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STABILITY FOR PERTURBED NONAUTONOMOUS DIFFERENTIAL EQUATIONS UNDER IMPULSIVE EFFECT

T. DONCHEV[1] AND D. KOLEV[2]

[1]Department of Mathematics
"Al.I. Cuza" University
Iaşi 700506, ROMANIA

[2]Department of Fundamental Sciences
PBZN Faculty
Academy of Ministry Interior
Sofia, BULGARIA

ABSTRACT: In this paper we study a class of non autonomous ODEs with a functional perturbation and impulsive effect. We establish a new sufficient condition which guarantees uniform stability of the zero-solution after "switching on" the impulsive effect.

Key Words: functional differential equations, stability, Lyapunov function, functional perturbation

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1. INTRODUCTION

In many applications there are Functional Differential Equations (FDE) having a general form

$$\left|\begin{array}{l} \dot{x} = F(t, x, \lambda_t(x)), \quad t \in (t_i, t_{i+1}), \quad i = 0, 1, 2, \ldots, \quad \dot{x} \equiv dx/dt \\ x(t) = \varphi(t), \quad t \in [t_0 - \alpha, t_0] \ \ \text{(initial data)}, \\ x(t_i) = x(t_i - 0) + \triangle_i \ \ (i = 0, 1, 2, \ldots) \ \ \text{(impulsive effect)}, \end{array}\right. \tag{1}$$

$$0 \le t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < t_{i+1} < \cdots .$$

Here the initial function is

$$\varphi : [-\alpha, t_0] \to \mathbb{R}^n, \ \alpha > 0, \quad \varphi \in C, \quad |\varphi| \le c_\varphi, \quad t_0 > 0,$$

c_φ =const, $\lambda_t(x)$ is a linear functional,

$$\lambda_t : \ C([t - \alpha, t], \mathbb{R}^n) \to \mathbb{R}^n, \quad t \ge t_0,$$

i.e. it is a n-dimensional vector whose components are functionals, and F is continuous function w.r.t. its arguments. We accept in this paper that that $t_0 = 0$.

Remark. We understand $\lambda_t(x)$ as a n–dimensional functional of the function $x(\cdot)$, i.e. $\lambda_t(x) \equiv \lambda(x(\cdot))$. In the case of delay FDE for instance we have $\lambda_t(x) = x(t + \theta)$, $\theta \in [-\alpha, 0]$.

There are some applications with a functional in integral form. There are also integro-differential equations describing different phenomena in Physics, Biology, and Engineering sciences with functionals like the above mentioned, [3]. Typical examples of applications of impulsive DEs are considered in the models of genetic network with impulsive control (see, e.g., [8]). An other application is the impulsive control of a financial model investigated in (see, e.g., [9]). All mentioned models include the impulsive ODEs of the form (1).

A functional differential equation (FDE) with delay has the form $\dot{x} = f(t, x(t), x(t - \sigma))$, where f is some function or operator w.r.t. t (very often it is the time), and λ_t in this case in the right-hand side of (1) is a functional which yield the delay, and particularly can be linear, that is, $\lambda_t(x) = x(t - \sigma)$. There are a lot of applications when the DEs have both delay and impulses.

For the stability and boundedness of FDEs we refer the reader to [2, 3, 5, 6, 7].

The goal of our consideration is by the second method of Lyapunov and some requirements imposed on (1) as it is in [1] to prove the stability of the zero-solution to the impulsive perturbed problem. The method that we use here is similar to those used in [1] and [4].

In the next section we give some preliminaries, notations and known facts to the FDEs as well as some requirements imposed on the functions in the right-hand side of (1). Here assume that the requirement for continuity and Lipschitzean of the vector field in the right-hand side of the considered equations is satisfied.

In the third section we prove a criterion tor existence of uniformly stable solution of (1).

2. PRELIMINARIES

Consider the system of nonautonomous ODEs without impulses (continuous case):

$$\left| \begin{array}{l} \dot{x} = f(t,x), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \quad \dot{x} \equiv dx/dt \\ \\ x(0) = x_0 \ \text{ (initial value).} \end{array} \right. \tag{2}$$

After perturbing the right-hand side of (2) obtain

$$F \equiv f(t,x) + \varepsilon R(\lambda_t(x)), \tag{3}$$

which is continuous, and $\varphi : [-\alpha, 0] \to \mathbb{R}^n$, also continuous and bounded, $|\varphi| \le C_0 = \text{const}$. So the quantity εR is the perturbation summand. Here ε is a small parameter, and εR is differentiable in the sense of Gâteaux n-dimensional vector-function.

Suppose that the unperturbed ODE (2) has the form

$$\dot{\tilde{x}} = f(t,\tilde{x}), \quad f(t,\tilde{x}) = A(t)\tilde{x} + \tilde{f}(t,\tilde{x}), \tag{4}$$

where $A(t)$ is a smooth linear $n \times n$ matrix operator, and $\tilde{f}(t,\tilde{x})$ is the nonlinearity. The initial value problem for (4)

$$\left| \begin{array}{l} \dot{\tilde{x}} = A(t)\tilde{x} + \tilde{f}(t,\tilde{x}), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \\ \\ x(0) = x_0 \ \text{ (initial value)} \end{array} \right. \tag{5}$$

has a solution $\tilde{x} = \tilde{x}(t)$. Here the initial data contain an initial function $\varphi(s), \quad s \in [-\alpha, 0]$, continuous on its closed domain. Therefore, $|\varphi(s)| \leq C_\varphi$, where C_φ is a positive constant, and $\varphi(s) \geq 0$ for $s \in [-\alpha, 0]$.

Let the following hypothesis, like in [1] as well [4], be satisfied:

H1. The linear functional λ_t is bounded, that is, $\|\lambda_t\| < c_1$ ($c_1 = \text{const} > 0$). The linear operator $A(t)$ is also bounded

$$\|A(t)\| < M \quad \forall t \in [0, \infty), \tag{6}$$

where $M > 0$ is a constant.

Introduce the following classes of Lipschitzean functions having the same constant $L > 0$: $Lip(L; \mathbb{R} \times \mathbb{R}^n)$ and $Lip(L; \mathbb{R}^n)$. Then assume that both f and R from (3) are Lipschitzean with the same constant L, i.e.,

$$\begin{aligned} &(a) \qquad f \in Lip(L; \mathbb{R} \times \mathbb{R}^n), \\ &(b) \qquad R \in Lip(L; \mathbb{R}^n). \end{aligned} \tag{7}$$

Suppose that $R(0) = f(t, 0) = 0$ ($t \in \mathbb{R}$). Hence the function $\tilde{f}$ is also Lipschitzean with a constant $M + L$, and $\tilde{f}(t, 0) = 0$.

Let

$$V : \mathbb{R} \times C \to \mathbb{R},$$

be a scalar function such that there exists the directional derivative $\dot{V}$,

$$\dot{V}(\xi + \tau h)|_{\tau=0} = \overline{\lim_{t \to 0+}} \left\{ \frac{1}{h} \Big[V(\xi + \tau h) - V(\xi) \Big] \right\}.$$

It is the upper right-hand derivative along the solution of (2). Note that some authors prefer to use the notation $\dot{V}_{(1)}(\xi)$.

H2. The system (2) has stable equilibrium state $x = 0$ which is guaranteed by the existence of Lyapunov function $V(t, x)$ such that

$$\frac{\partial V}{\partial t} + \nabla V \cdot f(t, x) \leq 0 \tag{8}$$

There exists a pair of positively definite functions $\tilde{a}(\|x\|)$ and $\tilde{b}(\|x\|)$ such that

$$\tilde{a}(\|x\|) \leq V(t, x) \leq \tilde{b}(\|x\|). \tag{9}$$

Note that the above condition is similar to one considered in [1] and [4].

Introduce the quantity $\Phi(t,x) \equiv \langle \nabla V,\ R(\lambda_t(x))\rangle$, which is differentiable in the sense of Gâteaux. Here by $\langle\cdot,\cdot\rangle$ we have denoted the scalar product.

Note that the origin O is an equilibrium point for the systems (2) and (4), hence $f(t,0) = 0 = \tilde{f}(t,0)$.

Define the variational system

$$\dot{z} = A(t)z. \tag{10}$$

If construct an initial value problem with initial condition $z(0) = x_0$, then obtain a solution $z = z(t)$.

3. PERTURBED PROBLEM

We consider two types perturbations of the problem (5). First we have continuous perturbation by a quantity $\varepsilon R(\lambda_t(x))$, and after this "switch on" impulsive perturbation that leads to an impulsive initial value problem (1).

3.1. THE PERTURBATIONS WITHOUT IMPULSES

Consider the right hand side of (4) perturbed by the quantity $\varepsilon R(\lambda_t(x))$. Here we trace the result in [1] concerning the stability of zero solution of (1) with $F = Ax + \tilde{f}(t,x) + \varepsilon R(\lambda_t(x))$ provided that the conditions **H1, H2** hold true. So we read

$$\left|\begin{array}{l} \dot{x} = f(t,x) + \varepsilon R(\lambda_t(x)), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \quad \dot{x} \equiv dx/dt \\ \\ x(0) = x_0 \ \ \text{(initial value).} \end{array}\right. \tag{11}$$

The norm of λ_t will be $\|\lambda_t\|$. Then arises the question if $\|\lambda_s - \lambda_\tau\|$ has maximal value for s, τ in some set.

Consider the set of linear functionals $\{\lambda_s\}_{s\in\Omega}$, where s is some real parameter in the set $\Omega \subset \mathbb{R}$. Define the set

$$S_k \equiv \{m_k(s,\tau) = \|\lambda_s - \lambda_\tau\| :\ 0 \le s \le \tau,\ k\alpha \le \tau < (k+1)\alpha\}, \tag{12}$$

where $k = 1, 2, \ldots$. Here the pairs of real parameters (s,τ) sweep the intervals given in (12), apparently, depending on $k = 1, 2, \ldots$.

For S_k there exists a binary relation

$$m_k(s,\tau) \prec m_k(r,\tau)$$

between certain pairs $m_k(s,\tau), m_k(r,\tau) \in S_k$ with the properties:

$$\begin{aligned} &(i) \quad m_k(s,\tau) \prec m_k(s,\tau);\\ &(ii) \quad \text{if } m_k(a,\tau) \prec m_k(b,\tau) \text{ and } m_k(b,\tau) \prec m_k(a,\tau),\\ &\qquad\qquad \text{then } m_k(a,\tau) = m_k(b,\tau);\\ &(iii) \quad \text{if } m_k(a,\tau) \prec m_k(b,\tau) \text{ and } m_k(b,\tau) \prec m_k(c,\tau),\\ &\qquad\qquad \text{then } m_k(a,\tau) \prec m_k(c,\tau)\\ &\qquad\qquad \text{(transitivity).} \end{aligned} \tag{13}$$

Then in this case, S_k is partially ordered (semi-ordered) by the relation "$\prec$".

Apparently, for every $m_k(a,\tau)$ and $m_k(b,\tau)$ there is $m_k(c,\tau)$ such that $m_k(a,\tau) \prec m_k(c,\tau)$, and $m_k(b,\tau) \prec m_k(c,\tau)$. Then $m_k(c,\tau)$ is an upper bound for $m_k(a,\tau)$ and $m_k(b,\tau)$.

Taking into account all these notes we state the following:

Lemma 1. *S_k is partially ordered set with the property that every linearly ordered subset of S_k has an upper bound in S_k. Then S_k contains at least one maximal element, which denote*

$$M_k \equiv \max_{s\in[0,\tau]} \|\lambda_s - \lambda_\tau\|$$

($k\alpha \le \tau < (k+1)\alpha$). It is true for $k = 1, 2, \ldots$.

Lemma 2. *Consider the perturbed problem (11). The functions f and R satisfy the Lipschitz condition (8) with the same constant L in the domain $\|x\| < C_x$ ($C_x = const > 0$). Then the solution $x(t)$ of (11) satisfies the estimate $\|x\| \le \|x_0\| e^{L\theta_k(t)}$, where $x_0 = x(0)$, and*

$$\theta_k(t) \equiv [1 + \varepsilon(M_k + c_1)]t, \quad M_k \equiv \max_{[0,\tau]} \|\lambda_s - \lambda_\tau\|,$$
$$k\alpha \le t \le \tau \le (k+1)\alpha \quad (k = 1, 2, \ldots).$$

Lemma 3. *Consider the unperturbed problem (5). The function f satisfy the Lipschitz condition with the constant L in the domain $\|x\| < C_x$ ($C_x = const$). Then the solution $\tilde{x}(t)$ of (5) satisfies the estimate $\|\tilde{x}\| \leq \|x_0\| e^{L\tilde{c}t}$, where $0 \leq t \leq l \leq k\alpha$ ($k = 1, 2, \ldots$), $x_0 = x(0)$, and $\tilde{c} = 1 + c_1\varepsilon$.*

Lemma 4. *For the problem (10) the following inequality hold:*

$$\|z\| \leq \|x_0\| e^{Mt}.$$

Lemma 5. *Let $x(t)$ and $\tilde{x}(t)$ be solutions of (11) and (5), respectively. Assume that the requirements of Lemma 2 are satisfied. Then the following estimate hold:*

$$\|x(t) - \tilde{x}(t)\| \leq \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \big(e^{L\theta_k(\tau)} - 1\big) e^{Lt} \quad (k = 1, 2, \ldots),$$

where $\theta_k(\tau)$ is the same in Lemma 2.

For the proof of the above stated Lemma 1-Lemma 5 the reader is referred to [1] and [4].

The following result can be seen in [1]

Theorem 6. *Let the following conditions be satisfied:*

1. The hypotheses H1, H2 for system (11)*;*

2. In the set $\omega \equiv \{(t, x) : t \in [-\alpha, \infty),\ \|x\| < C_x\}$, ($\alpha > 0$, $C_x > 0$)

$$|\Phi| \leq M\|x\|^d, \qquad \left|\frac{\partial \Phi}{\partial x}\right| \leq M\|x\|^{d-1}$$

for some $d \geq 1$; here $\dfrac{\partial \Phi}{\partial x}$ is the linear continuous functional of Gâteaux.

3. $\exists \tau > 0$, $\exists \delta > 0$, such that $\forall \alpha \geq 0$ and $\|x_0\| < \varepsilon_0$ for some $\varepsilon_0 > 0$, $x_0 \equiv \varphi(0)$, and

$$\int_0^\tau \Phi(z(t) dt \leq -\delta \|x_0\|^q, \quad 0 < q < d,$$

where $z(t)$ is the solution of the variational system (10) *with initial data $z(0) = x_0$.*

Then the zero solution of (11) *is uniformly stable in Lyapunov sense.*

The proof of Theorem 6 follows making use of the results in Lemma 1-Lemma 5.

3.2. IMPULSIVE PERTURBATIONS

After the impulsive perturbation being ”switched on“ then get the following initial value problem:

$$\left| \begin{array}{l} \dot{x} = Ax + f(t,x) + \varepsilon R(\lambda_t(x)), \quad t \in (t_i, t_{i+1}), \quad i = 0,1,2,\ldots, \quad \dot{x} \equiv dx/dt \\ x(t) = \varphi(t), \quad t \in [-\alpha, 0] \quad \text{(initial data)}, \\ x(t_i) = x(t_i - 0) + \triangle_i \quad (i = 0,1,2,\ldots) \quad \text{(impulsive effect)}, \end{array} \right. \tag{14}$$

where the impulsive condition is the same like that in (1), and the discrete differences $\{\triangle_i\}_{i=1}^{k}$ $(k = 1,2,\ldots)$, $\triangle_i \in \mathbb{R}^n$ $(i = 1,2,\ldots)$. Here we have n dimensional vectors $\triangle_i = (\triangle_{1i}, \triangle_{2i}, \ldots, \triangle_{ni})^T$ with real components.

Theorem 7. *Let the conditions of Theorem 6 be satisfied, and the difference between discrete impulses be $t_{i+1} - t_i = \alpha > 0$. In addition assume that the functions f, R have at least one side limits in there domains. Then the distance between impulsive perturbed solution $\xi = \xi(t)$ of (14) and the solution of the continuously perturbed problem (11) can be estimated by inequality:*

$$\|x - \xi\| \le \|\chi_{k-1}\| \left[1 + \tilde{c}L(e^t - 1)\right] \quad (t > 0), \tag{15}$$

where χ_{k-1} is the $n \times k - 1$ matrix $(k = 2,3,\ldots)$

$$\chi_{k-1} = \begin{pmatrix} \triangle_{11} & \triangle_{12} & \cdots & \triangle_{1,k-1} & \cdots \\ \triangle_{21} & \triangle_{22} & \cdots & \triangle_{2,k-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \triangle_{n1} & \triangle_{n2} & \cdots & \triangle_{n,k-1} & \cdots \end{pmatrix}.$$

Moreover, for sufficiently small norm $\|\chi_{k-1}\|$ the distance between both solutions x and ξ becomes very small.

Proof. 1) Consider the solutions in the first interval $t \in (0, t_1)$. Then we have $\|x - \xi\| = 0$.

2) Consider $t \in (t_1, t_2)$. Then

$$\|x-\xi\| \le \|x(t_1)-\xi(t_1^-)\| + \int_{t_1}^{t} \|f(s,x(s)) - f(s,\xi(s))\|ds+$$

$$+\varepsilon \int_{t_1}^{t} \|R(\lambda_s(x)) - R(\lambda_s(\xi))\|ds.$$

Note that $\xi(t_1) = \xi(t_1^-) + \triangle_1$, where $\triangle_1 = (\triangle_{11}, \triangle_{21}, \ldots, \triangle_{n1})^T$. Therefore,

$$\|x-\xi\| \le \|x(t_1)-\xi(t_1^-)\| + \|\triangle_1\| + L\int_{t_1}^{t} \|x-\xi\|ds + Lc_1\varepsilon \int_{t_1}^{t} \|x-\xi\|ds \le$$

$$\le \|\triangle_1\| + L\tilde{c}\int_{t_1}^{t} \|x-\xi\|ds.$$

3) Consider $t \in (t_2, t_3)$. Then by the same method obtain

$$\|x-\xi\| \le \|x(t_2)-\xi(t_2^-)\| + \|\triangle_1\| + L\int_{t_2}^{t} \|x-\xi\|ds + Lc_1\varepsilon \int_{t_2}^{t} \|x-\xi\|ds \le$$

$$\le \|\triangle_1\| + \|\triangle_2\| + L\tilde{c}\int_{t_1}^{t} \|x-\xi\|ds.$$

k) By the same method we obtain for $t \in (t_{k-1}, t_k)$,

$$\|x-\xi\| \le \|\chi_{k-1}\| + L\tilde{c}\int_{t_2}^{t} \|x-\xi\|ds.$$

Now we apply the Grönwall inequality for discontinuous functions or at least having one sided limits in their domains. Thus derive the inequality

$$\|x-\xi\| \le \sum_{i=1}^{k-1} \|\triangle_i\| \Big[1 + L\tilde{c}(e^t - 1)\Big],$$

and hence

$$\|x-\xi\| \le \|\chi_{k-1}\| \Big[1 + L\tilde{c}(e^t - 1)\Big],$$

where $\|\chi_{k-1}\|$ is the any norm of χ_{k-1}.

□

Corollary 8. *The zero-solution of the impulse perturbed equation is uniformly stable.*

The proof is almost obvious. Using for this purpose the proof of Theorem 6 in [1] it turns out that the estimate (15) allows to control the quantity $\|x-\xi\|$.

Using the above approach we may get other estimates,

$$\|\tilde{x}-\xi\| \leq \|\tilde{x}-x\| + \|x-\xi\| \leq$$
$$\leq \|\chi_{k-1}\| \left[1 + L\tilde{c}(e^t - 1)\right] + \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \left(e^{L\theta_k(\tau)} - 1\right) e^{Lt},$$

where $t \leq \tau < t_k$.

Conclusion.

As a conclusion we mention a known model of the stability of a genetic network under impulsive control (see, e.g., [8]). Many genetic networks are represented in the form of multiple additive terms.

$$\dot{y} = Ay(t) + \sum_{i=1}^{l} B_i f_i(y(t)),$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n$ represents the concentration of proteins, RNAs and chemical complexes. The quantities A, B_i are matrices in $\mathbb{R}^{n\times n}$. Here

$$f_i(y(t)) = (f_{i1}(y_1(t)), f_{i2}(y_2(t)), \ldots, f_{in}(y_n(t)))^T,$$

where $f_{ij}(y_j(t))$ are monotonic regulatory functions, which usually are of the Michaelis-Menten or Hill form, [8]. Thus the theory discussed here can be applied to this mathematical model.

The above stated methods as well those in [1] can be used for qualitative estimations of many other mathematical models (see, e.g, the Reference in [8], [9]). The asymptotic stability can be considered through the above stated approach.

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