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HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS FOR THE DESCRIPTION OF MASS-LESS PARTICLES

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ABSTRACT: Hyperbolic systems of partial differential equations are very important for physical applications because they satisfy the causality principle and lead to finite propagation of shocks waves. Many of this cases have been considered in Extended Thermodynamics. Here a new example of physical application is obtained by applying the new theory of the two blocks of balance equations to the case of mass-less particles, in the relativistic context.

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1. INTRODUCTION

Many physical problems are described by means of a set of partial differential equations of the type

$$\partial_\alpha \mathcal{A}^{\alpha A} = P^A \,, \tag{1}$$

where the summation convention over repeated greek indexes is used and they run from 0 to 3, the symbol ∂_0 denotes $\frac{1}{c}$ multiplied by the derivative with respect to time with c the light velocity, ∂_i denotes the derivative with respect to the spatial component x^i, and P^A are the production terms.

A part of the components of $\mathcal{A}^{\alpha A}$ are independent variables, while the remaining ones are constitutive functions. Systems of this type have been widely studied in the framework of Extended Thermodynamics (see for example [1]-[5] in the classical context and [6]-[8] in the relativistic case).

In these theories the constitutive functions are restricted by the relativity principle and by imposing a supplementary conservation law

$$\partial_\alpha h^\alpha = \sigma \,, \tag{2}$$

which must hold for every solution of eqs. (1). By imposing Liu's theorem [9], this is equivalent to assume the existence of Lagrange multipliers λ_A such that

$$\begin{aligned} &\partial_\alpha h^\alpha - \lambda_A \partial_\alpha \mathcal{A}^{\alpha A} = 0, \\ &\sigma = \lambda_A P^A, \end{aligned} \tag{3}$$

holds for every value of the independent variables. A bright idea described in [10] is to define the four-potential $h'^\alpha = -h^\alpha + \lambda_A \mathcal{A}^{\alpha A}$, so that eq. $(3)_1$ becomes $d\,h'^\alpha = \mathcal{A}^{\alpha A}\, d\,\lambda_A$, from which it follows $\mathcal{A}^{\alpha A} = \frac{\partial h'^\alpha}{\partial \lambda_A}$ if the Lagrange multipliers are taken as independent variables. In this way eqs. (1) become

$$\frac{\partial^2 h'^\alpha}{\partial \lambda_B \partial \lambda_A}\, \partial_\alpha \lambda_B = P^A$$

and the symmetry of the coefficients of $\partial_\alpha \lambda_B$ guarantees hyperbolicity and velocities of propagation of shock waves which are not greater than the speed of light if $t_\alpha\, h'^\alpha$ is a convex function of the Lagrange multipliers for every time-like congruence t_α (see also [11] for a proof of this property). Obviously, this property is very important for the causality principle and for the Einstein' s relativity principle.

In the last years the applicability of this theory has been extended to a wider set of materials by using, in the non relativistic context, a two block family of partial differential equations, the mass-block and the energy-block (see for example [12]-[24]). In the year 2017 the relativistic counterpart of this approach has been published in [25].

In the present article the aim is pursued to apply the same methodology to describe the ultrarelativistic case. This can be obtained with two different methods:

- By taking the limit of [25] for $\gamma \to 0$, where $\gamma = \frac{mc^2}{k_B T}$, m is the particle mass, T the absolute temperature and k_B the Boltzmann constant. The authors of [25] are planning to do this investigation in the next future.
- By considering $m = 0$ from the beginning, which is the case of mass-less particles. For this purpose an investigation independent of [25] has to be followed even if with similar passages.

This aim is reached in the present article. The result is important because calculations are simpler than in the general case; consequently, they may be considered a text of [25]. Moreover, in this article a function $\varphi(\mathcal{I})$ appears, which is not determined except for the fact that it must converge to the expression $\mathcal{I}^a$ of the classical counterpart of the theory, where $a = \frac{1}{2} f^i - 1$ with $f^i \geq 0$ internal degree of freedom due to the internal motion (rotation and vibration). Here a new expression is proposed for $\varphi(\mathcal{I})$ which guarantees integrability of the integrals appearing in the constitutive functions; moreover, in the monatomic limit (for $a \to -1$) these constitutive functions become the same of [26], where the new methodology of the two blocks of equations was not yet assumed. This is a strong confirmation of the present article and of [25].

In Section 2 we will see the form of eqs. (1) for the present physical situation, that is eqs. (4).

In Subsection 2.1 we will see that at equilibrium we have

$$\begin{aligned} V^\alpha &= nU^\alpha, \\ T_E^{\alpha\beta} &= p\left(h^{\alpha\beta} + \frac{3}{c^2}U^\alpha U^\beta\right), \\ A_E^{\alpha\beta\gamma} &= A_1^0\left(c^2 h^{(\alpha\beta}U^{\gamma)} + U^\alpha U^\beta U^\gamma\right), \end{aligned}$$

with $p = nk_BT$, A_1^0 given by (28) and $h^{\alpha\beta} = g^{\alpha\beta} + \frac{1}{c^2}U^\alpha U^\beta$.

In Section 3 the first order approximation is found; in particular, the independent variables will be considered $V^\alpha = nU^\alpha$, T, q^α, $t^{<\alpha\beta>_3}$ (here $< \cdots >_3$ denotes the 3-dimensional traceless part of a tensor) and ΔA. After that, the expressions of $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$ in terms of these variables are given by eq. (33) and (57) with B_3, B_8, D_3 given by (40), (42), (53), (38).

In Section 4, the monatomic limit of the present results will be considered and found that they converge to the results of ref. [26].

Finally, conclusions will be drawn.

2. THE PRESENT MODEL

The field equations (1) for the present physical model are

$$\partial_\alpha V^\alpha = 0; \quad \partial_\alpha T^{\alpha\beta} = 0; \quad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma} . \tag{4}$$

They are similar to eqs. (13) of [25], except that here it is not taken the traceless part of the last equation because we will see that $A^{\alpha\beta}{}_\beta = 0$.

The expressions of the functions appearing in the left hand sides are expressed in terms of only one unknown, the distribution function f; they are:

$$\begin{aligned} V^\alpha =& c \int_{\Re^3} \int_0^{+\infty} f p^\alpha \varphi(\tilde{\mathcal{I}})\, d\vec{P}\, d\tilde{\mathcal{I}} \ , \\ T^{\alpha\beta} =& c \int_{\Re^3} \int_0^{+\infty} f \left(1 + \frac{\tilde{\mathcal{I}}}{c^2}\right) p^\alpha p^\beta \, \varphi(\tilde{\mathcal{I}})\, d\vec{P}\, d\tilde{\mathcal{I}} \, , \\ A^{\alpha\beta\gamma} =& c \int_{\Re^3} \int_0^{+\infty} f \left(1 + 2\frac{\tilde{\mathcal{I}}}{c^2}\right) p^\alpha p^\beta p^\gamma \varphi(\tilde{\mathcal{I}})\, d\vec{P}\, d\tilde{\mathcal{I}} \, , \end{aligned} \tag{5}$$

Here p^μ is the four-momentum which in the general case satisfy the relation $p^\mu p_\mu = m^2 c^2$; this here becomes $p^\mu p_\mu = 0$.

These equations are the counterparts of eq. (16) of [25]; we have omitted, with respect to [25], a factor m^{2+a} in the definition of V^α, a factor m^{1+a} in the definitions of $T^{\alpha\beta}$, a factor m^a in the definition of $A^{\alpha\beta\gamma}$ as considering them new definitions; moreover, here $\tilde{\mathcal{I}}$ is corresponding to the expression $\frac{\mathcal{I}}{m}$ of [25]. It represents the energy of the internal modes of a molecule in order to take into account the exchange of energy between translational modes and internal modes in binary collisions. Moreover, we have $d\vec{P} = \frac{1}{p^0} dp^1 dp^2 dp^3$.

As consequence of these definitions we have that $T^\alpha{}_\alpha = 0$, $A^{\alpha\beta}{}_\beta = 0$.

This last approach was earlier followed in [26]-[31] for monatomic gases. In this article this last scheme is followed for polyatomic gases and $\tilde{\mathcal{I}}$ is replaced with $\mathcal{I}$ in order not to use an heavy notation.

The independent components of V^α and $T^{\alpha\beta}$ are 13 because of the condition $T^\alpha{}_\alpha = 0$; as in [26], to face this difficulty, we take as independent variable also one of the components of $A^{\alpha\beta\gamma}$ and in particular $A = \frac{1}{n^3 c^6} A^{\alpha\beta\gamma} U_\alpha U_\beta U_\gamma$ where n and U_α are defined by $V_\alpha = n U_\alpha$, $U_\alpha U^\alpha = c^2$.

As in [25], we obtain now the distribution function f by imposing the Maximum Entropy Principle (MEP). In particular, we define the functional

$$\begin{aligned} \mathcal{L} =& U_\alpha \left\{ -k_B c \int_{\Re^3} \int_0^{+\infty} f \ln f p^\alpha \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} + \right. \\ &+ \lambda \left[V^\alpha - c \int_{\Re^3} \int_0^{+\infty} f p^\alpha \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} \right] + \\ &+ \lambda_\beta \left[T^{\alpha\beta} - c \int_{\Re^3} \int_0^{+\infty} f \left(1 + \frac{\mathcal{I}}{c^2}\right) p^\alpha p^\beta \, \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} \right] + \\ &\left. + \Sigma_{\beta\gamma} \left[A^{\alpha\beta\gamma} - c \int_{\Re^3} \int_0^{+\infty} f \left(1 + 2\frac{\mathcal{I}}{c^2}\right) p^\alpha p^\beta p^\gamma \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} \right] \right\} . \end{aligned} \tag{6}$$

Obviously, the multiplier $\Sigma_{\beta\gamma}$ is symmetric and traceless. This functional $\mathcal{L}$ has to be maximized by the distribution function. So we have $\frac{\delta \mathcal{L}}{\delta f} = 0$, that is,

$$\begin{aligned} \int_{\Re^3} \int_0^{+\infty} \Bigg[-k_B c (\ln f + 1) - \lambda c - c \lambda_\beta p^\beta \left(1 + \frac{\mathcal{I}}{c^2}\right) - \\ c \Sigma_{\beta\gamma} p^\beta p^\gamma \left(1 + 2\frac{\mathcal{I}}{c^2}\right) \Bigg] U_\alpha p^\alpha \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} = 0 . \end{aligned} \tag{7}$$

It follows that we have

$$f = \exp\left(-1 - \frac{\chi}{k_B}\right) , \tag{8}$$

with

$$\chi = \lambda + \left(1 + \frac{\mathcal{I}}{c^2}\right) \lambda_\beta p^\beta + \left(1 + \frac{2\mathcal{I}}{c^2}\right) \Sigma_{\beta\gamma} p^\beta p^\gamma . \tag{9}$$

This result is the same of eq. (51) of [25] but with m replaced by 1.

2.1. RESULTS AT EQUILIBRIUM

The distribution function (8) calculated at equilibrium becomes

$$f = e^{-1 - \frac{1}{k_B}\left[\lambda + \left(1 + \frac{\mathcal{I}}{c^2}\right) \lambda_\beta p^\beta\right]} . \tag{10}$$

Now we can put $\lambda_\beta = \frac{U_\beta}{T}$ with $U_\alpha U^\alpha = c^2$. After that, we can perform our calculations in the reference frame where $U_\beta \equiv (c\,,\, 0\,,\, 0\,,\, 0)$ and to calculate the integrals, we put

$$p^1 = p^0 \sin\vartheta \cos\varphi, \quad p^2 = p^0 \sin\vartheta \sin\varphi, \quad p^3 = p^0 \cos\vartheta, \tag{11}$$
$$p^0 \in [0, +\infty[, \;\; \vartheta \in [0, \pi[\;\; \varphi \in [0, 2\pi[$$

from which it follows $d\vec{P} = p^0 \sin\vartheta \, d\, p^0 \, d\,\vartheta \, d\,\varphi$.

By using these expressions, eq. $(5)_1$ becomes

$$V^i = 0 \quad , \quad V^0 = nU^0 \quad (\text{from which} \quad V^\alpha = nU^\alpha \,) \,\text{with} \tag{12}$$

$$n = 4\pi e^{-1-\frac{\lambda}{k_B}} \int_0^{+\infty} \int_0^{+\infty} e^{-\left(1+\frac{\mathcal{I}}{c^2}\right)\frac{c}{k_B T} p^0} (p^0)^2 \phi(\mathcal{I}) \, dp^0 \, d\mathcal{I} \,. \tag{13}$$

But for whatever number $m \geq 1$, with an integration by parts, we have

$$\int_0^{+\infty} e^{-\left(1+\frac{\mathcal{I}}{c^2}\right)\frac{c}{k_B T} p^0} (p^0)^m \, dp^0 = \left| - \frac{k_B T}{\left(1+\frac{\mathcal{I}}{c^2}\right) c} e^{-\left(1+\frac{\mathcal{I}}{c^2}\right)\frac{c}{k_B T} p^0} (p^0)^m \right|_0^{+\infty}$$
$$+ + m \frac{k_B T}{\left(1+\frac{\mathcal{I}}{c^2}\right) c} \int_0^{+\infty} e^{-\left(1+\frac{\mathcal{I}}{c^2}\right)\frac{c}{k_B T} p^0} (p^0)^{m-1} \, dp^0$$
$$= m! \left(\frac{k_B T}{\left(1+\frac{\mathcal{I}}{c^2}\right) c} \right)^{m+1} \quad , \tag{14}$$

where the last passage has been obtained by iterating the procedure. By using this result for $m = 2$ in the above expression of n, we obtain

$$n = 8\pi e^{-1-\frac{\lambda}{k_B}} \left(\frac{k_B T}{c} \right)^3 \int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^3} \, d\mathcal{I} \,. \tag{15}$$

We can consider this result as a change of variables from λ to n.

Similarly, eq. $(5)_2$ becomes

$$T^{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \quad , \quad T^{11} = T^{22} = T^{33} = \frac{1}{3} T^{00} = p \tag{16}$$

from which

$$T^{\alpha\beta} = p \left(h^{\alpha\beta} + 3 \frac{U^\alpha U^\beta}{c^2} \right),$$

with

$$p = \frac{4}{3} \pi c e^{-1-\frac{\lambda}{k_B}} \int_0^{+\infty} \int_0^{+\infty} e^{-\left(1+\frac{\mathcal{I}}{c^2}\right)\frac{c}{k_B T} p^0} \left(1 + \frac{\mathcal{I}}{c^2} \right) (p^0)^3 \phi(\mathcal{I}) \, dp^0 \, d\mathcal{I} \,. \tag{17}$$

By using eq. (14) this expression becomes

$$p = 8\pi c e^{-1-\frac{\lambda}{k_B}} \left(\frac{k_B T}{c}\right)^4 \int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^3} \, d\mathcal{I} \,, \tag{18}$$

or

$$p = n k_B T \tag{19}$$

which is the same of $(40)_1$ of [25].

Now we have to discuss on the integrals which appear in these expressions; in Appendix A it will be proved that there is no integrability if we simply assume that $\phi(\mathcal{I}) = \mathcal{I}^a$, while it is surely integrable if we assume that $\phi(\mathcal{I}) = \mathcal{I}^a e^{-\mathcal{I}\frac{b}{c^2}} \psi\left(\frac{\mathcal{I}}{c^2}\right)$ with b a positive number and $\psi\left(\frac{\mathcal{I}}{c^2}\right)$ a polynomial function such that $\psi(0) = 1$. A possibility is to take

$$\phi(\mathcal{I}) = \mathcal{I}^a e^{-\mathcal{I}\frac{b}{c^2}} \left(1 + \frac{\mathcal{I}}{c^2}\right)^m \,, \tag{20}$$

where a , b, m are constants such that $a > -1$, $b > 0$, $m \geq 7$ and m an integer number. In this case we will see that not only integrability is assured, but we can also calculate explicitly the integrals and we will see that, as limiting case for $a \to -1$ we will obtain the results of [26] for monatomic gases. Moreover, it is evident that the non relativistic limit of (20) is exactly that of the non relativistic case [12] , [15]. This fact allows to conclude that a is related to the internal degree of freedom of the internal motion, as indicated in the Introduction.

Now we calculate the integrals by using the expression (20) for $\phi(\mathcal{I})$. To this end we will use the following property which holds for all $a > -1$

$$\int_0^{+\infty} e^{-\mathcal{I}\frac{b}{c^2}} \mathcal{I}^a \, d\mathcal{I} = \frac{1}{\left(\frac{b}{c^2}\right)^{a+1}} \quad \Gamma(a+1) \,, \tag{21}$$

where $\Gamma(\alpha)$ is the Gamma function defined for all $\alpha > 0$ as

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} \, d\, x \,, \tag{22}$$

which satisfies the recursive relation $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ and reduces to $\Gamma(\alpha) = (\alpha - 1)!$ when α is a positive integer.

Eq. (21) is an immediate consequence of this definition, with the simple substitution $\mathcal{I} = \frac{c^2}{b}\, x$.

More particulars will be given in Appendix B; but, for the sequel, it is useful to note that by using the property $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ we obtain

$$\begin{aligned}\Gamma(s+a+1) &= (s+a)(s+a-1)\cdots(a+1)\Gamma(a+1)\\ &= (s+a+1)(s+a)\cdots(a+2)(a+1)\frac{1}{s+a+1}\Gamma(a+1).\end{aligned} \tag{23}$$

If we accept the expression (20) for $\phi(\mathcal{I})$ and use (21), we obtain

$$\begin{aligned}\int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^r}\, d\mathcal{I} &= \int_0^{+\infty} e^{-\mathcal{I}\frac{b}{c^2}} \left(1+\frac{\mathcal{I}}{c^2}\right)^{m-r} \mathcal{I}^a\, d\mathcal{I} \\ &= \sum_{s=0}^{m-r} \begin{pmatrix} m-r \\ s \end{pmatrix} \frac{1}{c^{2s}} \int_0^{+\infty} e^{-\mathcal{I}\frac{b}{c^2}} \mathcal{I}^{s+a}\, d\mathcal{I} \\ &= \sum_{s=0}^{m-r} \begin{pmatrix} m-r \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \quad \Gamma(s+a+1)\,.\end{aligned} \tag{24}$$

Let us conclude this subsection by determining the triple tensor; from eqs. $(5)_3$ and (10) we deduce

$$\begin{aligned}A^{\alpha\beta\gamma} =& c \int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[\lambda+\left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]} \left(1+2\frac{\mathcal{I}}{c^2}\right) p^\alpha p^\beta p^\gamma \varphi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} \\ =& A_1^0 \left(U^\alpha U^\beta U^\gamma + c^2 h^{(\alpha\beta} U^{\gamma)}\right),\end{aligned} \tag{25}$$

where the last expression is suggested by the Representation Theorems [32, 33] and from the propery $A^{\alpha\beta}{}_\beta = 0$.

By contracting it with $U_\alpha U_\beta U_\gamma$, we obtain

$$A_1^0 = \frac{1}{c^5} \int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[\lambda+\left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]} \left(1+2\frac{\mathcal{I}}{c^2}\right) (U_\alpha p^\alpha)^3\, \varphi(\mathcal{I})\, d\vec{P}\, d\mathcal{I} \tag{26}$$

$$\begin{aligned}&= \frac{4\pi}{c^2} e^{-1-\frac{\lambda}{k_B}} \int_0^{+\infty}\int_0^{+\infty} e^{-\frac{c}{k_B T}\left(1+\frac{\mathcal{I}}{c^2}\right)p^0} \left(1+2\frac{\mathcal{I}}{c^2}\right) \left(p^0\right)^4 \varphi(\mathcal{I})\, d p^0\, d\mathcal{I} \\ &\overset{*}{=} \frac{4\pi}{c^2} e^{-1-\frac{\lambda}{k_B}} \cdot (4!) \left(\frac{k_B T}{c}\right)^5 \int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^5} \left(1+2\frac{\mathcal{I}}{c^2}\right) d\mathcal{I} \\ &= \frac{4\pi}{c^2} e^{-1-\frac{\lambda}{k_B}} \cdot (4!) \left(\frac{k_B T}{c}\right)^5 \int_0^{+\infty} \left(2\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^4} - \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^5}\right) d\mathcal{I} \\ &\overset{**}{=} \frac{4\pi}{c^2} e^{-1-\frac{\lambda}{k_B}} \cdot (4!) \left(\frac{k_B T}{c}\right)^5 \left[2\sum_{s=0}^{m-4} \begin{pmatrix} m-4 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \quad \Gamma(s+a+1)\right.\end{aligned}$$

$$- \sum_{s=0}^{m-5} \left(\begin{array}{c} m-5 \\ s \end{array} \right) \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \quad \Gamma(s+a+1) \Bigg] ,$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (14), while in that denoted by $\stackrel{**}{=}$ we have used (24).

By using (15) we can rewrite this as

$$\frac{A_1^0}{n} = 12 \left(\frac{k_B T}{c^2}\right)^2 \frac{2\sum_{s=0}^{m-4} \left(\begin{array}{c} m-4 \\ s \end{array} \right) \frac{\Gamma(s+a+1)}{b^s} - \sum_{s=0}^{m-5} \left(\begin{array}{c} m-5 \\ s \end{array} \right) \frac{\Gamma(s+a+1)}{b^s}}{\sum_{s=0}^{m-3} \left(\begin{array}{c} m-3 \\ s \end{array} \right) \frac{\Gamma(s+a+1)}{b^s}} \tag{27}$$

$$= 12 \left(\frac{k_B T}{c^2}\right)^2 \frac{1 + 2\sum_{s=1}^{m-4} \left(\begin{array}{c} m-4 \\ s \end{array} \right) \frac{1}{b^s} \frac{\Gamma(s+a+1)}{\Gamma(a+1)} - \sum_{s=1}^{m-5} \left(\begin{array}{c} m-5 \\ s \end{array} \right) \frac{1}{b^s} \frac{\Gamma(s+a+1)}{\Gamma(a+1)}}{1 + \sum_{s=1}^{m-3} \left(\begin{array}{c} m-3 \\ s \end{array} \right) \frac{1}{b^s} \frac{\Gamma(s+a+1)}{\Gamma(a+1)}}$$

But, thanks to eq. (23), this expression can be rewritten as

$$A_1^0 = 12n \left(\frac{k_B T}{c^2}\right)^2 \cdot \tag{28}$$

$$\left[1 + (a+1)\sum_{s=1}^{m-3} \left(\begin{array}{c} m-3 \\ s \end{array} \right) \frac{1}{b^s} (s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} \right]^{-1} \cdot$$

$$\left[1 + 2(a+1)\sum_{s=1}^{m-4} \left(\begin{array}{c} m-4 \\ s \end{array} \right) \frac{1}{b^s} (s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} - \right.$$

$$\left. (a+1)\sum_{s=1}^{m-5} \left(\begin{array}{c} m-5 \\ s \end{array} \right) \frac{1}{b^s} (s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} \right] .$$

Finally, from eq. (25) and from the definition $A = \frac{1}{n^3 c^6} A^{\alpha\beta\gamma} U_\alpha U_\beta U_\gamma$, we find the expression of the variable A at equilibrium, that is

$$A_E = \frac{1}{n^3} A_1^0 .$$

3. THE FIRST ORDER APPROXIMATION

From eq. (8) it follows

$$f - f_E = \frac{-1}{k_B} e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]}$$

$$\times \left[\lambda - \lambda_E + \left(1 + \frac{\mathcal{I}}{c^2}\right)\left(\lambda_\mu - \frac{U_\mu}{T}\right) p^\mu + \left(1 + \frac{2\mathcal{I}}{c^2}\right)\Sigma_{\mu\nu} p^\mu p^\nu\right] . \quad (29)$$

By substituting it in (5) and multiplying all the equations by $-k_B$, we obtain

$$V_E^\alpha(\lambda - \lambda_E) + T_E^{\alpha\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) + A_E^{\alpha\mu\nu}\Sigma_{\mu\nu} = 0\,, \quad (30)$$

$$T_E^{\alpha\beta}(\lambda - \lambda_E) + A_{11}^{\alpha\beta\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) + A_{12}^{\alpha\beta\mu\nu}\Sigma_{\mu\nu} = = -k_B\left(t^{<\alpha\beta>_3} + \frac{2}{c^2}\,U^{(\alpha} q^{\beta)}\right),$$

$$A_E^{\alpha\beta\gamma}(\lambda - \lambda_E) + A_{12}^{\alpha\beta\gamma\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) + A_{22}^{\alpha\beta\gamma\mu\nu}\Sigma_{\mu\nu} = -k_B(A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma})\,,$$

where we have called

$$A_{11}^{\alpha\beta\mu} = \int_{\Re^3}\int_0^{+\infty} c e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]}\left(1 + \frac{\mathcal{I}}{c^2}\right)^2 \times p^\alpha p^\beta p^\mu \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,, \quad (31)$$

$$A_{12}^{\alpha\beta\mu\nu} = \int_{\Re^3}\int_0^{+\infty} c e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]}\left(1 + \frac{\mathcal{I}}{c^2}\right)\left(1 + 2\frac{\mathcal{I}}{c^2}\right) \times p^\alpha p^\beta p^\mu p^\nu \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,$$

$$A_{22}^{\alpha\beta\gamma\mu\nu} = \int_{\Re^3}\int_0^{+\infty} c e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]}\left(1 + 2\frac{\mathcal{I}}{c^2}\right)^2 \times p^\alpha p^\beta p^\gamma p^\mu p^\nu \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\,,$$

and we have taken into account the decomposition (2.7) of [26], that is, $T^{\alpha\beta} - T_E^{\alpha\beta} = t^{<\alpha\beta>_3} + \frac{2}{c^2}\,U^{(\alpha} q^{\beta)}$.

This is a consequence of the definitions

$$t^{<\alpha\beta>_3} = T^{\mu\nu}\left(h_\mu^\alpha h_\nu^\beta - \frac{1}{3}h^{\alpha\beta}h^{\mu\nu}\right) \quad (32)$$

$$q^\alpha = -\, h_\mu^\alpha U_\nu T^{\mu\nu}\,,$$

$$e = \frac{1}{c^2} U_\mu U_\nu T^{\mu\nu} = 3p,$$

which are equivalent to

$$T^{\alpha\beta} = t^{<\alpha\beta>_3} + \tfrac{2}{c^2}\,U^{(\alpha} q^{\beta)} + 3p\left(\tfrac{U^\alpha U^\beta}{c^2} + \tfrac{1}{3}h^{\alpha\beta}\right) \quad (33)$$

and to the fact that we have ${T^\alpha}_\alpha = 0$ in the present case of an ultrarelativistic gas.

It may appear unsatisfactory the fact that in this model the dynamic pressure seems not present; but we recall that we have taken as independent variable also

$$A = \frac{1}{n^3 c^6} A^{\alpha\beta\gamma} U_\alpha U_\beta U_\gamma \,. \tag{34}$$

Well, from eqs. (48) and (56) of [25] it follows

$$A = \frac{1}{n^3} A_1^0 - \frac{3}{n^3 c^2} \frac{N_1^\pi}{D_1^\pi} \, \pi \,, \tag{35}$$

so that $A - A_E$ is a linear homogeneous function of the dynamic pressure π; in this way it plays a role also in the present theory.

It is convenient now to determine the expression of the new tensors $A_{11}^{\alpha\beta\mu}$, $A_{12}^{\alpha\beta\mu\nu}$, $A_{22}^{\alpha\beta\gamma\mu\nu}$. From the Representation Theorems and from the fact that they are symmetric and traceless, it follows that they have the following expressions

$$A_{11}^{\alpha\beta\mu} = B_5 \left(U^\alpha U^\beta U^\mu + c^2 h^{(\alpha\beta} U^{\mu)} \right) \,, \tag{36}$$
$$A_{12}^{\alpha\beta\mu\nu} = B_3 \left(\frac{1}{5} c^4 h^{(\alpha\beta} h^{\mu\nu)} + 2c^2 h^{(\alpha\beta} U^\mu U^{\nu)} + U^\alpha U^\beta U^\mu U^\nu \right) \,,$$
$$A_{22}^{\alpha\beta\gamma\mu\nu} = B_8 \left(c^4 h^{(\alpha\beta} h^{\mu\nu} U^{\nu)} + \frac{10}{3} c^2 h^{(\alpha\beta} U^\gamma U^\mu U^{\nu)} + U^\alpha U^\beta U^\gamma U^\mu U^\nu \right) \,,$$

(We preferred to use the same names of [25]). We have now to deduce the values of the scalars B_5, B_3, B_8.

By comparing (31) with (36) and after that contracting them with $U_\alpha U_\beta U_\mu$, $U_\alpha U_\beta U_\mu U_\nu$, $U_\alpha U_\beta U_\gamma U_\mu U_\nu$, respectively, we obtain

$$B_5 = \frac{1}{c^5} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]} \left(1 + \frac{\mathcal{I}}{c^2}\right)^2 (U_\alpha p^\alpha)^3 \, \varphi(\mathcal{I}) \, d\vec{P} \, d\mathcal{I} \,, \tag{37}$$
$$B_3 = \frac{1}{c^7} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]} \left(1 + \frac{\mathcal{I}}{c^2}\right)\left(1 + 2\frac{\mathcal{I}}{c^2}\right) (U_\alpha p^\alpha)^4 \, \varphi(\mathcal{I}) \, d\vec{P} \, d\mathcal{I} \,,$$
$$B_8 = \frac{1}{c^9} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[\lambda + \left(1+\frac{\mathcal{I}}{c^2}\right)\lambda_\beta p^\beta\right]} \left(1 + 2\frac{\mathcal{I}}{c^2}\right)^2 (U_\alpha p^\alpha)^5 \, \varphi(\mathcal{I}) \, d\vec{P} \, d\mathcal{I} \,,$$

and, by calculating the integrals,

$$B_5 = \frac{4\pi}{c^2} e^{-1-\frac{\lambda}{k_B}} \cdot (4!) \left(\frac{k_B T}{c}\right)^5 \int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1 + \frac{\mathcal{I}}{c^2}\right)^3} \, d\mathcal{I} \stackrel{*}{=} 12n \left(\frac{k_B T}{c}\right)^2 \,. \tag{38}$$

where in the passage denoted with $\overset{*}{=}$ we have used (15). (Note that these calculations are similar to those used to determine A_1^0).

Similarly, we have

$$B_3 = \frac{4\pi}{c^3} e^{-1-\frac{\lambda}{k_B}} \int_0^{+\infty}\int_0^{+\infty} e^{-\frac{c}{k_B T}\left(1+\frac{\mathcal{I}}{c^2}\right)p^0} \left(p^0\right)^5 \left(1+\frac{\mathcal{I}}{c^2}\right)\left(1+2\frac{\mathcal{I}}{c^2}\right)\varphi(\mathcal{I})\, d\,p^0\, d\mathcal{I} = \tag{39}$$

$$\overset{*}{=} \frac{4\pi}{c^3} e^{-1-\frac{\lambda}{k_B}} \cdot (5!) \left(\frac{k_B T}{c}\right)^6 \int_0^{+\infty} \frac{\phi(\mathcal{I})\left(1+2\frac{\mathcal{I}}{c^2}\right)}{\left(1+\frac{\mathcal{I}}{c^2}\right)^5}\, d\mathcal{I} =$$

$$= \frac{4\pi}{c^3} e^{-1-\frac{\lambda}{k_B}} \cdot (5!) \left(\frac{k_B T}{c}\right)^6 \int_0^{+\infty} \left(2\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^4} - \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^5}\right) d\mathcal{I} =$$

$$\overset{**}{=} \frac{4\pi}{c^3} e^{-1-\frac{\lambda}{k_B}} \cdot (5!) \left(\frac{k_B T}{c}\right)^6 \left[2\sum_{s=0}^{m-4}\begin{pmatrix} m-4 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \Gamma(s+a+1) - \sum_{s=0}^{m-5}\begin{pmatrix} m-5 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \Gamma(s+a+1)\right],$$

where in the passage denoted with $\overset{*}{=}$ we have used (14), while in that denoted by $\overset{**}{=}$ we have used (24).

By using (15) and (23) we can rewrite this as

$$B_3 = 60n\left(\tfrac{k_B T}{c^2}\right)^3 \cdot \tag{40}$$

$$\left[1+(a+1)\textstyle\sum_{s=1}^{m-3}\begin{pmatrix} m-3 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1}\right]^{-1} \cdot$$

$$\left[1+2(a+1)\textstyle\sum_{s=1}^{m-4}\begin{pmatrix} m-4 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} - (a+1)\textstyle\sum_{s=1}^{m-5}\begin{pmatrix} m-5 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1}\right].$$

Finally, we have

$$B_8 = \frac{4\pi}{c^4} e^{-1-\frac{\lambda}{k_B}} \int_0^{+\infty}\int_0^{+\infty} e^{-\frac{c}{k_B T}\left(1+\frac{\mathcal{I}}{c^2}\right)p^0} \left(p^0\right)^6 \left(1+2\frac{\mathcal{I}}{c^2}\right)^2 \varphi(\mathcal{I})\, d\,p^0\, d\mathcal{I} \tag{41}$$

$$\overset{*}{=} \frac{4\pi}{c^4} e^{-1-\frac{\lambda}{k_B}} \cdot (6!) \left(\frac{k_B T}{c}\right)^7 \int_0^{+\infty} \frac{\phi(\mathcal{I})\left(1+2\frac{\mathcal{I}}{c^2}\right)^2}{\left(1+\frac{\mathcal{I}}{c^2}\right)^7}\, d\mathcal{I}$$

$$= \frac{4\pi}{c^4} e^{-1-\frac{\lambda}{k_B}} \cdot (6!) \left(\frac{k_B T}{c}\right)^7$$

$$\int_0^{+\infty} \left(4\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^5} - 4\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^6} + \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^7}\right) d\mathcal{I}$$

$$\stackrel{**}{=} \frac{4\pi}{c^4} e^{-1-\frac{\lambda}{k_B}} \cdot (6!) \left(\frac{k_B T}{c}\right)^7 \left[4\sum_{s=0}^{m-5} \begin{pmatrix} m-5 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}}\right.$$

$$\Gamma(s+a+1) - 4\sum_{s=0}^{m-6} \begin{pmatrix} m-6 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \quad \Gamma(s+a+1) +$$

$$\left.\sum_{s=0}^{m-7} \begin{pmatrix} m-7 \\ s \end{pmatrix} \frac{1}{c^{2s}} \frac{1}{\left(\frac{b}{c^2}\right)^{s+a+1}} \quad \Gamma(s+a+1)\right],$$

where in the passage denoted with $\stackrel{*}{=}$ we have used (14), while in that denoted by $\stackrel{**}{=}$ we have used (24).

By using (15) and (23) we can rewrite this as

$$B_8 = 360n \left(\frac{k_B T}{c^2}\right)^4 \cdot \tag{42}$$

$$\left[1 + (a+1)\sum_{s=1}^{m-3} \begin{pmatrix} m-3 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1}\right]^{-1} \cdot$$

$$\left[1 + 4(a+1)\sum_{s=1}^{m-5} \begin{pmatrix} m-5 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} - \right.$$

$$4(a+1)\sum_{s=1}^{m-6} \begin{pmatrix} m-6 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1} +$$

$$\left.(a+1)\sum_{s=1}^{m-7} \begin{pmatrix} m-7 \\ s \end{pmatrix} \frac{1}{b^s}(s+a+1)(s+a)\cdots(a+2)\frac{1}{s+a+1}\right].$$

We note here the reason to choose $m \geq 7$.

3.1. THE SOLUTION OF SYSTEM (30)

Coming back to the system (30), we note that it is equivalent to the following ones:

$$U_\alpha V_E^\alpha(\lambda - \lambda_E) + U_\alpha T_E^{\alpha\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) + U_\alpha A_E^{\alpha\mu\nu}\Sigma_{\mu\nu} = 0\,, \tag{43}$$

$$U_\alpha U_\beta T_E^{\alpha\beta}(\lambda - \lambda_E) + U_\alpha U_\beta A_{11}^{\alpha\beta\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) + U_\alpha U_\beta A_{12}^{\alpha\beta\mu\nu}\Sigma_{\mu\nu} = 0\,,$$

$$U_\alpha U_\beta U_\gamma A_E^{\alpha\beta\gamma}(\lambda-\lambda_E)+U_\alpha U_\beta U_\gamma A_{12}^{\alpha\beta\gamma\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)+U_\alpha U_\beta U_\gamma A_{22}^{\alpha\beta\gamma\mu\nu}\Sigma_{\mu\nu}$$
$$=-k_B n^3 c^6 \Delta A\,,$$
$$h^\delta_\alpha T_E^{\alpha\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)+h^\delta_\alpha A_E^{\alpha\mu\nu}\Sigma_{\mu\nu}=0\,, \tag{44}$$
$$h^\delta_\alpha U_\beta A_{11}^{\alpha\beta\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)+h^\delta_\alpha U_\beta A_{12}^{\alpha\beta\mu\nu}\Sigma_{\mu\nu}$$
$$=k_B\, q^\delta\,,$$
$$U_\alpha U_\beta h^\delta_\gamma A_{12}^{\alpha\beta\gamma\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)+U_\alpha U_\beta h^\delta_\gamma A_{22}^{\alpha\beta\gamma\mu\nu}\Sigma_{\mu\nu}=-k_B\, U_\alpha U_\beta h^\delta_\gamma(A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma})\,,$$
$$h_\alpha^{<\delta}h_\beta^{\vartheta>_3}A_{12}^{\alpha\beta\mu\nu}\Sigma_{\mu\nu}=-k_B\, t^{<\delta\vartheta>_3}\,, \tag{45}$$
$$U_\alpha h_\beta^{<\delta}h_\gamma^{\vartheta>_3}A_{22}^{\alpha\beta\gamma\mu\nu}\Sigma_{\mu\nu}=-k_B\, U_\alpha h_\beta^{<\delta}h_\gamma^{\vartheta>_3}(A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma})\,,$$
$$0=h_\alpha^{<\delta}h_\beta^{\vartheta}h_\gamma^{\varphi>_3}(A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma})\,, \tag{46}$$

where in the last equation the 3-dimensional traceless part of a tensor is used; for example, for any given tensor $S^{(\delta\vartheta\varphi)}$ this is defined by

$$S^{<\delta\vartheta\varphi>_3}=S^{(\delta\vartheta\varphi)}-\frac{3}{5}h^{(\delta\vartheta}S^{\varphi)\mu\nu}h_{\mu\nu}$$
$$\text{from which it follows}\quad h^{<\delta\vartheta}h^{\phi>_3\psi}=0\,. \tag{47}$$

Moreover, in the right hand side of eq. $(44)_2$ the sign has been changed because

$$h^\delta_\alpha q^\alpha=-g^\delta_\alpha q^\alpha=-q^\delta.$$

The equivalence of eq. (30) with the system (43), (44), (45), (46) is evident. In fact, $(30)_1$ is equivalent to 2 equations obtained by contracting it respectively with U_α and h^δ_α, that is, eqs. $(43)_1$ and $(44)_1$.

Similarly, $(30)_2$ is equivalent to 3 equations obtained by contracting it respectively with $U_\alpha U_\beta$, $h^\delta_\alpha U_\beta$, $h_\alpha^{<\delta}h_\beta^{\vartheta>_3}$, that is, eqs. $(43)_2$, $(44)_2$ and $(45)_1$. (Contraction of $(30)_2$ with $h_{\alpha\beta}$, gives again $(43)_2$ because $h_{\alpha\beta}=-g_{\alpha\beta}+U_\alpha U_\beta\frac{1}{c^2}$ and $(30)_2$ has zero trace).

Finally, $(30)_3$ is equivalent to equations obtained by contracting it respectively with $U_\alpha U_\beta U_\gamma$, $U_\alpha U_\beta h^\delta_\gamma$, $U_\alpha h^\vartheta_\beta h^\varphi_\gamma$ (which can be further split in its trace and in its 3-dimensional traceless part; but the first of these is equivalent to $(43)_3$ because $(30)_3$ has zero trace) and $h^\delta_\alpha h^\vartheta_\beta h^\varphi_\gamma$ (which also can be further split in its trace and in its 3-dimensional traceless part; also here the first of these is equivalent to $(44)_3$ because $(30)_3$ has zero trace). The resulting equations are just $(43)_3$, $(44)_3$, $(45)_2$ and (46).

- Now we are ready to impose the system (43)-(46). Let us begin with $(43)_{1-3}$, even if it is useful only to determine the distribution function, while it isn' t influent for the determination of $A^{\alpha\beta\gamma}$.

It can be written as

$$\begin{gathered} nc^2(\lambda-\lambda_E)+eU^\mu\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+\tfrac{4}{3}c^2A_1^0U^\mu U^\nu\Sigma_{\mu\nu}=0\,, \\ ec^2(\lambda-\lambda_E)+A_1^0c^4U^\nu\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+\tfrac{4}{3}B_3c^4U^\mu U^\nu\Sigma_{\mu\nu}=0\,, \\ c^6A_1^0(\lambda-\lambda_E)+c^6B_3U^\mu\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+B_8c^6U^\mu U^\nu\Sigma_{\mu\nu}=-k_Bn^3c^6\Delta A\,, \end{gathered} \tag{48}$$

from which we obtain

$$\begin{gathered} \lambda-\lambda_E=-\frac{1}{D_1}n^3c^6k_B\Delta A\begin{vmatrix} e & \frac{4}{3}A_1^0c^2 \\ A_1^0c^4 & \frac{4}{3}B_3c^4 \end{vmatrix}, \\ U^\mu\left(\lambda_\mu-\frac{U_\mu}{T}\right)=\frac{1}{D_1}n^3c^6k_B\Delta A\begin{vmatrix} nc^2 & \frac{4}{3}A_1^0c^2 \\ ec^2 & \frac{4}{3}B_3c^4 \end{vmatrix}, \\ U^\mu U^\nu\Sigma_{\mu\nu}=-\frac{1}{D_1}n^3c^6k_B\Delta A\begin{vmatrix} nc^2 & e \\ ec^2 & A_1^0c^4 \end{vmatrix}, \end{gathered} \tag{49}$$

with

$$D_1=\begin{vmatrix} nc^2 & e & \frac{4}{3}A_1^0c^2 \\ ec^2 & A_1^0c^4 & \frac{4}{3}B_3c^4 \\ A_1^0c^6 & B_3c^6 & B_8c^6 \end{vmatrix}. \tag{50}$$

The system (44) can be written as

$$\begin{gathered} \tfrac{1}{3}eh^{\delta\mu}\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+\tfrac{2}{3}A_1^0c^2h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=0\,, \\ \tfrac{1}{3}B_5c^4h^{\delta\mu}\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+\tfrac{2}{3}B_3c^4h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=-k_B\,q^\delta\,, \\ \tfrac{1}{3}c^6B_3h^{\delta\mu}\left(\lambda_\mu-\tfrac{U_\mu}{T}\right)+\tfrac{2}{3}B_8c^6h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=k_B\,U_\alpha U_\beta h^\delta_\gamma(A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma})\,. \end{gathered} \tag{51}$$

From the first two of these equations we obtain

$$h^{\delta\mu}\left(\lambda_\mu - \frac{U_\mu}{T}\right) = \frac{2}{3D_3} A_1^0 c^2 k_B\, q^\delta\,, \qquad (52)$$
$$h^{\delta\mu} U^\nu \Sigma_{\mu\nu} = \frac{-1}{3D_3} e\, k_B\, q^\delta\,,$$

with

$$D_3 = \begin{vmatrix} p & \frac{2}{3} A_1^0 c^2 \\ \frac{1}{3} B_5 c^4 & \frac{2}{3} B_3 c^4 \end{vmatrix}\,. \qquad (53)$$

By substituting them in the third equation, we obtain

$$U_\alpha U_\beta h^\delta_\gamma (A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}) = -\frac{q^\delta}{D_3}\begin{vmatrix} \frac{1}{3} e & \frac{2}{3} A_1^0 c^2 \\ \frac{1}{3} B_3 c^6 & \frac{2}{3} B_8 c^6 \end{vmatrix}\,. \qquad (54)$$

The system (45) can be written as

$$\tfrac{2}{15} B_3 c^4 h^{\mu<\delta} h^{\vartheta>_3\nu} \Sigma_{\mu\nu} = -k_B\, t^{<\delta\vartheta>_3}\,, \qquad (55)$$
$$\tfrac{2}{15} B_8 c^6 h^{\mu<\delta} h^{\vartheta>_3\nu} \Sigma_{\mu\nu} = -k_B\, U_\alpha h_\beta^{<\delta} h_\gamma^{\vartheta>_3} (A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma})\,,$$

from which we deduce

$$h^{\mu<\delta} h^{\vartheta>_3\nu} \Sigma_{\mu\nu} = -\tfrac{15}{2} \tfrac{k_B}{B_3 c^4}\, t^{<\delta\vartheta>_3}\,, \qquad (56)$$
$$U_\alpha h_\beta^{<\delta} h_\gamma^{\vartheta>_3} (A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}) = \tfrac{B_8}{B_3}\, c^2 t^{<\delta\vartheta>_3}\,.$$

We can use now all these results and see that, from (54), $(56)_2$, (34) and (46) it follows

$$A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma} = 3\frac{\begin{vmatrix} p & \frac{2}{3} A_1^0 c^2 \\ \frac{1}{3} B_3 c^6 & \frac{2}{3} B_8 c^6 \end{vmatrix}}{D_3 c^2}\left(\tfrac{1}{5} h^{(\alpha\beta} q^{\gamma)} + \tfrac{1}{c^2} q^{(\alpha} U^\beta U^{\gamma)}\right) + \qquad (57)$$
$$3\tfrac{B_8}{B_3}\, t^{(<\alpha\beta>_3} U^{\gamma)} + n^3 \Delta A \left(c^2 h^{(\alpha\beta} U^{\gamma)} + U^\alpha U^\beta U^\gamma\right)\,.$$

4. THE MONATOMIC LIMIT

The constitutive functions of the monatomic case can be obtained by taking the limit of the present one for $a \to -1$. Well, from eq. (28) it easily follows that

$$\lim_{a\to -1} A_1^0 = 12n \left(\frac{k_B T}{c^2}\right)^2 . \tag{58}$$

So we recover in this limit the result (5.5) of [26] (Se also the second line of its page 284), taking into account that the comparison of (5.4) of [26] and of the present (25) gives $C_0^1 = 2A_1^0$.

Similarly, from eqs. (40), (42) we obtain respectively

$$\lim_{a\to -1} B_3 = 60n \left(\frac{k_B T}{c^2}\right)^3 \quad ; \quad \lim_{a\to -1} B_8 = 360n \left(\frac{k_B T}{c^2}\right)^4 . \tag{59}$$

Moreover, we have eq. (38). So we can now compare eq. (57) with (6.8) of [26].

- The coefficient of ΔA is the same (We have to take into account that in (6.8) of [26] we find reported the expression of $\tilde{A}^{\alpha\beta\gamma}$; thanks to $(1.3)_1$ of the same article, we have to consider its sum and of $An^3U^\alpha U^\beta U^\gamma$ to obtain $A^{\alpha\beta\gamma}$).

- The coefficient of $t^{(<\alpha\beta>_3 U^{\gamma})}$ is here $3\frac{B_8}{B_3}$ which, thanks to (59) yields

$$\lim_{a\to -1} 3\frac{B_8}{B_3} = 18\frac{k_B T}{c^2} , \tag{60}$$

  just as in (6.8) of [26]; (We have to take into account that in this equation of [26] there is a printing mistake: The last q_γ has to be substituted by U_γ.)

- The coefficient of $-h^{(\alpha\beta}q^{\gamma)} - \frac{5}{c^2}q^{(\alpha}U^\beta U^{\gamma)}$ is here

$$-\frac{3}{5}\frac{1}{c^2}\frac{\begin{vmatrix} \frac{1}{3}e & \frac{2}{3}A_1^0 c^2 \\ \frac{1}{3}B_3 c^6 & \frac{2}{3}B_8 c^6 \end{vmatrix}}{\begin{vmatrix} \frac{1}{3}e & \frac{2}{3}A_1^0 c^2 \\ \frac{1}{3}B_5 c^4 & \frac{2}{3}B_3 c^4 \end{vmatrix}} . \tag{61}$$

By using $e = 3p$, (19), (58), $(38)_2$ and (59), we see that the limit of this coefficient for $a \to -1$ is $-6\frac{k_B T}{c^2}$; this is the same value in (6.8) of [26] only if the arbitrary function $A_2(\xi_E)$ appearing in this article is zero. But this function arised in (4.8) of [26] from an integration and it is well known that in the kinetic approach this function is zero.

CONCLUSIONS

The field equations have been obtained for the physical problem of mass-less particles in the relativistic context. They constitute a set of hyperbolic partial differential equations which predict finite speeds of propagation of shocks wave. Although the method used is the same of [25], an independent investigation was needed. As a bonus, some problems remained open in [25], have been here solved such as the problem of integrability of some integrals; here this property was not only proved, but the integrals have been also explicitly calculated. Object of a future research may be to take the limit of the results in [25] for $\gamma \to 0$ and to compare the results with the present ones.

A. INTEGRABILITY CONSIDERATIONS

We have seen in (15), (39) and similar equations, that the integrals

$$\int_0^{+\infty} \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} \, d\mathcal{I}$$

have to be considered. Now, we see that it is not possible that $\phi(\mathcal{I}) = \mathcal{I}^a$. In fact, when $a > m-1$ we can choose $\beta \in]m-a\,, 1[$ and have

$$\lim_{\mathcal{I}\to+\infty} \frac{\phi(\mathcal{I})\mathcal{I}^\beta}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} = +\infty \,. \tag{62}$$

This means that,

$$\forall H > 0\,,\ \exists \mathcal{I}^* \ :\ \text{for}\, \mathcal{I} > \mathcal{I}^* \text{ we have } \frac{\phi(\mathcal{I})\mathcal{I}^\beta}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} > H\,, \text{ that is} \tag{63}$$

$$\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} > \frac{H}{\mathcal{I}^\beta}\,, \quad \text{so that} \quad \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} \tag{64}$$

is greater than a divergent function for $\mathcal{I}$ going to infinity. Consequently, it is not integrable.

Instead of this, integrability is assured if $\phi(\mathcal{I}) = \mathcal{I}^a e^{-\mathcal{I}\frac{b}{c^2}} \psi\left(\frac{\mathcal{I}}{c^2}\right)$ with b a positive number and $\psi\left(\frac{\mathcal{I}}{c^2}\right)$ a polynomial function such that $\psi(0) = 1$.

In fact, for every number β (also greater than 1), we have

$$\lim_{\mathcal{I}\to+\infty} \frac{\phi(\mathcal{I})\mathcal{I}^\beta}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} = \lim_{\mathcal{I}\to+\infty} e^{-\mathcal{I}\frac{b}{c^2}} \psi\left(\frac{\mathcal{I}}{c^2}\right) \frac{\mathcal{I}^{a+\beta-m}}{\left(\frac{1}{\mathcal{I}}+\frac{1}{c^2}\right)^m} = 0\,. \tag{65}$$

This means that,

$$\forall\varepsilon > 0\,,\ \exists\mathcal{I}^*\ :\ \text{for}\,\mathcal{I} > \mathcal{I}^*\ \text{we have}\ \frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m} < \frac{\varepsilon}{\mathcal{I}^\beta}\,. \tag{66}$$

It follows that $\frac{\phi(\mathcal{I})}{\left(1+\frac{\mathcal{I}}{c^2}\right)^m}$ is integrable because it is smaller than an integrable function.

B. PROPERTY OF THE GAMMA FUNCTION

We want here to show simply the following property of the Gamma function

$$\Gamma(a+1) = \frac{1}{a+2}\frac{1}{a+1}[(a+2)(a+1)(a)(a-1)\cdots(a+2-[a+1])] \int_0^{+\infty} e^{-\mathcal{J}}\mathcal{J}^{a+1-[a+1]}\,d\mathcal{J}\,. \tag{67}$$

(We don' t simplify here $(a+2)$ and $(a+1)$ so that the first factor in square brackets is greater than the last one).

Proof. Let us prove (67) with the iterative procedure.

- If $a \in]-1\,,0[$ then the last factor in square brackets is equal to $(a+2)$, so that in the square brackets there is only $(a+2)$ and (67) becomes

$$\Gamma(a+1) = \frac{1}{a+1}\int_0^{+\infty} e^{-\mathcal{J}}\mathcal{J}^{a+1}\,d\mathcal{J}$$
$$= \frac{1}{a+1}\left|-e^{-\mathcal{J}}\mathcal{J}^{a+1}\right|_0^{+\infty} + \int_0^{+\infty} e^{-\mathcal{J}}\mathcal{J}^{a}\,d\mathcal{J} \tag{68}$$

which coincides with the value given by (22). There is only to verify the integrability of the function $e^{-x}x^a$. To this end, let us take whatever number β (also greater than 1), and have

$$\lim_{x\to+\infty} e^{-x}x^{a+\beta} = 0\,. \tag{69}$$

This means that,

$$\forall\varepsilon > 0\,,\ \exists x^*\ :\ \text{for}\, x > x^*\,\text{we have}\quad e^{-x}x^a < \frac{\varepsilon}{x^\beta}\,. \tag{70}$$

It follows that $e^{-x}x^a$ is integrable because it is smaller than an integrable function.

- If $a = 0$, then the last factor in square brackets is equal to 1, so that (67) becomes

$$\Gamma(1) = \int_0^{+\infty} e^{-\mathcal{J}}\, d\,\mathcal{J} = \left|-e^{-\mathcal{J}}\right|_0^{+\infty} = 1\,. \tag{71}$$

This is the same result given by (22).

- Let us now suppose, for the iterative procedure, that (67) holds when $a \in]h\,,\, h+1]$ with $h \geq -1$ and let us prove it for the case when $a \in]h+1\,,\, h+2]$. From (22) we have

$$\begin{aligned}\Gamma(a+1) =& \left|-e^{-x}x^a\right|_0^{+\infty} + a\int_0^{+\infty} e^{-x}x^{a-1}\, d\,x \\ =& a\,\Gamma(a) \\ =& a\,\frac{1}{a+1}\frac{1}{a}[(a+1)(a)(a-1)(a-2)\cdots(a+1-[a])] \\ & \int_0^{+\infty} e^{-\mathcal{J}}\,\mathcal{J}^{a-[a]}\, d\,\mathcal{J}\,,\end{aligned} \tag{72}$$

where in the last two passages we have used (22) and (67) respectively. the result is equal to (67) because $a\,\frac{1}{a} = \frac{1}{a+2}\,(a+2)$, $a+1-[a] = a+2-[a+1]$, $a-[a] = a+1-[a+1]$. This completes the proof.

We note that the recurrence relation $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ is now a consequence of eq. (67) because for $a > 1$ we have $(a-1)\,\frac{1}{a-1} = \frac{1}{a+1}\,(a+1)$, $a-[a-1] = a+1-[a]$, $a-1-[a-1] = a-[a]$.

Similarly, from (67) and (71) it follows that, if a is an integer number, we have $\Gamma(a+1) = a!$ from which the above written expression $\Gamma(\alpha) = (\alpha-1)!$. $\square$

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