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ON $\mathcal{N}$-STRUCTURED p-IDEAL OF BF-ALGEBRA

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ABSTRACT: The idea of $\mathcal{N}$-structured p-ideal of a BF-algebra is proposed. The relationship between $\mathcal{N}$-structured-ideal and $\mathcal{N}$-structured p-ideal is studied. Some interesting and elegant results are also discussed.

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1. INTRODUCTION

After the concept of fuzzy sets by Zadeh [8], many research works have been done using fuzzy subsets in the various classes of abstract algebraic structures. Fuzzy BF-subalgebras[2] and Fuzzy Ideals of BF-Algebras[7] are few of them. Recently N-Subalgebras of BF-algebras were introduced by A.R. Hadipour et al [3] and also Y.B. Jun et al [4] discussed about N-ideals BCK/BCI-algebras.

Inspired by the concepts in [5] and [6] the authors applied the notion of $\mathcal{N}$-structure to the Ideals of BF-algebras and Filter of CI-algebras. In this paper, these concepts are intended to p-ideal of BF-algebras and $\mathcal{N}$-structured p-ideal

($\mathcal{N}_p$-ideal) is proposed. The nature of the homomorphic images of $\mathcal{N}_p$-ideal of a BF-algebra is also analysed.

The paper is organised as follows: Section 2 provides the preliminaries. In Section 3, $\mathcal{N}$-structured p-ideal is discussed and in Section 4, Homomorphism on $\mathcal{N}$-structured p-ideal is studied. Section 5 gives the conclusion.

2. PRELIMINARIES

In this section, some basic definitions and results that are required in the sequel are recalled. For any two elements $a, b \in [-1, 0]$, we use the notation $a \vee b$ to represent $max(a, b)$.

2.1. BASIC RESULTS ON BF-ALGEBRAS

Definition 2.1. [1] A BF-algebra is a non-empty set X with a constant 0 and a single binary operation $*$ which satisfies the following axioms:

1. $x * x = 0$.
2. $x * 0 = x$.
3. $0 * (x * y) = y * x, \ \forall x, y \in X$.

Example 2.2. Let $X = \{0, 1, 2, 3, 4\}$ be a set which comprises the following table.

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then, $(X, *, 0)$ is BF-algebra.

Definition 2.3. [1] A BG-algebra is a non-empty set X with a constant 0 and a single binary operation $*$ satisfying the following axioms:

1. $x * x = 0$.

2. $x * 0 = x$.

3. $(x * y) * (0 * y) = x$ for all $x, y \in X$.

A binary relation $\leq$ in a BF-algebra X can be defined as $x \leq y$, if and only if $x * y = 0$.

A subset S of a BF-algebra X is called a subalgebra of X, if $x * y \in S$ for all $x, y \in S$.

An ideal of a BF-algebra X is a subset I of X consisting 0 such that, if $x * y \in I$ and $y \in I$, then $x \in I$.

An ideal I of of a BF-algebra$X$ is called closed, if $0 * x \in I \ \forall \ x \in I$.

A non-empty subset I of a BF-algebra X is p-ideal, if $\forall \ x, y, z \in X,\ (x * z) * (y * z) \in I$ and $y \in I \ \Rightarrow \ x \in I$.

A p-ideal I of X is called closed, if $0 * x \in I \ \forall x \in X$.

A fuzzy set μ in a BF-algebra X can be called as a fuzzy subalgebra of X, if it satisfies:

$$\mu(x * y) \geq \mu(x) \wedge \mu(y) \text{ for all } x, y \in X. \tag{2.1}$$

A fuzzy set μ in a BF-algebra X can be called as a fuzzy ideal of X, if it satisfies:

$$\mu(0) \geq \mu(x) \text{ for all } x \in X, \tag{2.2}$$

$$\mu(x) \geq \mu(x * y) \wedge \mu(y) \text{ for all } x, y \in X. \tag{2.3}$$

A fuzzy set μ in a BF-algebra X can be called as a fuzzy p-ideal of X, if it satisfies:

$$\mu(0) \geq \mu(0)(x) \text{ for all } x \in X, \tag{2.4}$$

$$\mu(x) \geq \mu((x * z) * (y * z)) \wedge \mu(y) \text{ for all } x, y, z \in X. \tag{2.5}$$

Definition 2.4. A function $f : X \rightarrow Y$ of BF-algebras is considered to be homomorphism of X, if $f(x * y) = f(x) * f(y) \ \forall \ x, y \in X$.

Remark 2.5. If $f : X \rightarrow Y$ is a homomorphism on BF-algebras, $f(0_X) = 0_Y$.

Definition 2.6. A function $f : X \rightarrow Y$ of BF-algebras is said to be anti-homomorphism of X if $f(x * y) = f(y) * f(x) \ \ \forall \ x, y \in X$.

Definition 2.7. **$\mathcal{N}$-fuction and $\mathcal{N}$-structure** Consider a non-empty Set S. Denote the collection of functions from S to $[-1, 0]$ by $F(S, [-1, 0])$. We say that a member of $F(S, [-1, 0])$ is a negative valued function from S to $[-1, 0]$. Briefly $\mathcal{N}$-function and by an $\mathcal{N}$-structure(NS) on S, we mean that an ordered pair (S, η) of S and $\mathcal{N}$-function η on S.

Definition 2.8. A NS (X, η) in a set X with an $\mathcal{N}$-function $\eta : X \to [-1, 0]$ is indicated to have **Inf property,** if for any subset T of X, there exists $x_0 \in T$ such that $\eta(x_0) = \inf_{t \in T} \eta(t)$.

Definition 2.9. Let $f : X \to Y$ be a function and let $A = (X; \eta_A)$ be an $\mathcal{N}$-structure on X. Then, the image of A under f is defined as $f(A) = (Y; \eta_{f(A)})$ such that

$$\eta_{f(A)}(y) = \begin{cases} \inf_{z \in f^{-1}(y)} \eta(z) & if\ f^{-1}(y) = \{x : f(x) = y\} \neq \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10. Let $f : X \to Y$ be a function and let $B = (Y; \eta_B)$ be an $\mathcal{N}$-structure on Y . Then, the inverse image of B under f is defined as

$$f^{-1}(B) = (X; \eta_{f^{-1}(B)})$$

such that $\eta_{f^{-1}(B)}(x) = \eta_{(B)}(f(x))\ \forall x \in X$.

3. $\mathcal{N}$-STRUCTURED P-IDEAL

In this section, $\mathcal{N}$-structured p-ideal of a BF-algebra is defined. It is also proved that any $\mathcal{N}$-structured p-ideal in X is a $\mathcal{N}$-structured-ideal and the sufficient condition is derived for the converse.

Definition 3.1. An $\mathcal{N}$-structure (X, η) in X is called a $\mathcal{N}$-structured subalgebra of X, if it satisfies:

$$\eta(x * y) \leq \eta(x) \vee \eta(y) \quad \forall\ x, y \in X.$$

Definition 3.2. An $\mathcal{N}$-structure (X, η) in $X$is called an $\mathcal{N}$-structured Ideal($\mathcal{N}$-Ideal) of X, if it satisfies:

1. $\eta(0) \leq \eta(x)$

2. $\eta(x) \leq \eta(x * y) \vee \eta(y) \quad \forall\, x, y \in X.$

Definition 3.3. An $\mathcal{N}$-structure (X, η) in X, is to be an $\mathcal{N}$-structured Closed-ideal ($\mathcal{NC}$-ideal) of X, if

1. $\eta(x) \leq \eta(x * y) \vee \eta(y)$

2. $\eta(0 * x) \leq \eta(x) \qquad \forall x, y \in X.$

Definition 3.4. An $\mathcal{N}$-structure (X, η) in X is called to be a $\mathcal{N}$-structured p-ideal ($\mathcal{N}_p$-ideal) of X, if

1. $\eta(0) \leq \eta(x)$

2. $\eta(x) \leq \eta((x * z) * (y * z)) \vee \eta(y) \quad \forall\, x, y, z \in X.$

Example 3.5. The BF-algebra $X = \{0, 1, 2, 3\}$ is considered with the Cayley table as given below.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The $\mathcal{N}$-structure (X, η) of X defined as

$$\eta(x) = \begin{cases} -1 \ ; \ x = 0, 1 \\ -0.6 \ ; \ x = 2, 3 \end{cases}$$

is a $\mathcal{N}_p$-ideal of X.

Definition 3.6. An $\mathcal{N}$-structure (X, η) in X is considered to be a$\mathcal{N}$-structured closed p-ideal ($\mathcal{NC}_p$-ideal) of X, if

1. $\eta(x) \leq \eta((x * z) * (y * z)) \vee \eta(y)$

2. $\eta(0 * x) \leq \eta(x) \quad \forall\, x, y, z \in X.$

Example 3.7. Consider the BF-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table given below.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The $\mathcal{N}$-structure (X, η) of X defined as

$$\eta(x) = \begin{cases} -0.5 \ ; \ x = 0, 1 \\ -0.1 \ ; \ x = 2, 3 \end{cases}$$

is a $\mathcal{NC}_p$-ideal of X.

Trivially, the following can be proved:

Proposition 3.8. Every $\mathcal{NC}_p$-ideal is a $\mathcal{N}_p$-ideal.

In general,the converse of the above proposition is not true from the following:

Example 3.9. Consider the BF-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table given below.

$*$	0	1	2	3
0	0	3	0	1
1	1	0	1	3
2	2	3	0	1
3	3	1	3	0

The $\mathcal{N}$-structure (X, η) of X defined as

$$\eta(x) = \begin{cases} -0.7 \ ; \ x = 0, 1 \\ -0.3 \ ; \ x = 2, 3 \end{cases}$$

is a $\mathcal{N}_p$-ideal of X but not $\mathcal{NC}_p$-ideal, since $\eta(0 * 1) > \eta(1)$.

Proposition 3.10. If (X, η) is $\mathcal{N}_p$-ideal of X with $x \leq y$ for any $x, y \in X$, then $\eta(x) \leq \eta(y)$.

That η is order-preserving.

Proof. Let $x, y, z \in X$ such that $x \leq y \leq z$. Then, by the partial ordering if $\leq$ is defined in X, $x * y = 0$ and $y * z = 0$.

Thus, $\eta(x) \leq \eta((x * y) * (y * z)) \vee \eta(y)$
$\leq \eta((0 * 0) \vee \eta(y)) = \eta(0) \vee \eta(y) = \eta(y)$.

It completes the proof.

Theorem 3.11. If (X, η) is $\mathcal{NC}_p$-ideal of X, then the set

$$K = \{x \in X \ ; \ \eta(x) = \eta(0)\}$$

is p-ideals of X.

Proof. Clearly, $0 \in K$. Hence, $K \neq \phi$.

Let $(x * y) * (y * z) \in K$ and $y \in K$.

$\Rightarrow \ \eta((x * y) * (y * z)) = \eta(y) = 0$

$\Rightarrow \ \eta(x) \leq \eta((x * y) * (y * z)) \vee \eta(y) = 0 \vee 0 = 0$.

But $\eta(0) \leq \eta(x) \Rightarrow \ \eta(x) = 0$.

Hence, K is p-ideal of X.

Theorem 3.12. Any $\mathcal{N}_p$-ideal of X is a $\mathcal{N}$-ideal of X.

Proof. It is trivial by putting $z = 0$ in the definition of $\mathcal{N}_p$-ideal.

The converse of the above theorem is need not be true.

Now, a sufficient condition is derived for a $\mathcal{N}$-ideal to be a $\mathcal{N}_p$-ideal as follows:

Theorem 3.13. Let A be a $\mathcal{N}$-ideal of X. If $\eta(x * y) \leq \eta((x * z) * (y * z))$ for all $x, y, z \in X$ then A is $\mathcal{N}_p$-ideal of X.

Proof. Let A be a $\mathcal{N}$-ideal of X and assign $x, y, z \in X$.

So, we have $\eta(0) \leq \eta(x)$.

Then,

$$\eta(x) \leq \eta(x * y) \vee \eta(y) \leq \eta((x * z) * (y * z)) \vee \eta(y).$$

Hence, A is $\mathcal{N}_p$-ideal of X.

The following theorem shows the arbitrary union of family of $\mathcal{NC}_p$-ideals of X is also an $\mathcal{NC}_p$-ideal of X.

Theorem 3.14. Let $\{\eta_i \ ; \ i \in I\}$ be the family of $\mathcal{NC}_p$-ideals of X. Then $\bigcup_i \eta_i$ $\mathcal{NC}_p$-ideal of X.

Proof. Let x and $y \in X$.

Since $\{\eta_i; i \in I\}$ be the family of $\mathcal{NC}_p$-ideals of X, $\forall i \in I$, we have
(i) $\eta_i(x) \leq \eta_i((x * z) * (y * z)) \vee \eta_i(y)$ and (ii) $\eta_i(0 * x) \leq \eta_i(x)$ $\forall x, y \in X$.
Now $\bigcup_i \eta_i(x) = Sup\{\eta_i(x) \ ; \ i \in I\}$

$$\begin{aligned}
&\leq Sup\{\eta_i((x * z) * (y * z)) \vee \eta_i(y) \ ; \ i \in I\} \\
&= Sup\{\eta_i((x * z) * (y * z)) \ i \in I\} \vee Sup\{\eta_i(y) \ ; \ i \in I\} \\
&= \bigcup_i \eta_i((x * z) * (y * z)) \ \vee \ \bigcup_i \eta_i(y)
\end{aligned}$$

and $\bigcup_i \eta_i(0 * x) = Sup\{\eta_i(0 * x); i \in I\} \leq Sup\{\eta_i(x); i \in I\} = \bigcup_i \eta_i(x)$.

Hence $\bigcup_i \eta_i$ $\mathcal{NC}_p$-ideal of X.

Remark 3.15. (X, η) is an $\mathcal{N}$-structure on any universe X, if and only if $-\eta$ are the fuzzy subset of X.

Theorem 3.16. An $\mathcal{N}$-structure (X, η) is an $\mathcal{N}_p$-ideal of X, if and only if the fuzzy subset $-\eta$ are fuzzy p-ideal of X.

Proof. Let $\mathcal{N}$-structure (X, η) be an $\mathcal{N}_p$-ideal of X.

Further, clearly $\eta(0) \leq \eta(x)$ and $\eta(x) \leq \eta((x*z)*(y*z)) \vee \eta(y)$ $\forall\, x, y, z \in X$. $\Rightarrow -\eta(0) \geq -\eta(x)$ and $-\eta(x) \geq -\eta((x * z) * (y * z)) \wedge -\eta(y)$.

Therefore, $-\eta$ is a fuzzy p- ideal of X.

Conversely, assume $-\eta$ is a fuzzy p- ideal of X.

Now $\forall\, x, y, z \in X$,
$-\eta(0) \geq -\eta(x)$ and $-\eta(x) \geq -\eta((x * z) * (y * z)) \wedge -\eta(y)$
$\Rightarrow \eta(0) \leq \eta(x)$ and $\eta(x) \leq \eta((x * z) * (y * z)) \vee \eta(y)$.
It fulfils the proof.

4. HOMOMORPHISM ON $\mathcal{N}$-STRUCTURED P-IDEAL

Here, the image and pre-image of $\mathcal{N}$-structured p-ideals under the action of homomorphism and anti-homomorphisms on BF-algebras are discussed.

Theorem 4.1. Let f be a homomorphism from BF-algebras X onto Y and $A = (X; \eta_A)$ be an $\mathcal{N}_p$-ideal of X with **Inf property,**. Then, the image of A, $f(A) = (Y; \eta_{f(A)})$ is an $\mathcal{N}_p$-ideal of Y.

Proof. Let $a, b, c \in Y$ with $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ and $z_0 \in f^{-1}(c)$ such that

$$\eta(x_0) = \inf_{t \in f^{-1}(a)} \eta(t) \ ; \ \eta(y_0) = \inf_{t \in f^{-1}(b)} \eta(t) \ ; \ \eta(z_0) = \inf_{t \in f^{-1}(c)} \eta(t).$$

Now, by the definitions $2.8, 2.9$ and 2.4, the following is framed

$$\eta_{f(A)}(0) = \inf_{t \in f^{-1}(a)} \eta(t) \leq \eta(0) \leq \eta(x_0) = \inf_{t \in f^{-1}(a)} \eta(t) = \eta_{f(A)}(a)$$

Also,

$$\begin{aligned}
\eta_{f(A)}((a * c) * (b * c)) \vee \eta_{f(A)}(b) &= \inf_{t \in f^{-1}((a*c)*(b*c))} \eta(t) \vee \inf_{t \in f^{-1}(b)} \eta(t) \\
&\leq \eta((x_0 * z_0) * (y_0 * z_0)) \vee \eta(y_0) \\
&\leq \eta(x_0) \\
&= \inf_{t \in f^{-1}(a)} \eta(t) \\
&= \eta_{f(A)}(a)
\end{aligned}$$

Hence, the image $f(A)$ is an $\mathcal{N}_p$-ideal of Y.

Theorem 4.2. Let f be a homomorphism from BF-algebras X onto Y and $A = (X; \eta_A)$ be an $\mathcal{NC}_p$-ideal of X with **Inf property**. Then, the image of A, $f(A) = (Y; \eta_{f(A)})$ is an $\mathcal{NC}_p$-ideal of Y.

Proof. Let $x \in Y$ with $x_0 \in f^{-1}(x)$ such that $\eta(x_0) = \inf_{t \in f^{-1}(x)} \eta(t)$.

Then, we have

$\eta_{f(A)}(x) = \inf_{t \in f^{-1}(x)} \eta(t) \leq \eta(x_0) \leq \eta(0 * x_0) = \inf_{t \in f^{-1}(0*x)} \eta(t) = \eta_{f(A)}(0 * x)$.

Hence, by the above theorem, the image $f(A)$ is considered as a Bipolar valued fuzzy closed p-ideal of Y.

Theorem 4.3. Let f be a homomorphism from BF-algebras X onto Y and $B = (X; \eta_B)$ be an $\mathcal{N}_p$-ideal of Y. Then, the inverse image of B, $f^{-1}(B)$ is an $\mathcal{N}_p$-ideal of X.

Proof. Let $x, y \in X$.

Now, it is clear that $\eta_{f^{-1}(B)}(0) = \eta_B(f(0)) \leq \eta_B(f(x)) = \eta_{f^{-1}(B)}(x)$

$$\begin{aligned} \text{Then } \eta_{f^{-1}(B)}(x) &= \eta_B(f(x)) \\ &\leq \eta_B((f(x) * f(z)) * (f(y) * f(z))) \vee \eta_B(f(y)) \\ &= \eta_{f^{-1}(B)}((x * z) * (y * z)) \vee \eta_{f^{-1}(B)}(y). \end{aligned}$$

Then, the inverse image of B, $f^{-1}(B)$ is an $\mathcal{N}_p$-ideal of X.

Theorem 4.4. Let f be a homomorphism from BF-algebras X onto Y and $B = (X; \eta_B)$ be an $\mathcal{NC}_p$-ideal of Y. Then, the inverse image of B, $f^{-1}(B)$ is an $\mathcal{NC}_p$-ideal of X.

Proof. Let $x \in X$. Then, we have

$\eta_{f^{-1}(B)}(0 * x) = \eta_B(f(0 * x)) = \eta_B(f(0) * f(x)) \leq \eta_B(f(x)) = \eta_{f^{-1}(B)}(x)$.

Hence, through the above theorem, the inverse image $f^{-1}(B)$ becomes an $\mathcal{NC}_p$-ideal of X.

In the same way, the following can be proved.

Theorem 4.5. Let f be an anti-homomorphism from X onto Y and $A = (X; \eta_A)$ be an $\mathcal{N}_p$-ideal of X with **Inf property**. Then, the image of A, $f(A)$ is a an $\mathcal{N}_p$-ideal of Y.

Theorem 4.6. Let f be an anti-homorphism from X onto Y and $B = (X; \eta_B)$ be an $\mathcal{N}_p$-ideal of Y. Then, the inverse image of B, $f^{-1}(B)$ is an $\mathcal{N}_p$-ideal of X.

Theorem 4.7. Let f be an anti-homomorphism from X onto Y and $A = (X; \eta_A)$ be an $\mathcal{NC}_p$-ideal of X with **Inf property**. Then, the image of A, $f(A)$ is a an $\mathcal{NC}_p$-ideal of Y.

Theorem 4.8. Let f be an anti-homorphism from X onto Y and $B = (X; \eta_B)$ be an $\mathcal{NC}_p$-ideal of Y. Then, the inverse image of B, $f^{-1}(B)$ is an $\mathcal{NC}_p$-ideal of X.

5. CONCLUSION

An investigation on the $\mathcal{N}$-structured p-ideal of BF-algebras is done and several interesting results are observed. The surprising point is that, Andrzej Walendziak[1] says that the structure of BF algebra becomes a BG-algebra and the proof is followed directly from the definition. Hence, it is concluded that all the results prove here for BF-algebras can directly be carried over to BG-algebras.

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