Communications in Applied Analysis, **21**, No. 3 (2017), 405-424 ISSN: 1083-2564

ON THE SOLUTION OF EQUATIONS AND APPLICATIONS ON BANACH SPACE VALUED FUNCTIONS AND FRACTIONAL VECTOR CALCULI

GEORGE A. ANASTASSIOU[1] AND IOANNIS K. ARGYROS[2]

[1]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, USA

[2]Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA

ABSTRACT: The aim of this paper is to solve equations on Banach space using iterative methods under generalized conditions. The differentiability of the operator involved is not assumed and its domain is not necessarily convex. Several applications are suggested including Banach space valued functions of abstract fractional calculus, where all integrals are of Bochner-type.

AMS Subject Classification: 65G99, 65H10, 26A33, 46B25, 47J25, 47J05
Key Words: iterative method, Banach space, semi-local convergence, fractional calculus, Bochner-type integral

Received: March 19, 2017; **Accepted:** May 30, 2017;
Published: June 2, 2017. **doi:** 10.12732/caa.v21i3.5
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

Let B_1, B_2 denote Banach spaces and let Ω stand for an open subset of B_1.

Let also $U(z,\rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z,\rho)$ denote the closure of $U(z,\rho)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can be brought in a form like

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[16], where $F : \Omega \to B_2$ is a continuous operator. The solution x^* of equation (1.1) is sought in closed form, but this can be achieved only in special cases. That is why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving equation (1.1). We can divide these methods in two categories.

Explicit Methods: Newton's method [6, 7, 11, 15, 16]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot,\cdot;F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [7, 15, 16].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from B_1 into B_2. Other explicit methods can be found in [7], [11], [15], [16] and the references there in.

Implicit Methods: [6, 9, 11, 16]:

$$F(x_n) + A_n(x_{n+1} - x_n) = 0 \tag{1.5}$$

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.6}$$

where $A_n = A(x_{n+1}, x_n) = A(F)(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$. We also write $A(F)(x,x) = A(x,x) = A(x)$ for each $x \in \Omega$.

There is a plethora on local as well as semi-local convergence results for explicit methods [1]-[8], [10]-[16]. However, the research on the convergence of implicit methods has received little attention. Authors, usually consider the fixed point problem

$$P_z(x) = x, \tag{1.7}$$

where

$$P_z(x) = x + F(z) + A(x,z)(x - z) \tag{1.8}$$

or

$$P_z(x) = z - A(x,z)^{-1} F(z) \tag{1.9}$$

for methods (1.5) and (1.6), respectivelly, where $z \in \Omega$ is given. If P is a contraction operator mapping a closed set into itself, then according to the contraction mapping principle [11], [15], [16], P_z has a fixed point x_z^* which can be found using the method of succesive substitutions or Picard's method [16] defined for each fixed n by

$$y_{k+1,n} = P_{x_n}(y_{k,n}), \quad y_{0,n} = x_n, \ \ x_{n+1} = \lim_{k \to +\infty} y_{k,n}. \tag{1.10}$$

Let us also consider the analogous explicit methods

$$F(x_n) + A(x_n, x_n)(x_{n+1} - x_n) = 0 \tag{1.11}$$

$$x_{n+1} = x_n - A(x_n, x_n)^{-1} F(x_n) \tag{1.12}$$

$$F(x_n) + A(x_n, x_{n-1})(x_{n+1} - x_n) = 0 \tag{1.13}$$

and

$$x_{n+1} = x_n - A(x_n, x_{n-1})^{-1} F(x_n). \tag{1.14}$$

In Section 2 of this paper, we present the semi-local convergence of method (1.5) and method (1.6). Section 3 contains the semi-local convergence of method (1.11), method (1.12), method (1.13) and method (1.14). Several applications to Abstract Fractional Calculus are suggested in Section 4 on Banach space valued functions, where all the integrals are of Bochner-type [7, 13].

2. SEMI-LOCAL CONVERGENCE FOR IMPLICIT METHODS

We present the semi-local convergence analysis of method (1.6) using conditions (S):

(s_1) $F : \Omega \subset B_1 \to B_2$ is continuous and $A(x,y) \in \mathcal{L}(B_1, B_2)$ for each $(x,y) \in \Omega \times \Omega$.

(s_2) There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(s_3) There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \to [0,+\infty)$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(x,y)(x-y)\| \leq \beta\psi(\|x-y\|, \|x-x_0\|, \|y-x_0\|)\|x-y\|.$$

(s_4) For each $x \in \Omega_0$ there exists $y \in \Omega_0$ such that

$$y = x - A(y,x)^{-1} F(x).$$

(s_5) For $x_0 \in \Omega_0$ and $x_1 \in \Omega_0$ satisfying (s_4) there exists $\eta \geq 0$ such that

$$\left\| A(x_1,x_0)^{-1} F(x_0) \right\| \leq \eta.$$

(s_6) Define $q(t) := \psi(\eta, t, t)$ for each $t \in [0,+\infty)$. Equation

$$t(1-q(t)) - \eta = 0$$

has positive solutions. Denote by s the smallest such solution.

(s_7) $\overline{U}(x_0, s) \subset \Omega$, where

$$s = \frac{\eta}{1-q_0} \quad \text{and} \quad q_0 = \psi(\eta, s, s).$$

Next, we present the semi-local convergence analysis for method (1.6) using the conditions (S) and the preceding notation.

Theorem 2.1. *Assume that the conditions (S) hold. Then, sequence* $\{x_n\}$ *generated by method (1.6) starting at* $x_0 \in \Omega$ *is well defined in* $U(x_0, s)$*, remains in* $U(x_0, s)$ *for each* $n = 0, 1, 2, ...$ *and converges to a solution* $x^* \in \overline{U}(x_0, s)$ *of equation* $F(x) = 0$*. Moreover, suppose that there exists a continuous and nondecreasing function* $\psi_1 : [0, +\infty)^4 \to [0, +\infty)$ *such that for each* $x, y, z \in \Omega_1$

$$\begin{aligned}&\|F(x) - F(y) - A(z, y)(x - y)\| \\ &\qquad \le \beta\psi_1(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|)\|x - y\|\end{aligned}$$

and $q_1 = \psi_1(\eta, s, s, s) < 1$.

Then, x^* *is the unique solution of equation* $F(x) = 0$ *in* $\overline{U}(x_0, s)$.

Proof. By the definition of s and (s_5), we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on k. Suppose that $\|x_k - x_{k-1}\| \le q_0^{k-1}\eta$ and $\|x_k - x_0\| \le s$.

We get by (1.6), $(s_2) - (s_5)$ in turn that

$$\begin{aligned}\|x_{k+1} - x_k\| &= \left\|A_k^{-1}F(x_k)\right\| \\ &= \left\|A_k^{-1}(F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1}))\right\| \\ &\le \left\|A_k^{-1}\right\| \|F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \\ &\le \beta^{-1}\beta\psi(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|y_k - x_0\|)\|x_k - x_{k-1}\| \\ &\le \psi(\eta, s, s)\|x_k - x_{k-1}\| \\ &= q_0\|x_k - x_{k-1}\| \\ &\le q_0^k\|x_1 - x_0\| \\ &\le q_0^k\eta \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\|x_{k+1} - x_0\| &\le \|x_{k+1} - x_k\| + ... + \|x_1 - x_0\| \\ &\le q_0^k\eta + ... + \eta = \frac{1 - q_0^{k+1}}{1 - q_0}\eta < \frac{\eta}{1 - q_0} = s.\end{aligned}$$

The induction is completed. Moreover, we have by (2.1) that for $m = 0, 1, 2, ...$

$$\|x_{k+m} - x_k\| \le \frac{1 - q_0^m}{1 - q_0}q_0^k\eta.$$

It follows from the preceding inequation that sequence $\{x_k\}$ is complete in a Banach space B_1 and as such it converges to some $x^* \in \overline{U}(x_0, s)$ (since $\overline{U}(x_0, s)$ is a closed ball). By letting $k \to +\infty$ in (2.1) we get $F(x^*) = 0$. To show the uniqueness part, let $x^{**} \in U(x_0, s)$ be a solution of equation $F(x) = 0$. By using (1.6) and the hypothesis on ψ_1, we obtain in turn that

$$\begin{aligned}\|x^{**} - x_{k+1}\| &= \left\|x^{**} - x_k + A_k^{-1} F(x_k) - A_k^{-1} F(x^{**})\right\| \\ &\leq \left\|A_k^{-1}\right\| \|F(x^{**}) - F(x_k) - A_k(x^{**} - x_k)\| \\ &\leq \beta^{-1}\beta\psi_1(\|x^{**} - x_k\|, \|x_{k-1} - x_0\|, \|x_k - x_0\| \\ &\quad , \|x^{**} - x_0\|)\|x^{**} - x_k\| \\ &\leq q_1 \|x^{**} - x_k\| \leq q_1^{k+1} \|x^{**} - x_0\|,\end{aligned}$$

so $\lim_{k\to+\infty} x_k = x^{**}$. We have shown that $\lim_{k\to+\infty} x_k = x^*$, so $x^* = x^{**}$. □

Remark 2.2. (1) The equation in (s_6) is used to determine the smallness of η. It can be replaced by a stronger condition as follows. Choose $\mu \in (0, 1)$. Denote by s_0 the smallest positive solution of equation $q(t) = \mu$. Notice that if function q is strictly increasing, we can set $s_0 = q^{-1}(\mu)$. Then, we can suppose instead of (s_6) :

(s_6') $\eta \leq (1 - \mu) s_0$

which is a stronger condition than (s_6).

However, we wanted to leave the equation in (s_6) as uncluttered and as weak as possible.

(2) Condition (s_2) can become part of condition (s_3) by considering

$(s_3)'$ There exists a continuous and nondecreasing function $\varphi : [0, +\infty)^3 \to [0, +\infty)$ such that for each $x, y \in \Omega_1$

$$\left\|A(x, y)^{-1}[F(x) - F(y) - A(x, y)(x, y)]\right\| \\ \leq \varphi(\|x - y\|, \|x - x_0\|, \|y - x_0\|)\|x - y\|.$$

Notice that

$$\varphi(u_1, u_2, u_3) \leq \psi(u_1, u_2, u_3)$$

for each $u_1 \geq 0$, $u_2 \geq 0$ and $u_3 \geq 0$. Similarly, a function φ_1 can replace ψ_1 for the uniqueness of the solution part. These replacements are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in (s_6), error bounds and the precision of s.

(3) Suppose that there exist $\beta > 0$, $\beta_1 > 0$ and $L \in \mathcal{L}(B_1, B_2)$ with $L^{-1} \in \mathcal{L}(B_2, B_1)$ such that

$$\left\|L^{-1}\right\| \leq \beta^{-1}$$

$$\|A(x,y) - L\| \leq \beta_1$$

and

$$\beta_2 := \beta^{-1}\beta_1 < 1.$$

Then, it follows from the Banach lemma on invertible operators [11], and

$$\left\|L^{-1}\right\| \|A(x,y) - L\| \leq \beta^{-1}\beta_1 = \beta_2 < 1$$

that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$. Let $\beta = \frac{\beta^{-1}}{1-\beta_2}$. Then, under these replacements, condition (s_2) is implied, therefore it can be dropped from the conditions (S).

(4) Clearly method (1.5) converges under the conditions (S), since (1.6) implies (1.5).

(5) We wanted to leave condition (s_4) as uncluttered as possible, since in practice equations (1.6) (or (1.5)) may be solvable in a way avoiding the already mentioned conditions of the contraction mapping principle. However, in what follows we examine the solvability of method (1.5) under a stronger version of the contraction mapping principle using the conditions (V) :

$(v_1) = (s_1)$.

(v_2) There exist functions $w_1 : [0,+\infty)^4 \to [0,+\infty)$, $w_2 : [0,+\infty)^4 \to [0,+\infty)$ continuous and nondecreasing such that for each $x, y, z \in \Omega$

$$\|I + A(x,z) - A(y,z)\| \leq w_1(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|)$$

$$\|A(x,z) - A(y,z)\| \leq w_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|) \|x-y\|$$

and

$$w_1(0,0,0,0) = w_2(0,0,0,0) = 0.$$

Set

$$h(t,t,t,t) = \begin{cases} w_1(2t,t,t,t) + w_2(2t,t,t,t)(t + \|x_0\|), & z \neq x_0 \\ w_1(2t,t,t,0) + w_2(2t,t,t,0)\|x_0\|, & z = x_0. \end{cases}$$

(v_3) There exists $\tau > 0$ satisfying

$$h(t,t,t,t) < 1$$

and

$$h(t,t,0,t)\,t + \|F(x_0)\| \le t$$

(v_4) $\overline{U}(x_0,\tau) \subseteq D$.

Theorem 2.3. *Suppose that the conditions* (V) *are satisfied. Then, equation (1.5) is uniquely solvable for each* $n = 0, 1, 2, \ldots$. *Moreover, if* $A_n^{-1} \in \mathcal{L}(B_2, B_1)$, *the equation (1.6) is also uniquely solvable for each* $n = 0, 1, 2, \ldots$

Proof. The result is an application of the contraction mapping principle. Let $x, y, z \in U(x_0, \tau)$. By the definition of operator P_z, (v_2) and (v_3), we get in turn that

$$\|P_z(x) - P_z(y)\| = \|(I + A(x,z) - A(y,z))(x-y) - (A(x,z) - A(y,z))\,z\|$$

$$\le \|I + A(x,z) - A(y,z)\|\,\|x-y\| + \|A(x,z) - A(y,z)\|\,\|z\|$$

$$\le [w_1(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|) +$$

$$w_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|, \|z-x_0\|)(\|z-x_0\| + \|x_0\|)]\,\|x-y\|$$

$$\le h(\tau,\tau,\tau,\tau)\,\|x-y\|$$

and

$$\|P_z(x) - x_0\| \le \|P_z(x) - P_z(x_0)\| + \|P_z(x_0) - x_0\|$$

$$\le h(\|x-x_0\|, \|x-x_0\|, 0, \|z-x_0\|)\,\|x-x_0\| + \|F(x_0)\|$$

$$\le h(\tau,\tau,0,\tau)\,\tau + \|F(x_0)\| \le \tau.$$

□

Remark 2.4. Section 2 and Section 3 have an interest independent of Section 4. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract Fractional Calculus as illustrated in Section 4. By specializing function ψ, we can apply the results of say Theorem 2.1 in the examples suggested in Section 4. In particular for (4.8), we choose $\psi(u_1,u_2,u_3) = \frac{c u_1^{p-1}}{\beta p}$ for $u_1 \ge 0$, $u_2 \ge 0$, $u_3 \ge 0$ and c, p are given in Section 4. Similar choices for the other examples of Section 4.

3. SEMI-LOCAL CONVERGENCE FOR EXPLICIT METHODS

A specialization of Theorem 2.1 can be utilized to study the semi-local convergence of the explicit methods given in the introduction of this study. In particular, for the study of method (1.12) (and consequently of method (1.11)), we use the conditions (S') :

(s_1') $F : \Omega \subset B_1 \to B_2$ is continuous and $A(x,x) \in \mathcal{L}(B_1, B_2)$ for each $x \in \Omega$.

(s_2') There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,x)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $x \in \Omega_0$ and

$$\left\| A(x,x)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(s_3') There exist continuous and nondecreasing functions $\psi_0 : [0,+\infty)^3 \to [0,+\infty)$, $\psi_2 : [0,+\infty)^3 \to [0,+\infty)$ with $\psi_0(0,0,0) = \psi_2(0,0,0) = 0$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(y,y)(x-y)\| \leq$$

$$\beta\psi_0(\|x-y\|, \|x-x_0\|, \|y-x_0\|)\|x-y\|$$

and

$$\|A(x,y) - A(y,y)\| \leq \beta\psi_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|).$$

Set $\psi = \psi_0 + \psi_2$.

(s_4') There exist $x_0 \in \Omega_0$ and $\eta \geq 0$ such that $A(x_0,x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\left\| A(x_0,x_0)^{-1} F(x_0) \right\| \leq \eta.$$

$(s_5') = (s_6)$

$(s_6') = (s_7)$.

Next, we present the following semi-local convergence analysis of method (1.12) using the (S') conditions and the preceding notation.

Proposition 3.1. *Suppose that the conditions (S') are satisfied. Then, sequence $\{x_n\}$ generated by method (1.12) starting at $x_0 \in \Omega$ is well defined in $U(x_0, s)$, remains in $U(x_0, s)$ for each $n = 0, 1, 2, ...$ and converges to a unique solution $x^* \in \overline{U}(x_0, s)$ of equation $F(x) = 0$.*

Proof. We follow the proof of Theorem 2.1 but use instead the analogous estimate

$$\|F(x_k)\| = \|F(x_k) - F(x_{k-1}) - A(x_{k-1}, x_{k-1})(x_k - x_{k-1})\| \leq$$

$$\|F(x_k) - F(x_{k-1}) - A(x_k, x_{k-1})(x_k - x_{k-1})\| +$$

$$\|(A(x_k, x_{k-1}) - A(x_{k-1}, x_{k-1}))(x_k - x_{k-1})\| \leq$$

$$[\psi_0(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|) +$$

$$\psi_2(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|)] \|x_k - x_{k-1}\| =$$

$$\psi(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|) \|x_k - x_{k-1}\|.$$

The rest of the proof is identical to the one in Theorem 2.1 until the uniqueness part for which we have the corresponding estimate

$$\|x^{**} - x_{k+1}\| = \left\|x^{**} - x_k + A_k^{-1}F(x_k) - A_k^{-1}F(x^{**})\right\| \leq$$

$$\left\|A_k^{-1}\right\| \|F(x^{**}) - F(x_k) - A_k(x^{**} - x_k)\| \leq$$

$$\beta^{-1}\beta\psi_0(\|x^{**} - x_k\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|) \leq$$

$$q\|x^{**} - x_k\| \leq q^{k+1}\|x^{**} - x_0\|.$$

□

Remark 3.2. Comments similar to the ones given in Section 2 can follows but for method (1.13) and method (1.14) instead of method (1.5) and method (1.6), respectively.

4. APPLICATIONS TO X-VALUED FRACTIONAL AND VECTOR CALCULI

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions f of real domain $[a, b]$. All integrals here are of Bochner-type, see [13]. The derivatives of f are defined similarly to numerical ones, see [16], pp. 83-86 and p. 93.

We want to solve numerically

$$f(x) = 0. \tag{4.1}$$

I) Application to X-valued Fractional Calculus

Let $p \in \mathbb{N} - \{1\}$ such that $p - 1 < \nu < p$, where $\nu \notin \mathbb{N}$, $\nu > 0$, i.e. $\lceil \nu \rceil = p$ ($\lceil \cdot \rceil$ ceiling of the number), $a < b$, $f \in C^p([a, b], X)$.

We define the following X-valued left Caputo fractional derivatives (see [3])

$$\left(D_{*y}^{\nu} f\right)(x) := \frac{1}{\Gamma(p-\nu)} \int_y^x (x-t)^{p-\nu-1} f^{(p)}(t)\,dt, \tag{4.2}$$

when $x \geq y$, and

$$\left(D_{*x}^{\nu} f\right)(y) := \frac{1}{\Gamma(p-\nu)} \int_x^y (y-t)^{p-\nu-1} f^{(p)}(t)\,dt, \tag{4.3}$$

when $y \geq x$, where Γ is the gamma function.

We define also the X-valued linear operator

$$(A_1(f))(x,y) := \begin{cases} \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^{k-1} + \left(D_{*y}^{\nu} f\right)(x) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}, & x > y, \\ \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^{k-1} + \left(D_{*x}^{\nu} f\right)(y) \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}, & y > x, \\ f^{(p-1)}(x), & x = y. \end{cases} \tag{4.4}$$

By X-valued left fractional Caputo Taylor's formula (see [3]), we get that

$$f(x) - f(y) = \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} D_{*y}^{\nu} f(t)\,dt, \text{ for } x > y, \tag{4.5}$$

and

$$f(y) - f(x) = \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{1}{\Gamma(\nu)} \int_x^y (y-t)^{\nu-1} D_{*x}^{\nu} f(t)\,dt, \text{ for } x < y. \tag{4.6}$$

Immediately, we observe that (by [11], p. 3)

$$\|(A_1(f))(x,x) - (A_1(f))(y,y)\| = \left\| f^{(p-1)}(x) - f^{(p-1)}(y) \right\| \tag{4.7}$$

$$\leq \left\| f^{(p)} \right\|_{\infty} |x-y|, \ \forall\, x, y \in [a, b],$$

We would like to prove that

$$\|f(x) - f(y) - (A_1(f))(x,y)(x-y)\| \leq c \frac{|x-y|^p}{p}, \tag{4.8}$$

for any $x, y \in [a, b]$ and some constant $0 < c < 1$.

When $x = y$, the last condition (4.8) is trivial.

We assume $x \neq y$. We distinguish the cases:

1) $x > y$: We observe that

$$\|f(x) - f(y) - (A_1(f))(x,y)(x-y)\| = \tag{4.9}$$

$$\left\| \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} D_{*y}^{\nu} f(t)\, dt - \right.$$

$$\left. \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k - \left(D_{*y}^{\nu} f\right)(x) \frac{(x-y)^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_{*y}^{\nu} f\right)(t)\, dt - \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_{*y}^{\nu} f\right)(x)\, dt \right\|$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\nu)} \left\| \int_y^x (x-t)^{\nu-1} \left(\left(D_{*y}^{\nu} f\right)(t) - \left(D_{*y}^{\nu} f\right)(x) \right) dt \right\|$$

(by [7])

$$\leq \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left\| \left(D_{*y}^{\nu} f\right)(t) - \left(D_{*y}^{\nu} f\right)(x) \right\| dt \tag{4.10}$$

(assume that

$$\left\| \left(D_{*y}^{\nu} f\right)(t) - \left(D_{*y}^{\nu} f\right)(x) \right\| \leq \lambda_1 |t-x|^{p-\nu}, \tag{4.11}$$

for any $t, x, y \in [a,b] : x \geq t \geq y$, where $\lambda_1 < \Gamma(\nu)$, i.e. $\rho_1 := \frac{\lambda_1}{\Gamma(\nu)} < 1$)

$$\leq \frac{\lambda_1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} (x-t)^{p-\nu}\, dt = \tag{4.12}$$

$$\frac{\lambda_1}{\Gamma(\nu)} \int_y^x (x-t)^{p-1}\, dt = \frac{\lambda_1}{\Gamma(\nu)} \frac{(x-y)^p}{p} = \rho_1 \frac{(x-y)^p}{p}. \tag{4.13}$$

We have proved that

$$\|f(x) - f(y) - (A_1(f))(x,y)(x-y)\| \leq \rho_1 \frac{(x-y)^p}{p}, \tag{4.14}$$

where $0 < \rho_1 < 1$, and $x > y$.

2) $x < y$: We observe that

$$\|f(x) - f(y) - (A_1(f))(x,y)(x-y)\| = \tag{4.15}$$

$$\left\|f(y)-f(x)-(A_1(f))(x,y)(y-x)\right\| =$$

$$\left\|\sum_{k=1}^{p-1}\frac{f^{(k)}(x)}{k!}(y-x)^k+\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_{*x}^{\nu}f\right)(t)\,dt-\right.$$

$$\left.\sum_{k=1}^{p-1}\frac{f^{(k)}(x)}{k!}(y-x)^k-\left(D_{*x}^{\nu}f\right)(y)\frac{(y-x)^{\nu}}{\Gamma(\nu+1)}\right\| =$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_{*x}^{\nu}f\right)(t)\,dt-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_{*x}^{\nu}f\right)(y)\,dt\right\| = \tag{4.16}$$

$$\frac{1}{\Gamma(\nu)}\left\|\int_x^y (y-t)^{\nu-1}\left(\left(D_{*x}^{\nu}f\right)(t)-\left(D_{*x}^{\nu}f\right)(y)\right)dt\right\| \leq$$

$$\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left\|\left(D_{*x}^{\nu}f\right)(t)-\left(D_{*x}^{\nu}f\right)(y)\right\|dt \tag{4.17}$$

(we assume that

$$\left\|\left(D_{*x}^{\nu}f\right)(t)-\left(D_{*x}^{\nu}f\right)(y)\right\| \leq \lambda_2 |t-y|^{p-\nu}, \tag{4.18}$$

for any $t,y,x\in[a,b]: y\geq t\geq x$)

$$\leq \frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}(y-t)^{p-\nu}\,dt =$$

$$\frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)^{p-1}\,dt = \frac{\lambda_2}{\Gamma(\nu)}\frac{(y-x)^p}{p}. \tag{4.19}$$

Assuming also

$$\rho_2 := \frac{\lambda_2}{\Gamma(\nu)} < 1 \tag{4.20}$$

(i.e. $\lambda_2 < \Gamma(\nu)$), we have proved that

$$\left\|f(x)-f(y)-(A_1(f))(x,y)(x-y)\right\| \leq \rho_2\frac{(y-x)^p}{p}, \text{ for } x<y. \tag{4.21}$$

Conclusion: Choosing $\lambda := \max(\lambda_1,\lambda_2)$ and $\rho := \frac{\lambda}{\Gamma(\nu)} < 1$, we have proved that

$$\left\|f(x)-f(y)-(A_1(f))(x,y)(x-y)\right\| \leq \rho\frac{|x-y|^p}{p}, \text{ for any } x,y\in[a,b]. \tag{4.22}$$

II) Application from Banach space Mathematical Analysis

In [4], we proved the following general X-valued Taylor's formula:

Theorem 4.1. *Let $p \in \mathbb{N}$ and $f \in C^p([A,B], X)$, where $[A,B] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([A,B])$, strictly increasing, such that $g^{-1} \in C^p([g(A), g(B)])$. Let any $a, b \in [A,B]$. Then*

$$f(b) = f(a) + \sum_{k=1}^{p-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(b) - g(a))^k + R_p(a,b), \quad (4.23)$$

where

$$R_p(a,b) = \frac{1}{(p-1)!} \int_a^b (g(b) - g(s))^{p-1} \left(f \circ g^{-1}\right)^{(p)}(g(s))\, g'(s)\, ds \quad (4.24)$$

$$= \frac{1}{(p-1)!} \int_{g(a)}^{g(b)} (g(b) - t)^{p-1} \left(f \circ g^{-1}\right)^{(p)}(t)\, dt.$$

Theorem 4.1 will be applied next for $g(x) = e^x$. One can give similar applications for $g = \sin, \cos, \tan$, etc., over suitable intervals.

Proposition 4.2. *Let $f \in C^p([A,B], X)$, $p \in \mathbb{N}$. Then*

$$f(b) = f(a) + \sum_{k=1}^{p-1} \frac{\left[(f \circ \ln)^{(k)}(e^a)\right]}{k!} \cdot \left(e^b - e^a\right)^k + R_p(a,b), \quad (4.25)$$

where

$$R_p(a,b) = \frac{1}{(p-1)!} \int_{e^a}^{e^b} \left(e^b - t\right)^{p-1} (f \circ \ln)^{(p)}(t)\, dt$$

$$= \frac{1}{(p-1)!} \int_a^b \left(e^b - e^a\right)^{p-1} (f \circ \ln)^{(p)}(e^s) \cdot e^s ds, \quad \forall\, a, b \in [A,B].$$

We will use the following variant.

Theorem 4.3. *Let all as in Theorem 4.1. Then*

$$f(\beta) - f(\alpha) = \sum_{k=1}^{p} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(\alpha))}{k!} (g(\beta) - g(\alpha))^k + R_p^*(\alpha, \beta), \quad (4.26)$$

where

$$R_p^*(\alpha, \beta) =$$

$$\frac{1}{(p-1)!} \int_\alpha^\beta (g(\beta) - g(s))^{p-1} \left(\left(f \circ g^{-1}\right)^{(p)}(g(s)) - \left(f \circ g^{-1}\right)^{(p)}(g(\alpha))\right) g'(s)\, ds \quad (4.27)$$

$$= \frac{1}{(p-1)!} \int_{g(\alpha)}^{g(\beta)} (g(\beta) - t)^{p-1} \left(\left(f \circ g^{-1}\right)^{(p)} (t) - \left(f \circ g^{-1}\right)^{(p)} (g(\alpha)) \right) dt,$$

$\forall\ \alpha, \beta \in [A, B]$.

Proof. Easy. □

Remark 4.4. Call $l = f \circ g^{-1}$. Then $l, l', ..., l^{(p)}$ are continuous from $[g(A), g(B)]$ into $f([A, B])$.

Next we estimate $R_p^*(\alpha, \beta)$: We assume that

$$\left\| \left(f \circ g^{-1}\right)^{(p)} (t) - \left(f \circ g^{-1}\right)^{(p)} (g(\alpha)) \right\| \le K |t - g(\alpha)|, \tag{4.28}$$

$\forall\ t, g(\alpha) \in [g(A), g(B)]$, where $K > 0$.

We distinguish the cases:

i) if $g(\beta) > g(\alpha)$, then

$$\left\| R_p^*(\alpha, \beta) \right\| \le$$

$$\frac{1}{(p-1)!} \int_{g(\alpha)}^{g(\beta)} (g(\beta) - t)^{p-1} \left\| \left(f \circ g^{-1}\right)^{(p)} (t) - \left(f \circ g^{-1}\right)^{(p)} (g(\alpha)) \right\| dt \le$$

$$\frac{K}{(p-1)!} \int_{g(\alpha)}^{g(\beta)} (g(\beta) - t)^{p-1} (t - g(\alpha))^{2-1} dt = \tag{4.29}$$

$$\frac{K}{(p-1)!} \frac{\Gamma(p)\Gamma(2)}{\Gamma(p+2)} (g(\beta) - g(\alpha))^{p+1} =$$

$$\frac{K}{(p-1)!} \frac{(p-1)!}{(p+1)!} (g(\beta) - g(\alpha))^{p+1} = K \frac{(g(\beta) - g(\alpha))^{p+1}}{(p+1)!}. \tag{4.30}$$

We have proved that

$$\left\| R_p^*(\alpha, \beta) \right\| \le K \frac{(g(\beta) - g(\alpha))^{p+1}}{(p+1)!}, \tag{4.31}$$

when $g(\beta) > g(\alpha)$.

ii) if $g(\alpha) > g(\beta)$, then

$$\left\| R_p^*(\alpha, \beta) \right\| =$$

$$\frac{1}{(p-1)!} \left\| \int_{g(\beta)}^{g(\alpha)} (t - g(\beta))^{p-1} \left(\left(f \circ g^{-1}\right)^{(p)} (t) - \left(f \circ g^{-1}\right)^{(p)} (g(\alpha)) \right) dt \right\| \le$$

$$\frac{1}{(p-1)!}\int_{g(\beta)}^{g(\alpha)}(t-g(\beta))^{p-1}\left\|\left(f\circ g^{-1}\right)^{(p)}(t)-\left(f\circ g^{-1}\right)^{(p)}(g(\alpha))\right\|dt\leq \tag{4.32}$$

$$\frac{K}{(p-1)!}\int_{g(\beta)}^{g(\alpha)}(g(\alpha)-t)^{2-1}(t-g(\beta))^{p-1}dt=$$

$$\frac{K}{(p-1)!}\frac{\Gamma(2)\Gamma(p)}{\Gamma(p+2)}(g(\alpha)-g(\beta))^{p+1}= \tag{4.33}$$

$$\frac{K}{(p-1)!}\frac{(p-1)!}{(p+1)!}(g(\alpha)-g(\beta))^{p+1}=K\frac{(g(\alpha)-g(\beta))^{p+1}}{(p+1)!}. \tag{4.34}$$

We have proved that

$$\left\|R_p^*(\alpha,\beta)\right\|\leq K\frac{(g(\alpha)-g(\beta))^{p+1}}{(p+1)!}, \tag{4.35}$$

whenever $g(\alpha)>g(\beta)$.

Conclusion: It holds

$$\left\|R_p^*(\alpha,\beta)\right\|\leq K\frac{|g(\alpha)-g(\beta)|^{p+1}}{(p+1)!}, \tag{4.36}$$

$\forall\ \alpha,\beta\in[A,B]$.

Both sides of (4.36) equal zero when $\alpha=\beta$.

We define the following X-valued linear operator:

$$(A_3(f))(x,y):=\begin{cases}\sum_{k=1}^{p}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1}, & \text{when } g(x)\neq g(y),\\ f^{(p-1)}(x), & x=y,\end{cases} \tag{4.37}$$

for any $x,y\in[A,B]$.

Easily, we see that ([11], p. 3)

$$\|(A_3(f))(x,x)-(A_3(f))(y,y)\|=\left\|f^{(p-1)}(x)-f^{(p-1)}(y)\right\| \tag{4.38}$$

$$\leq\left\|f^{(p)}\right\|_\infty|x-y|,\ \ \forall\ x,y\in[A,B].$$

Next we observe that (case of $g(x)\neq g(y)$)

$$\|f(x)-f(y)-(A_3(f))(x,y)\cdot(g(x)-g(y))\|=$$

$$\left\|\sum_{k=1}^{p}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k+R_p^*(y,x) \tag{4.39}\right.$$

$$-\left(\sum_{k=1}^{p}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1}(g(x)-g(y))\right)\Bigg\| =$$

$$\left\|R_p^*(y,x)\right\| \overset{(4.36)}{\leq} K\frac{|g(x)-g(y)|^{p+1}}{(p+1)!}, \tag{4.40}$$

$\forall\ x,y\in[A,B]: g(x)\neq g(y)$.

We have proved that

$$\|f(x)-f(y)-(A_3(f))(x,y)\cdot(g(x)-g(y))\|\leq K\frac{|g(x)-g(y)|^{p+1}}{(p+1)!}, \tag{4.41}$$

$\forall\ x,y\in[A,B]$

(the case $x=y$ is trivial).

We apply the above theory as follows:

II$_1$) We define

$$(A_{31}(f))(x,y):=\begin{cases}\sum_{k=1}^{p}\frac{(f\circ\ln)^{(k)}(e^y)}{k!}(e^x-e^y)^{k-1}, & x\neq y,\\ f^{(p-1)}(x), & x=y,\end{cases} \tag{4.42}$$

for any $x,y\in[A,B]$.

Furthermore it holds

$$\|f(x)-f(y)-(A_{31}(f))(x,y)\cdot(e^x-e^y)\|\leq K_1\frac{|e^x-e^y|^{p+1}}{(p+1)!}, \tag{4.43}$$

$\forall\ x,y\in[A,B]$, where we assumed that

$$\left\|(f\circ\ln)^{(p)}(t)-(f\circ\ln)^{(p)}(e^y)\right\|\leq K_1|t-e^y|, \tag{4.44}$$

$\forall\ t,e^y\in\left[e^A,e^B\right]$, $A<B$, with $K_1>0$.

II$_2$) Next let $f\in C^p\left(\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]\right)$, $p\in\mathbb{N}$, $\varepsilon>0$ small.

Here we define that

$$(A_{32}(f))(x,y):=\begin{cases}\sum_{k=1}^{p}\frac{\left(f\circ\sin^{-1}\right)^{(k)}(\sin y)}{k!}(\sin x-\sin y)^{k-1}, & \text{when } x\neq y,\\ f^{(p-1)}(x), & x=y,\end{cases} \tag{4.45}$$

for any $x,y\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]$.

We assume that

$$\left\|\left(f\circ\sin^{-1}\right)^{(p)}(t)-\left(f\circ\sin^{-1}\right)^{(p)}(\sin y)\right\|\leq K_2|t-\sin y|, \tag{4.46}$$

$\forall\ t, \sin y \in \left[\sin\left(-\frac{\pi}{2}+\varepsilon\right), \sin\left(\frac{\pi}{2}-\varepsilon\right)\right]$, where $K_2 > 0$.

It holds

$$\|f(x) - f(y) - (A_{32}(f))(x,y)\cdot(\sin x - \sin y)\| \leq K_2 \frac{|\sin x - \sin y|^{p+1}}{(p+1)!}, \tag{4.47}$$

$\forall\ x, y \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$.

II$_3$**)** Next let $f \in C^p([\varepsilon, \pi-\varepsilon])$, $p \in \mathbb{N}$, $\varepsilon > 0$ small.

Here we define

$$(A_{33}(f))(x,y) := \begin{cases} \sum_{k=1}^{p} \frac{\left(f\circ\cos^{-1}\right)^{(k)}(\cos y)}{k!}(\cos x - \cos y)^{k-1}, & \text{when } x \neq y, \\ f^{(p-1)}(x), \quad x = y, \end{cases} \tag{4.48}$$

for any $x, y \in [\varepsilon, \pi-\varepsilon]$.

We assume that

$$\left\|\left(f\circ\cos^{-1}\right)^{(p)}(t) - \left(f\circ\cos^{-1}\right)^{(p)}(\cos y)\right\| \leq K_3\,|t - \cos y|, \tag{4.49}$$

$\forall\ t, \cos y \in [\cos\varepsilon, \cos(\pi-\varepsilon)]$, where $K_3 > 0$.

It holds

$$\|f(x) - f(y) - (A_{33}(f))(x,y)\cdot(\cos x - \cos y)\| \leq K_3 \frac{|\cos x - \cos y|^{p+1}}{(p+1)!}, \tag{4.50}$$

$\forall\ x, y \in [\varepsilon, \pi-\varepsilon]$.

Finally we give:

II$_4$**)** Let $f \in C^p\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]\right)$, $p \in \mathbb{N}$, $\varepsilon > 0$ small.

We define

$$(A_{34}(f))(x,y) := \begin{cases} \sum_{k=1}^{p} \frac{\left(f\circ\tan^{-1}\right)^{(k)}(\tan y)}{k!}(\tan x - \tan y)^{k-1}, & \text{when } x \neq y, \\ f^{(p-1)}(x), \quad x = y, \end{cases} \tag{4.51}$$

for any $x, y \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$.

We assume that

$$\left\|\left(f\circ\tan^{-1}\right)^{(p)}(t) - \left(f\circ\tan^{-1}\right)^{(p)}(\tan y)\right\| \leq K_4\,|t - \tan y|, \tag{4.52}$$

$\forall\ t, \tan y \in \left[\tan\left(-\frac{\pi}{2}+\varepsilon\right), \tan\left(\frac{\pi}{2}-\varepsilon\right)\right]$, where $K_4 > 0$.

It holds that

$$\|f(x) - f(y) - (A_{34}(f))(x,y) \cdot (\tan x - \tan y)\| \leq K_4 \frac{|\tan x - \tan y|^{p+1}}{(p+1)!}, \tag{4.53}$$

$\forall\, x, y \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right]$.

REFERENCES

[1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.

[2] S. Amat, S. Busquier, S. Plaza, Chaotic dynamics of a third-order Newton-type method, *J. Math. Anal. Applic.*, **366**, no. 1 (2010), 164-174.

[3] G.A. Anastassiou, A strong fractional calculus theory for Banach space valued functions, *Nonlinear Functional Analysis and Applications*, Korea, 2017, Accepted for publication.

[4] G.A. Anastassiou, Principles of general fractional analysis for Banach space valued functions, 2017, Submitted.

[5] I.K. Argyros, A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space, *J. Math. Anal. Appl.*, **298** (2004), 374-397.

[6] I.K. Argyros, A. Magrenan, *Iterative Methods and their Dynamics with Applications*, CRC Press, New York, 2017.

[7] Bochner integral, *Encyclopedia of Mathematics*, url: http://www.ency clopedia ofmath.org/index.php?title=Bochner_integral& oldid=38659.

[8] M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, **37** (1962), 74-79.

[9] J.A. Ezquerro, J.M. Gutierrez, M.A. Hernandez, N. Romero, M.J. Rubio, The Newton method: From Newton to Kantorovich, *Spanish, Gac. R. Soc. Mat. Esp.*, **13** (2010), 53-76.

[10] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1982.

[11] G.E. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces,* Academic Press, New York, London, 1972.

[12] A. Magrenan, A new tool to study real dynamics: The convergence plane, *Appl. Math. Comput.*, **248** (2014), 215-224.

[13] J. Mikusinski, *The Bochner Integral*, Academic Press, New York, 1978.

[14] F.A. Potra, V. Ptak, *Nondiscrete Induction and Iterative Processes*, Pitman Publ., London, 1984.

[15] P.D. Proinov, New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, **26** (2010), 3-42.

[16] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.