdots\ x_{k+1}(t)=(\Phi x_k)(t),\ t\in[T,\infty)_{\mathbb{T}},$$

where Φ is defined by

$$(\Phi x)(t)=R(t)\Big(\frac{l}{2}+\int_T^t f(s,x^\sigma(s),x^\tau(s))\Delta s\Big)+\int_t^\infty R^\sigma(s)f(s,x^\sigma(s),x^\tau(s))\Delta s$$

for $t\in[T,\infty)_{\mathbb{T}}$. Recall that (A2), it is easy to see $0\le x_k(t)\le x_{k+1}(t)$ for $t\in[T,\infty)_{\mathbb{T}}$ and $k=1,2,\cdots$. Besides,

$$x_2(t)=(\Phi x_1)(t)=\frac{l}{2}R(t)\le lR(t)<\infty,\quad t\in[T,\infty)_{\mathbb{T}},$$

consequently, we can take $T^*\ge T$ such that

$$x_2^\sigma(t)\le lR^\sigma(t),\ x_2^\tau(t)\le lR^\tau(t)<\infty,\ t\in[T^*,\infty)_{\mathbb{T}}.$$

So,

$$\begin{aligned}(\Phi x_k)(t)\le& R(t)\Big(\frac{l}{2}+\int_{T^*}^t f(s,lR^\sigma(s),lR^\tau(s))\Delta s+\int_t^\infty f(s,lR^\sigma(s),lR^\tau(s))\Delta s\Big)\\ \le& R(t)\Big(\frac{l}{2}+\int_{T^*}^\infty f(s,lR^\sigma(t),lR^\tau(t))\Delta s\Big)\\ \le& lR(t)<\infty,\end{aligned}$$

for $t\in[T^*,\infty)_{\mathbb{T}},\ k\ge 2$. Hence, by Lebegue's dominated convergence theorem, we know there exists x such that $\Phi x=x$. Furthermore, it is clear that $x(t)\to 0$ as $t\to\infty$, i.e., $x(t)\in P_0$. □

Theorem 3.3. *Let* (1.3) *hold. Then a necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution* $x(t)\in P_c^\gamma$ *is that* (3.5) *holds for some* $l>0$ *and* $d>0$,

$$\int_{t_0}^\infty f(t,d,d)\Delta t<\infty. \quad (3.8)$$

Proof. (Sufficiency) Let $x(t)$ be an eventually positive solution of (1.1) with $x(t) \in P_c^{\gamma}$. It follows from Lemma 3.2 that (3.5) holds, and furthermore, there exist $d > 0$ and $T_0 \geq t_0$ such that $x^{\sigma}(t) \geq d$, $x^{\tau}(t) \geq d$ for $t \in [T_0, \infty)_{\mathbb{T}}$. Recall that (A2), we have

$$\int_{T_0}^{\infty} f(t,d,d)\Delta t \leq \int_{T_0}^{\infty} f(t, x^{\sigma}(t), x^{\tau}(t))\Delta t$$
$$= r(T_0)|x^{\Delta}(T_0)|^{\alpha-1}x^{\Delta}(T_0) - \gamma < \infty.$$

Thus, (3.8) holds.

(Necessity) We need consider two cases:

Case 1. Since (3.5) holds, it follows from Lemma 3.2 that $\lim_{t\to\infty} x(t) = c > 0$, and note that (3.8) holds with $d > 0$, then we choose $T \in [t_0, \infty)_{\mathbb{T}}$ so large such that $d/2 \leq x(t) \leq d$ and

$$\int_{T}^{\infty} f(t,d,d)\Delta t \leq \Big(\frac{d}{2R(T)}\Big)^{\alpha}.$$

We define a bounded, convex and closed subset Ω of $C([T,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ as

$$\Omega = \Big\{x \in C([T,\infty)_{\mathbb{T}}, \mathbb{R}^+) : \frac{d}{2} \leq x(t) \leq d,\ t \in [T,\infty)_{\mathbb{T}}\Big\},$$

and an operator

$$(\Phi x)(t) = d - \int_{t}^{\infty} \Big(\frac{1}{r(s)}\int_{s}^{\infty} f(u, x^{\sigma}(u), x^{\tau}(u))\Delta u\Big)^{1/\alpha}\Delta s, \quad t \in [T,\infty)_{\mathbb{T}}.$$

Then, under the conditions (3.8), similar to the proof of Theorem 2.2, we can show that Φ has a fixed point x which satisfies $\lim\limits_{t\to\infty} x(t) = d > 0$ and

$$r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) = \int_{t}^{\infty} f(s, x^{\sigma}(s), x^{\tau}(s))\Delta s,\ t \in [T_0,\infty)_{\mathbb{T}}.$$

Consequently, $\lim\limits_{t\to\infty} r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t) = 0$. Thus, $x(t) \in P_d^0$.

Case 2. Similar to the case 1, since (3.5) holds, by Lemma 3.2, $\lim\limits_{t\to\infty} x(t) = c > 0$, and note that (3.8) holds with $d > 0$, then we choose $T \in [t_0, \infty)_{\mathbb{T}}$ so large such that $d/2 \leq x(t) \leq d$ and

$$\int_{T}^{\infty} f(t,d,d)\Delta t < (2^{\alpha}-1)d \quad \text{and} \quad \int_{T}^{\infty} \frac{\Delta t}{r^{1/\alpha}(s)} < \frac{1}{4}.$$

We define a bounded, convex and closed subset Ω of $C([T,\infty)_{\mathbb{T}},\mathbb{R}^+)$ as

$$\Omega=\Big\{x\in C([T,\infty)_{\mathbb{T}},\mathbb{R}^+):\frac{d}{2}\le x(t)\le d,\ t\in[T,\infty)_{\mathbb{T}}\Big\},$$

and an operator

$$(\Phi x)(t)=d-\int_t^\infty\Big(\frac{1}{r(s)}\Big(d+\int_s^\infty f(u,x^\sigma(u),x^\tau(u))\Delta u\Big)\Big)^{1/\alpha}\Delta s,\quad t\in[T,\infty)_{\mathbb{T}}.$$

Then under the conditions (3.8), similar to the proof of Theorem 2.2, we can show that Φ has a fixed point x which satisfies $\lim_{t\to\infty}x(t)=d>0$ and

$$r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)=d+\int_t^\infty f(s,x^\sigma(s),x^\tau(s))\Delta s,\ t\in[T_0,\infty)_{\mathbb{T}}.$$

Consequently, $\lim_{t\to\infty}r(t)|x^\Delta(t)|^{\alpha-1}x^\Delta(t)=d$. Thus, $x(t)\in P_d^d$. □

As an direct applications of Theorems 3.1 and 3.3, we have the following results.

Theorem 3.4. *Let (1.3) hold. Then a necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution $x(t)\in P_c^{-\infty}$ is that (3.5) holds for some $l>0$, and for any $d>0$,*

$$\int_{t_0}^\infty f(t,d,d)\Delta t=\infty. \tag{3.9}$$

4. APPLICATIONS

Let $\mathbb{T}=\mathbb{Z}$ and consider the second order difference equation

$$\Delta\big(r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)\big)+f(n,x(n+1),x(n\pm n_0))=0,\ n\ge n_0, \tag{4.1}$$

where $\Delta x(n)=x(n+1)-x(n)$, $n_0\in Z^+$ and $\alpha>0$ is a constant. Here, we assume that

(A1) $r(n)>0$ for all $n\in\mathbb{Z}$;

(A2) $f\in C(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$ with $uf(n,u,v)>0$ for $n\ge N\ge n_0$, $N\in\mathbb{N}$, $uv>0$, $f(n,u,v)$ is non-decreasing in u and v for each fixed $n\ge N$.

Define

$$R(n,n_0)=\sum_{s=n_0}^{n}\frac{1}{r^{1/\alpha}(s)},\quad R(n)=\sum_{s=n}^{\infty}\frac{1}{r^{1/\alpha}(s)},\quad n>n_0.$$

By Theorems 2.1-2.4 and 3.1-3.4, we have

Theorem 4.1. *Let $R(n_0)=\infty$ hold. Then any eventually positive solution $x(n)$ of Eq.(4.1) must belong to one of the following classes:*

$$P_c^0=\{x(n)\in P\mid \lim_{n\to\infty}x(n)=c>0,\ \lim_{n\to\infty}r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)=0\};$$
$$P_\infty^0=\{x(n)\in P\mid \lim_{n\to\infty}x(n)=+\infty,\ \lim_{n\to\infty}r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)=0\};$$
$$P_\infty^\gamma=\{x(n)\in P\mid \lim_{n\to\infty}x(n)=+\infty,\ \lim_{n\to\infty}r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)=\gamma>0\}.$$

Theorem 4.2. *Let $R(n_0)=\infty$ hold. A necessary and sufficient condition for Eq.(4.1) to have an eventually positive solution $x(n)\in P_c^0$ is that for some $l>0$, $N_0\ge n_0$,*

$$\sum_{n=N_0}^{\infty}\Big(\frac{1}{r(n)}\sum_{s=n}^{\infty}f(s,l,l)\Big)^{1/\alpha}<\infty. \tag{4.2}$$

Theorem 4.3. *Let $R(n_0)=\infty$ hold. A necessary and sufficient condition for Eq.(4.1) to have an eventually positive solution $x(n)\in P_\infty^\gamma$ is that for some $l>0$, $N_0\ge n_0$,*

$$\sum_{n=N_0}^{\infty}f(n,lR(n+1,N_0),lR(n\pm n_0,N_0))<\infty. \tag{4.3}$$

Theorem 4.4. *Let $R(n_0)=\infty$ hold. If for any l, $d>0$ such that for some $N_0\ge n_0$,*

$$\sum_{n=N_0}^{\infty}\Big(\frac{1}{r(n)}\sum_{s=n}^{\infty}f(s,l,l)\Big)^{1/\alpha}=\infty, \tag{4.4}$$

and

$$\sum_{n=N_0}^{\infty}f\big(n,dR(n+1,N_0),dR(n\pm n_0,N_0)\big)=\infty, \tag{4.5}$$

then Eq.(4.1) has an eventually positive solution $x(n)\in P_\infty^0$.

Theorem 4.5. *Let $R(n_0)<\infty$ hold. Then any eventually positive solution of Eq.(4.1) must belong to one of the following classes:*

$$P_0=\{x(n)\in P\mid \lim_{n\to\infty}x(n)=0\};$$

$$P_c^{\gamma} = \{x(n) \in P \mid \lim_{n\to\infty} x(n) = c > 0,\ \lim_{n\to\infty} r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n) = \gamma\};$$
$$P_c^{-\infty} = \{x(n) \in P \mid \lim_{n\to\infty} x(n) = c > 0,\ \lim_{n\to\infty} r(n)|\Delta x(n)|^{\alpha-1}\Delta x(n) = -\infty\}.$$

Theorem 4.6. *Let $R(n_0) < \infty$ hold. If for some $l > 0$,*

$$\sum_{n=n_0}^{\infty} f(n, lR(n+1), lR(n \pm n_0)) < \infty, \tag{4.6}$$

then Eq.(4.1) with $\alpha = 1$ has an eventually positive solution $x(n) \in P_0$.

Theorem 4.7. *Let $R(n_0) < \infty$ hold. Then a necessary and sufficient condition for Eq.(1.1) to have an eventually positive solution $x(n) \in P_c^{\gamma}$ is that for some $l > 0$,*

$$\sum_{n=n_0}^{\infty} \Big(\frac{1}{r(n)} \sum_{s=n_0}^{n} f(s, l, l)\Big)^{1/\alpha} < \infty, \tag{4.7}$$

and for some $d > 0$,

$$\sum_{n=n_0}^{\infty} f(n, d, d) < \infty. \tag{4.8}$$

Theorem 4.8. *Let $R(n_0) < \infty$ hold. Then a necessary and sufficient condition for Eq.(4.1) to have an eventually positive solution $x(n) \in P_c^{-\infty}$ is that (4.7) holds for some $l > 0$, and for any $d > 0$,*

$$\sum_{n=n_0}^{\infty} f(n, d, d) = \infty. \tag{4.9}$$

Example 4.1. Consider the following second order difference equation

$$\Delta\big(n^{\mu}|\Delta x(n)|^{\alpha-1}\Delta x(n)\big) + \frac{c}{n^{\nu}}|x(n+1)|^{\beta-1}x(n+1)|x(n-1)|^{\gamma-1}x(n-1) = 0, \tag{4.10}$$

where α, β, γ, μ, ν, $c > 0$. Here, corresponding to Eq.(4.1), we have

$$r(n) = n^{\mu}, \quad \tau(n) = n-1, \quad f(n, u, v) = \frac{c}{n^{\nu}}|u|^{\beta-1}u|v|^{\gamma-1}v.$$

Note that

$$\sum_{n=1}^{\infty} \frac{1}{r^{\alpha}(n)} = \sum_{n=1}^{\infty} \frac{1}{n^{\mu/\alpha}}.$$

By Theorems 4.2-4.4, we have

(1) Eq.(4.10) has an eventually positive solution $x(n) \in P_c^0$ if and only if $1+\alpha-\nu<\mu\leq\alpha$.

(2) Eq.(4.10) has an eventually positive solution $x(n) \in P_\infty^\gamma$ if and only if $\mu<\alpha$ and $\frac{\alpha(1+\alpha+\beta)}{\alpha+\beta+\gamma}<\nu<\alpha$.

(3) If $\mu+\nu\leq\alpha$ and $\nu+\frac{\mu}{\nu}(\beta+\gamma)\leq 1$, then Eq.(4.10) has an eventually positive solution $x(t) \in P_\infty^0$.

Proof. Here, we only give the proof of the case (2), the others are similar. Note that, for some $l>0$,

$$\sum_{n=1}^{\infty} f(n,lR(n+1,1)),lR(n-1,1)) = cl^{\beta+\gamma}\sum_{n=1}^{\infty}\frac{1}{n^\nu}\Big(\sum_{s=1}^{n+1}\frac{1}{s^{\mu/\alpha}}\Big)^\beta\Big(\sum_{s=1}^{n-1}\frac{1}{s^{\mu/\alpha}}\Big)^\gamma$$
$$\leq cl^{\beta+\gamma}\sum_{n=1}^{\infty}\frac{1}{n^\nu}\Big(\sum_{s=1}^{n+1}\frac{1}{s^{\nu/\alpha}}\Big)^{\beta+\gamma}$$
$$\leq cl^{\beta+\gamma}\Big(\frac{\alpha}{\alpha-\nu}\Big)^{\beta+\gamma}\sum_{n=1}^{\infty}\frac{1}{n^\nu}\Big(\frac{1}{(n+1)^{\nu/\alpha-1}}\Big)^{\beta+\gamma}$$
$$\leq cl^{\beta+\gamma}\Big(\frac{\alpha}{\alpha-\nu}\Big)^{\beta+\gamma}\sum_{n=1}^{\infty}\frac{1}{n^{\nu+(\beta+\gamma)(\nu/\alpha-1)}}<\infty,$$

i.e., (4.3) holds. Hence, by Theorem 4.3, the conclusion of the case (2) is true. □

On the other hand, by Theorems 4.6-4.8, we have

(4) If $\mu>\alpha$ and $\frac{\alpha(1+\alpha+\beta)}{\alpha+\beta+\gamma}<\nu<\alpha$, then Eq.(4.10) with $\alpha=1$ has an eventually positive solution $x(n) \in P_0$.

(5) Eq.(4.10) has an eventually positive solution $x(n) \in P_c^\gamma$ if and only if $\mu>\alpha$ and $\nu>1$.

(6) Eq.(4.10) has an eventually positive solution $x(n) \in P_c^{-\infty}$ if and only if $1+\alpha-\mu<\nu\leq 1$.

We may employ other types of time scales, e.g., $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, and $\mathbb{T}=\mathbb{N}_0^2$, etc., see [2, 3]. The details are left to the interesting reader.

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