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A NOTE ON CYCLIC SUBGROUP GRAPH OF A FINITE GROUP

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ABSTRACT: The cyclic subgroup graph $\Gamma_z(G)$ of a finite group G is a graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. This parameter is introduced in [4]. In this paper few parameters in cyclic subgroup graph of a finite group such as restrained triple connected domination number, strong triple connected domination mumber, perfect domination number, triple connected two domination number, two domination number, restrained domination number and strong domination number are studied.

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Key Words: cyclic subgroup graph, restrained triple connected, strong triple connected, perfect domination

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1. INTRODUCTION

Algebraic Graph theory has developed in various dimensions in the past two decades. In this paper, we relate a group with a graph by its subgroup structure.

By a graph $\Gamma = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. The cyclic subgroup graph $\Gamma_z(G)$ is a graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. A subset S of V of a nontrivial graph G is called a dominating set of G if every vertex in $V - S$ is adjacent to atleast one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A subset S of V of a nontrivial graph G is called a restrained dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S as well as another vertex in $V - S$. The restrained domination number $\gamma_r(G)$ of G is the minimum cardinality taken over all restrained dominating sets in G. A graph is said to be triple connected [8], if any three vertices lie on a path in G. A subset S of V of a nontrivial graph G is said to be triple connected dominating set [9], if S is a dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by γ_{tc}. A subset S of V of a nontrivial graph G is said to be restrained triple connected dominating set [5], if S is a restrained dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the restrained triple connected domination number and is denoted by γ_{rtc}. A set $D \subseteq V(G)$ is a strong dominating set of G, if for every vertex $x \in V(G) - D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $d(x, G) \leq d(y, G)$. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A subset S of V of a nontrivial graph G is said to be strong triple connected dominating set [6], if S is a strong dominating set and the induced sub graph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all strong triple connected dominating sets is called the strong triple connected domination number and is denoted by γ_{stc}. A dominating set S is said to be two dominating set if every vertex in $V - S$ is adjacent to atleast two vertices in S. The minimum cardinality taken over all two dominating sets is called the two domination number and is denoted by γ_2. A subset S of V of a nontrivial graph G is said to be triple connected two dominating set [7], if S is a two dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected two dominating sets is called the triple connected two domination number and is denoted by γ_{2tc}. A dominating set S is a perfect dominating set [3], if $|N(v) \cap S| = 1$ for each $v \in V - S$. The minimum cardinality taken over all perfect dominating sets is called the perfect domination number and is denoted by γ_p.

2. MAIN RESULTS

Definition 2.1. The cyclic subgroup graph $\Gamma_z(G)$ of a finite group G is a simple undirected graph in which the cyclic subgroups of G are the vertices and two distinct subgroups are adjacent if one of them is a subset of the other.

Theorem 2.2. *For any finite group G with $|G| \geq 3$, the restrained domination number of the cyclic subgroup graph, $\gamma_r(\Gamma_z(G)) = 1$.*

Proof. Let $A = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1 \rangle = \langle e \rangle = \langle 0 \rangle$. Since $\langle x_1 \rangle \subseteq \langle x_j \rangle$, $j = 2, 3, 4, \ldots$, $\langle x_1 \rangle$ is adjacent to every other vertices $\langle x_j \rangle$. Let $\langle x_2 \rangle = \langle 1 \rangle$. Since $\langle x_k \rangle \subseteq \langle x_2 \rangle, k = 1, 3, 4, \ldots, \langle x_k \rangle$ is adjacent to $\langle x_2 \rangle$. Let $S = \{\langle x_1 \rangle\}$. Here every vertex in $V - S$ is adjacent to $\langle x_1 \rangle$ in S and atleast one other vertex in $V - S$. Therefore S is a restrained dominating set. Since S contains only one vertex, S is the minimum restrained dominating set. Hence $\gamma_r(\Gamma_z(G)) \leq 1$. So that $\gamma_r(\Gamma_z(G)) = 1$. □

Theorem 2.3. *For any finite group G with $|G| \geq 5$, the restrained triple connected domination number of the cyclic subgroup graph, $\gamma_{rtc}(\Gamma_z(G)) = 3$.*

Proof. Let $A = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, \ldots\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1 \rangle = \langle e \rangle = \langle 0 \rangle$. Since $\langle x_1 \rangle \subseteq \langle x_j \rangle$, $j = 2, 3, 4, \ldots$, $\langle x_1 \rangle$ is adjacent to every other vertices $\langle x_j \rangle$. In particular take $\langle x_2 \rangle$, $\langle x_3 \rangle \in A$. So that $\langle x_1 \rangle$ is adjacent to $\langle x_2 \rangle$ and $\langle x_3 \rangle$. Therefore $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$ lie on a path. Let $S = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle\}$. Here $\langle S \rangle$ is triple connected and every vertex in $V - S$ is adjacent to atleast one vertex in S. Hence S is triple connected dominating set. Let $\langle x_4 \rangle = \langle 1 \rangle \in V - S$. Since $\langle x_k \rangle \subseteq \langle 1 \rangle$, $k = 1, 2, 3, 5, 6, \ldots$, every vertex $\langle x_k \rangle$ is adjacent to $\langle 1 \rangle$. So that every vertex in $V - S$ is adjacent to atleast one vertex in S and one other vertex in $V - S$. Therefore A is a restrained triple connected dominating set. Hence $\gamma_{rtc}(\Gamma_z(G)) \leq 3$. Since any triple connected set has atleast 3 vertices, $\gamma_{rtc}(\Gamma_z(G)) = 3$. □

Theorem 2.4. *For any finite group G, the strong domination number of the cyclic subgroup graph, $\gamma_s(\Gamma_z(G)) = 1$.*

Proof. Let $A = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1\rangle = \langle e\rangle = \langle 0\rangle$. Since $\langle x_1\rangle \subseteq \langle x_j\rangle$, $j = 2, 3, 4, \ldots$, $\langle x_1\rangle$ is adjacent to every other vertices $\langle x_j\rangle$. Therefore $deg(\langle x_1\rangle) \geq deg(\langle x_j\rangle)$, $j = 2, 3, \ldots$. Let $S = \{\langle x_1\rangle\}$. Then every vertex in $V - S$ is adjacent to $\langle x_1\rangle \in S$ with $deg(\langle x_1\rangle) \geq deg(\langle x_j\rangle)$, $j = 2, 3, \ldots$. So that S is a strong dominating set. Since S contains only one vertex, S is the minimum strong dominating set. Hence $\gamma_s(\Gamma_z(G)) \leq 1$. So that $\gamma_s(\Gamma_z(G)) = 1$. □

Theorem 2.5. *For any finite group G with $|G| \geq 3$, the strong triple connected domination number of the cyclic subgroup graph, $\gamma_{stc}(\Gamma_z(G)) = 3$.*

Proof. Let $A = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1\rangle = \langle e\rangle = \langle 0\rangle$. Since $\langle x_1\rangle \subseteq \langle x_j\rangle, j = 2, 3, 4, \ldots.$, $\deg(\langle x_1\rangle, \Gamma_z(G)) \geq deg(\langle x_j\rangle, \Gamma_z(G))$. Since $\langle x_k\rangle \subseteq \langle 1\rangle, k = 1, 3, 4, 5, \ldots, \langle x_k\rangle$ is adjacent to $\langle 1\rangle$, $\deg(\langle x_2\rangle, \Gamma_z(G)) \geq deg(\langle x_k\rangle, \Gamma_z(G))$. Choose a vertex $\langle x_3\rangle$ such that $deg(\langle x_3\rangle, \Gamma_z(G)) \geq deg(\langle x_i\rangle, \Gamma_z(G))$, $i = 4, 5, 6 \ldots$. .So that $\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle$ lie on a path. Let $S = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle\}$. Therefore $\langle S\rangle$ is triple connected and every vertex in $V - S$ is adjacent to atleast one vertex in S with higher degree. So that S is a strong triple connected dominating set. Hence $\gamma_{stc}(\Gamma_z(G)) \leq 3$. Since any triple connected set has atleast 3 vertices, $\gamma_{stc}(\Gamma_z(G)) = 3$. □

Theorem 2.6. *For any finite group G, the perfect domination number of the cyclic subgroup graph, $\gamma_p(\Gamma_z(G)) = 1$.*

Proof. Let $A = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1\rangle = \langle e\rangle = \langle 0\rangle$. Since $\langle e\rangle \subseteq \langle x_j\rangle$, $j = 2, 3, 4, \ldots$, $\langle e\rangle$ is adjacent to every other vertices. Hence $\langle e\rangle$ dominates every other vertices. Let $S = \{\langle e\rangle\}$ be a dominating set of $\Gamma_z(G)$. Therefore for every $\langle x_j\rangle \in V - S$, $|N(x_j) \cap S| = 1$. So that S is a perfect dominating set. Hence $\gamma_p(\Gamma_z(G)) = 1$. □

Theorem 2.7. *For any finite group G with $|G| \geq 2$, the two domination number of the cyclic subgroup graph, $\gamma_2(\Gamma_z(G)) = 2$.*

Proof. Let $A = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1\rangle =$

$\langle e \rangle = \langle 0 \rangle$. Since $\langle x_1 \rangle \subseteq \langle x_j \rangle$, $j = 2, 3, 4, \ldots$ $\langle x_1 \rangle$ is adjacent to every other vertices $\langle x_j \rangle$. Let $\langle x_2 \rangle = \langle 1 \rangle$. Since $\langle x_k \rangle \subseteq \langle x_2 \rangle$, $k = 2, 3, 4, \ldots$ $\langle x_k \rangle$ is adjacent to $\langle x_2 \rangle$.Therefore $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are adjacent. Let $S = \{\langle x_1 \rangle, \langle x_2 \rangle\}$. Hence every vertex in $V - S$ is adjacent to $\langle x_1 \rangle$ and $\langle x_2 \rangle \in S$. Therefore S is a two dominating set. Since S contains only two vertices, S is a minimum two dominating set. So that $\gamma_2(\Gamma_z(G)) \leq 2$. Since any two dominating set has atleast two vertices, $\gamma_2(\Gamma_z(G)) = 2$. □

Theorem 2.8. *For any finite group G with $|G| \geq 3$, the triple connected 2 domination number of the cyclic subgroup graph, $\gamma_{2tc}(\Gamma_z(G)) = 3$.*

Proof. Let $A = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, ...\}$ be the vertex set of $\Gamma_z(G)$. Let $\langle x_1 \rangle = \langle e \rangle = \langle 0 \rangle$. Since $\langle e \rangle \subseteq \langle x_j \rangle$, $j = 2, 3, 4, \ldots, \langle x_1 \rangle$ is adjacent to every other vertices. Let $\langle x_2 \rangle = \langle 1 \rangle$. Since $\langle x_i \rangle \subseteq \langle 1 \rangle$, $i = 1, 3, 4, \ldots$, every $\langle x_i \rangle$ is adjacent to $\langle x_2 \rangle$. Choose another vertex $\langle x_3 \rangle$, which is adjacent to both $\langle x_1 \rangle$ and $\langle x_2 \rangle$. Let $S = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle\}$. Therefore every vertex in $V - S$ is adjacent to atleast two vertices in S and the elements of S lie on a path. Therefore S is a triple connected two dominating set. Since S is the minimum triple connected two dominating set, $\gamma_{2tc}(\Gamma_z(G)) \leq 3$. Since any triple connected set has atleast 3 vertices, $\gamma_{2tc}(\Gamma_z(G)) = 3$. □

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