

**ANALYSIS OF NONLINEAR DELAY SYSTEMS WITH
APPLICATIONS IN BUMBLEBEE POPULATION MODELS**

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ABSTRACT: Bumblebees are ubiquitous creatures and crucial pollinators to a vast assortment of crops worldwide. These populations have been in decline in recent decades and researchers are seeking to understand why populations are decreasing and how to direct conservation efforts. Because of their reproductive patterns, bumblebee population dynamics can be modeled with delay differential equations (DDEs). We present non-linear, non-autonomous DDE models of bumblebee colonies and resources. We demonstrate that the models satisfy the conditions in [4] and complete the subsequent theoretical developments therein in order to rigorously justify families of approximate solutions.

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1. INTRODUCTION

The protection of bumblebee populations, among other pollinators, is vital to sustain global agricultural food production [26, 19], biodiversity and ecosystem functioning [18, 30]. It is now widely accepted that bumblebee diversity has dramatically declined in the past several decades [12, 13, 14]. Diminishing populations have been ascribed to habitat loss, resulting in many consequences, including loss of nest and flower resources [32, 41]. The buff-tailed bumblebee *Bombus terrestris* has been the subject of much study (see for example, [38, 31, 16, 17, 1]), as it is abundant in Europe and known to be an important pollinator [25].

Empirical research has concluded that forage resources (pollen and nectar) in the landscape affect overall bumblebee abundancy. Mathematical modeling based on empirical information on life history parameters can be a strong tool [35, 39, 40] to project population dynamics and identify vulnerable traits and life stages. Such mathematical modeling, especially in an iterative approach [10], can be used for projecting population abundance and understanding the importance of key life traits, such as survival, reproduction and seasonal reproductive switch times under contrasting scenarios. We have used delay differential equations to understand the various ways in which *B. terrestris* populations are dynamically affected by environmental pressures, including pesticide exposure and resource limitation. In [6, 7], we presented a delay differential equation (DDE) model to simulate the abundance of different bumblebee castes and in-nest resources over time, with dynamics including colony establishment, mortality, colony growth, reproduction, and queen hibernation.

Delay equations have been used in various applications, including biology, ecology, engineering (see [2, 15, 20, 23] for examples) and even honeybee population modeling [24]. We refer the reader to [37] for an introduction to DDEs and applications, as well as [27] for DDEs in ecology. In [6, 7], we explained why a DDE model may be preferable over the more commonly used ordinary differential equation (ODE) system for describing bumblebee and other population dynamics. In addition, we presented several models with the underlying assumptions and proposed ways in which pressures such as resource limitation and insecticide exposure can be reflected in the model. We also described a linear spline approximation method for obtaining numerical solutions to our models and provided model simulations in the absence of pressures. Here, we demonstrate that the models in [6, 7] satisfy the assumptions required in [4], and complete the necessary theory in order to obtain the approximate solution to the DDE system. The theory we complete here, first outlined in detail in [4] without proofs, has subsequently been discussed or referred to in various publications [3, 5, 8, 9, 28, 29, 33, 34] in the past 35 years, but proofs for the outline in [4] have not been presented. Since not all of the references cited above are readily accessible (including [4] itself), we present here the complete theory and verify that our models from [6, 7] satisfy the theory.

2. MODELS

2.1. BUMBLEBEE DDE MODELS

We present a general bumblebee model, versions of which were originally discussed in [6, 7], and which is 6-dimensional with state variables: in-nest nectar abundance $A(t)$, in-nest pollen abundance $B(t)$, queens $Q(t)$, workers $W(t)$, males $M(t)$ and gynes (daughter queens) $G(t)$. The version discussed here has been slightly modified from those developed in [6, 7] and is currently being used in pesticide studies. We define the first day of spring, when hibernating gynes emerge as new queens that found new colonies, at $t = T_S$. We therefore consider the time interval $t \in [T_S + 22, T_W]$, allowing for a 22-day delay before the first brood of worker bees emerges, where T_W is the end of the season.

The dynamics of a collection of hives are described by the DDE

$$\begin{aligned} \frac{dA}{dt} &= (b_{AW} - \mu_{AW})W - \mu_{AQ}Q - 2[l_W(t, Q_t) + l_M(t, Q_t) + l_G(t, Q_t)], \\ \frac{dB}{dt} &= (b_{BW} - \mu_{BW})W - \mu_{BQ}Q - [l_W(t, Q_t) + l_M(t, Q_t) + l_G(t, Q_t)], \\ \frac{dQ}{dt} &= -\mu_Q Q, \\ \frac{dW}{dt} &= b_W(t - 22)Q(t - 22)S_W(t, A_t, B_t, W_t) - \mu_W W, \\ \frac{dM}{dt} &= b_M(t - 26)Q(t - 26)S_M(t, A_t, B_t, W_t) - \mu_M M, \\ \frac{dG}{dt} &= b_G(t - 30)Q(t - 30)S_G(t, A_t, B_t, W_t) - \mu_G G, \end{aligned}$$

with initial conditions and histories for $\theta \in [-30, 0]$,

$$\begin{aligned} A_{T_S+22}(\theta) &= A_0, & W_{T_S+22}(\theta) &= \begin{cases} 0 & \theta < 0, \\ W_0 & \theta = 0, \end{cases} \\ B_{T_S+22}(\theta) &= B_0, & M_{T_S+22}(\theta) &= 0, \\ Q_{T_S+22}(\theta) &= Q_0 e^{-\mu_Q(T_S+22+\theta)}, & G_{T_S+22}(\theta) &= 0. \end{aligned}$$

As is usual in the formulation of delay equations, the notation Q_t , etc., refers to the function $Q_t(\theta) = Q(t + \theta), \theta \in [-30, 0]$, etc. Larval consumption in the equations for $\frac{dA}{dt}$ and $\frac{dB}{dt}$ is modelled as

$$\begin{aligned} l_W(t, Q_t) &= \int_{t-13}^{t-4} b_W(s)Q(s)P_0 e^{\bar{r}(t-s)} ds, \\ l_M(t, Q_t) &= \int_{t-15}^{t-4} b_M(s)Q(s)P_0 e^{\bar{r}(t-s)} ds, \\ l_G(t, Q_t) &= \int_{t-17}^{t-4} b_G(s)Q(s)P_0 e^{\bar{r}(t-s)} ds, \end{aligned}$$

where \bar{r} is a larval pollen consumption growth rate. The birth rates of workers, males, and gynes depend on predefined seasonal switch times T^* and T^{**} ; in formulating these terms, we enforce the conditions that for $t < T^{**}$, $b_M(t), b_G(t) = 0$ and for $t > T^{**}$, $b_W(t) = 0$. The probabilities of survival for

worker, male, and gyne larvae are given by

$$\begin{aligned}
 S_W(t, A_t, B_t, W_t) &= \frac{1}{9} \int_{t-18}^{t-9} L\left(\frac{W(s)}{\gamma_W Q(s)}\right) L\left(\frac{A(s)}{A_{max} Q(s)}\right) L\left(\frac{B(s)}{B_{max} Q(s)}\right) ds, \\
 S_M(t, A_t, B_t, W_t) &= \frac{1}{11} \int_{t-22}^{t-11} L\left(\frac{W(s)}{\gamma_M Q(s)}\right) L\left(\frac{A(s)}{A_{max} Q(s)}\right) L\left(\frac{B(s)}{B_{max} Q(s)}\right) ds, \\
 S_G(t, A_t, B_t, W_t) &= \frac{1}{13} \int_{t-26}^{t-13} L\left(\frac{W(s)}{\gamma_G Q(s)}\right) L\left(\frac{A(s)}{A_{max} Q(s)}\right) L\left(\frac{B(s)}{B_{max} Q(s)}\right) ds,
 \end{aligned}$$

for the function

$$L(x) = (1 - e^{-x}) / (1 + e^{-x}), \tag{1}$$

which maps the positive real line onto the interval $[0, 1]$. See [6, 7] for all model assumptions and an explanation of bumblebee reproduction and other seasonal dynamics, and Table 1 below for a description of all model variables and parameters.

2.2. GENERAL FORMULATION AND HYPOTHESIS

Now, consider the following DDE form, as given in [4]:

$$\begin{aligned}
 \frac{dx}{dt} &= f(t, x(t), x_t, x(t - \tau_1), \dots, x(t - \tau_\nu)) + f_2(t), \\
 x_0 &= \phi,
 \end{aligned} \tag{2}$$

for $0 \leq t \leq T$ and $f : [0, T] \times Z \times \mathbb{R}^{n \times \nu} \rightarrow \mathbb{R}^n$. We define $Z = \mathbb{R}^n \times L_2(-r, 0)$, $0 < \tau_1 < \dots < \tau_\nu = r$, $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$. We assume that $\phi \in H^1(-r, 0)$ (where $H^j(a, b)$ denotes the Sobolev space $W^{j,2}(a, b, \mathbb{R}^n)$ of \mathbb{R}^n -valued functions f such that $f \in L_2[a, b]$, $\partial^j f \in L_2[a, b]$), and $f_2 \in L_2(0, T)$. We further assume that f satisfies the hypotheses from [4]:

(H1) The function f satisfies a global Lipschitz condition

$$\begin{aligned}
 &\left\| f(t, \eta, \psi, y_1, \dots, y_\nu) - f(t, \xi, \tilde{\psi}, w_1, \dots, w_\nu) \right\|_{\mathbb{R}^n} \\
 &\leq K \left(\|\eta - \xi\|_{\mathbb{R}^n} + \left\| \psi - \tilde{\psi} \right\|_{L_2(-r, 0)} + \sum_{i=1}^{\nu} \|y_i - w_i\|_{\mathbb{R}^n} \right)
 \end{aligned}$$

for some fixed constant K and all $(t, \eta, \psi, y_1, \dots, y_\nu), (t, \xi, \tilde{\psi}, w_1, \dots, w_\nu) \in \mathbb{R}^n \times Z \times \mathbb{R}^{n \times \nu}$. (A more relaxed condition would also suffice, in which

Variable	Description	Units
t	time	days
$A(t)$	amount of nectar in colonies	ml
$B(t)$	amount of pollen in colonies	g
$Q(t)$	number of queens	individuals (queens)
$W(t)$	number of workers	individuals (workers)
$M(t)$	number of males	individuals (males)
$G(t)$	number of gynes	individuals (gynes)
Timeline		
T_S	first day of spring	
T^*	first day male/gyne eggs laid	
T^{**}	end of worker eggs laid	
T_W	beginning of winter	
Parameters		
μ_{AQ}	queen nectar consumption rate	$\frac{\text{ml}}{\text{day} \cdot \text{individual}(Q)}$
μ_{BQ}	queen pollen consumption rate	$\frac{\text{g}}{\text{day} \cdot \text{individual}(Q)}$
μ_Q	queen death rate	$\frac{1}{\text{day}}$
b_{AW}	worker nectar collection rate	$\frac{\text{ml}}{\text{day} \cdot \text{individual}(W)}$
b_{BW}	worker pollen collection rate	$\frac{\text{g}}{\text{day} \cdot \text{individual}(W)}$
μ_{AW}	worker nectar consumption rate	$\frac{\text{ml}}{\text{day} \cdot \text{individual}(W)}$
μ_{BW}	worker pollen consumption rate	$\frac{\text{g}}{\text{day} \cdot \text{individual}(W)}$
$b_W(t)$	worker birth rate	$\frac{\text{workers}}{\text{queen} \cdot \text{day}}$
μ_W	worker death rate	$\frac{1}{\text{day}}$
γ_W	worker-worker larvae survival coefficient	$\frac{\text{individual}(W)}{\text{individual}(Q)}$
$b_M(t)$	male birth rate	$\frac{\text{males}}{\text{queen} \cdot \text{day}}$
μ_M	male death rate	$\frac{1}{\text{day}}$
γ_M	worker-male larvae survival coefficient	$\frac{\text{individual}(W)}{\text{individual}(Q)}$
$b_G(t)$	gyne birth rate	$\frac{\text{gynes}}{\text{queen} \cdot \text{day}}$
μ_G	in-season gyne death rate	$\frac{1}{\text{day}}$
μ_{G_W}	hibernation gyne death rate	$\frac{1}{\text{day}}$
γ_G	worker-gyne larvae survival coefficient	$\frac{\text{individual}(W)}{\text{individual}(Q)}$
A_{max}	in-nest nectar maximum (per nest)	$\frac{\text{ml}}{\text{individual}(Q)}$
B_{max}	in-nest pollen maximum (per nest)	$\frac{\text{g}}{\text{individual}(Q)}$
P_0	initial larval pollen consumption	$\frac{\text{g}}{\text{individual}(W) \cdot \text{day}}$
\bar{r}	larval pollen consumption growth rate	$\frac{1}{\text{day}}$

Table 1: Model variables and parameters, with corresponding units.

$K = K(t)$, some uniformly bounded function on the interval $[0, T]$. We proceed with the assumption that K is constant.

(H2) The functions $f_2 : [0, T] \rightarrow \mathbb{R}^n$ and $f : [0, T] \times Z \times \mathbb{R}^{n \times \nu} \rightarrow \mathbb{R}^n$ are differentiable with all derivatives (ordinary and partial) dominated by integrable functions.

We define $\mathbf{x} = [A, B, Q, W, M, G]^T$. Then the system DDE can be written as

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= f(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)), \\ &= \begin{bmatrix} f_A(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \\ f_B(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \\ f_Q(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \\ f_W(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \\ f_M(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \\ f_G(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)) \end{bmatrix}. \end{aligned}$$

We note here that the the subscripts A, B, Q, \dots , etc., are used to denote the components of the $\mathbb{R}^n = \mathbb{R}^6$ function f . We can see from the model statement that (H2) is satisfied for the bumblebee DDE. We proceed to show that the Lipschitz condition (H1) is satisfied for the bumblebee model

2.3. LIPSCHITZ CONDITION FOR THE BUMBLEBEE MODEL

We introduce the notation that for a function $f(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30))$,

$$\begin{aligned} \delta f &= f(t, \tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}_t, \tilde{\mathbf{x}}(t - 22), \tilde{\mathbf{x}}(t - 26), \tilde{\mathbf{x}}(t - 30)) \\ &\quad - f(t, \mathbf{x}(t), \mathbf{x}_t, \mathbf{x}(t - 22), \mathbf{x}(t - 26), \mathbf{x}(t - 30)), \end{aligned}$$

and note that $\|\delta f\|_{\mathbb{R}^6}^2 = \delta f_A^2 + \delta f_B^2 + \delta f_Q^2 + \delta f_W^2 + \delta f_M^2 + \delta f_G^2$.

We first make the biologically reasonable assumption that for all times, $b_W(t) \leq b_{W,max} < \infty$ for some constant $b_{W,max}$. Then by noting that the

integrand in the larval consumption term is non-negative, we have

$$\begin{aligned}
 (l_W(t, Q_t) - l_W(t, \tilde{Q}_t))^2 &= \left(\int_{t-13}^{t-4} b_W(s)Q(s)P_0e^{\bar{r}(t-s)}ds \right. \\
 &\quad \left. - \int_{t-13}^{t-4} b_W(s)\tilde{Q}(s)P_0e^{\bar{r}(t-s)}ds \right)^2 \\
 &= \leq 9 \int_{t-13}^{t-4} \left(b_W(s)[Q(s) - \tilde{Q}(s)]P_0e^{\bar{r}(t-s)} \right)^2 ds \\
 &\leq 9 \int_{t-30}^t \left(b_W(s)[Q(s) - \tilde{Q}(s)]P_0e^{\bar{r}(t-s)} \right)^2 ds \\
 &\leq 9b_{W,max}^2P_0^2e^{60r} \int_{t-30}^t [Q(s) - \tilde{Q}(s)]^2 ds.
 \end{aligned}$$

Similar inequalities for l_M and l_G can be obtained under the assumption that there exist finite constants such that $b_M(t) \leq b_{M,max}$ and $b_G(t) \leq b_{G,max}$ for all t . For $K_l = 18P_0e^{30r} \max(b_{W,max}, b_{M,max}, b_{G,max})$, we obtain

$$\begin{aligned}
 \delta f_A^2 &\leq 3(b_{AW} - \mu_{AW})^2(W(t) - \tilde{W}(t))^2 + 3\mu_{AQ}^2(Q(t) - \tilde{Q}(t))^2 \\
 &\quad + 3K_l^2 \int_{t-30}^t [Q(s) - \tilde{Q}(s)]^2 ds \\
 &\leq 3(b_{AW} - \mu_{AW})^2(W(t) - \tilde{W}(t))^2 + 3\mu_{AQ}^2(Q(t) - \tilde{Q}(t))^2 \\
 &\quad + 3K_l^2 \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \delta f_B^2 &\leq 3(b_{BW} - \mu_{BW})^2(W(t) - \tilde{W}(t))^2 + 3\mu_{BQ}^2(Q(t) - \tilde{Q}(t))^2 \\
 &\quad + 3(K_l/2)^2 \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2.
 \end{aligned}$$

The function $L(x)$ of (1) satisfies the Lipschitz condition,

$$\begin{aligned}
 L(x) - L(\tilde{x}) &= \frac{1 - e^{-x}}{1 + e^{-x}} - \frac{1 - e^{-\tilde{x}}}{1 + e^{-\tilde{x}}} \\
 &= \frac{(1 - e^{-x})(1 + e^{-\tilde{x}}) - (1 - e^{-\tilde{x}})(1 + e^{-x})}{(1 + e^{-x})(1 + e^{-\tilde{x}})} \\
 &\leq (1 - e^{-x})(1 + e^{-\tilde{x}}) - (1 - e^{-\tilde{x}})(1 + e^{-x}) \\
 &= 2(e^{-\tilde{x}} - e^{-x}) \leq 2|x - \tilde{x}|.
 \end{aligned}$$

Moreover, since $L(x) \in [0, 1]$ for all $x > 0$, we have for positive arguments

$$L(x)L(y)L(z) - L(\tilde{x})L(\tilde{y})L(\tilde{z}) = [L(x) - L(\tilde{x})]L(y)L(x)$$

$$\begin{aligned}
 &+ [L(y) - L(\tilde{y})]L(\tilde{x})L(z) \\
 &+ [L(\tilde{z}) - L(z)]L(\tilde{x})L(\tilde{y}) \\
 &\leq 2|x - \tilde{x}| + 2|y - \tilde{y}| + 2|z - \tilde{z}|.
 \end{aligned}$$

Thus we find that the worker survivability term satisfies the inequality

$$\begin{aligned}
 \delta S_W \leq \frac{1}{9} \int_{t-18}^{t-9} &2 \left| \frac{W(s)}{\gamma_W Q(s)} - \frac{\tilde{W}(s)}{\gamma_W \tilde{Q}(s)} \right| + 2 \left| \frac{A(s)}{A_{max} Q(s)} - \frac{\tilde{A}(s)}{A_{max} \tilde{Q}(s)} \right| \\
 &+ 2 \left| \frac{B(s)}{B_{max} Q(s)} - \frac{\tilde{B}(s)}{B_{max} \tilde{Q}(s)} \right| ds.
 \end{aligned}$$

We note that the solution $Q(s)$ is exponentially decreasing, so $Q(s)$ is bounded above by Q_0 and below by $K_Q = Q_0 e^{-\mu_Q(T_W - T_S - 22)}$ for all s . Then for $K_W = W_0 e^{b_{W,max} Q_0(T_W - T_S - 22)}$, we have $W(s) \leq K_W$ for all s . We therefore have

$$\begin{aligned}
 \left| \frac{W(s)}{\gamma_W Q(s)} - \frac{\tilde{W}(s)}{\gamma_W \tilde{Q}(s)} \right| &= \frac{1}{\gamma_W} \left| \frac{W(s)\tilde{Q}(s) - \tilde{W}(s)Q(s)}{Q(s)\tilde{Q}(s)} \right| \\
 &\leq \frac{1}{\gamma_W K_Q^2} |W(s)\tilde{Q}(s) - \tilde{W}(s)Q(s)| \\
 &\leq \frac{1}{\gamma_W K_Q^2} (|\tilde{Q}(s)| |W(s) - \tilde{W}(s)| + |\tilde{W}(s)| |Q(s) - \tilde{Q}(s)|) \\
 &\leq \frac{1}{\gamma_W K_Q^2} (Q_0 |W(s) - \tilde{W}(s)| + K_W |Q(s) - \tilde{Q}(s)|).
 \end{aligned}$$

By similarly noting that for $K_A = A_0 e^{(b_{AW} - \mu_{AW})K_W(T_W - T_S - 22)}$ and $K_B = B_0 e^{(b_{BW} - \mu_{BW})K_W(T_W - T_S - 22)}$, we have $A(s) \leq K_A$ and $B(s) \leq K_B$ for all s , we obtain

$$\begin{aligned}
 \delta S_W^2 &\leq \left(\frac{2}{9} \int_{t-18}^{t-9} \left\{ \frac{|W(s) - \tilde{W}(s)|}{\gamma_W K_Q^2 / Q_0} + \frac{|A(s) - \tilde{A}(s)|}{A_{max} K_Q^2 / Q_0} + \frac{|B(s) - \tilde{B}(s)|}{B_{max} K_Q^2 / Q_0} \right. \right. \\
 &\quad \left. \left. + \frac{K_W + K_A + K_B}{\gamma_W K_Q^2} |Q(s) - \tilde{Q}(s)| \right\} ds \right)^2 \\
 &\leq \frac{4}{81} \left(\int_{t-30}^t \left\{ \frac{|W(s) - \tilde{W}(s)|}{\gamma_W K_Q^2 / Q_0} + \frac{|A(s) - \tilde{A}(s)|}{A_{max} K_Q^2 / Q_0} + \frac{|B(s) - \tilde{B}(s)|}{B_{max} K_Q^2 / Q_0} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. + \frac{\frac{K_W}{\gamma_W} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \left| Q(s) - \tilde{Q}(s) \right| \right\} ds \Big)^2 \\
 \leq & \frac{120}{81} \int_{t-30}^t \left\{ \frac{|W(s) - \tilde{W}(s)|}{\gamma_W K_Q^2 / Q_0} + \frac{|A(s) - \tilde{A}(s)|}{A_{max} K_Q^2 / Q_0} + \frac{|B(s) - \tilde{B}(s)|}{B_{max} K_Q^2 / Q_0} \right. \\
 & \left. + \frac{\frac{K_W}{\gamma_W} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \left| Q(s) - \tilde{Q}(s) \right| \right\}^2 ds \\
 \leq & \frac{840}{81} \int_{t-30}^t \left(\frac{|W(s) - \tilde{W}(s)|^2}{(\gamma_W K_Q^2 / Q_0)^2} + \frac{|A(s) - \tilde{A}(s)|^2}{(A_{max} K_Q^2 / Q_0)^2} + \frac{|B(s) - \tilde{B}(s)|^2}{(B_{max} K_Q^2 / Q_0)^2} \right. \\
 & \left. + \frac{(\frac{K_W}{\gamma_W} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}})^2}{(K_Q^2)^2} \left| Q(s) - \tilde{Q}(s) \right|^2 \right) ds \\
 \leq & \frac{280}{27} \max \left(\frac{Q_0}{\gamma_W K_Q^2}, \frac{Q_0}{A_{max} K_Q^2}, \frac{Q_0}{B_{max} K_Q^2}, \right. \\
 & \left. \frac{\frac{K_W}{\gamma_W} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \right)^2 \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2.
 \end{aligned}$$

In the second to last inequality above we have used the easily verifiable inequality

$$(a + b + c + d)^2 \leq 7(a^2 + b^2 + c^2 + d^2).$$

Computing the quantity δf_W^2 , we find

$$\begin{aligned}
 \delta f_W^2 &= (b_W(t - 22)Q(t - 22)S_W(t, A_t, B_t, W_t, Q_t) - \mu_W W(t) \\
 &\quad - b_W(t - 22)\tilde{Q}(t - 22)S_W(t, \tilde{A}_t, \tilde{B}_t, \tilde{W}_t, \tilde{Q}_t) + \mu_W \tilde{W}(t))^2 \\
 &= 2b_W(t - 22)^2 \left(Q(t - 22)\delta S_W + S(t, \tilde{A}_t, \tilde{B}_t, \tilde{W}_t)(Q(t - 22) - \tilde{Q}(t - 22)) \right)^2 \\
 &\quad + 2\mu_W^2 (\tilde{W}(t) - W(t))^2 \\
 &\leq 4b_{W,max}^2 Q_0^2 \delta S_W^2 + 4b_{W,max}^2 \left| Q(t - 22) - \tilde{Q}(t - 22) \right|^2 \\
 &\quad + 2\mu_W^2 \left| W(t) - \tilde{W}(t) \right|^2 \\
 &\leq K_{S_W} \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2 + 4b_{W,max}^2 \|\mathbf{x}(t - 22) - \tilde{\mathbf{x}}(t - 22)\|_{\mathbb{R}^6}^2 \\
 &\quad + 2\mu_W^2 \left| W(t) - \tilde{W}(t) \right|^2,
 \end{aligned}$$

where

$$K_{S_W} = \frac{1120}{27} b_{W,max}^2 Q_0^2 \max \left(\frac{Q_0}{\gamma_W K_Q^2}, \frac{Q_0}{A_{max} K_Q^2}, \frac{Q_0}{B_{max} K_Q^2}, \frac{\frac{K_W}{\gamma_W} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \right)^2.$$

We similarly have

$$\delta f_M^2 \leq K_{S_M} \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2 + 4b_{M,max}^2 \|\mathbf{x}(t - 26) - \tilde{\mathbf{x}}(t - 26)\|_{\mathbb{R}^6}^2 + 2\mu_M^2 \left| M(t) - \tilde{M}(t) \right|^2,$$

for

$$K_{S_M} = \frac{3360}{121} b_{M,max}^2 Q_0^2 \max \left(\frac{Q_0}{\gamma_M K_Q^2}, \frac{Q_0}{A_{max} K_Q^2}, \frac{Q_0}{B_{max} K_Q^2}, \frac{\frac{K_W}{\gamma_M} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \right)^2$$

and

$$\delta f_G^2 \leq K_{S_G} \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2 + 4b_{G,max}^2 \|\mathbf{x}(t - 30) - \tilde{\mathbf{x}}(t - 30)\|_{\mathbb{R}^6}^2 + 2\mu_G^2 \left| G(t) - \tilde{G}(t) \right|^2,$$

for

$$K_{S_G} = \frac{3360}{169} b_{G,max}^2 Q_0^2 \max \left(\frac{Q_0}{\gamma_G K_Q^2}, \frac{Q_0}{A_{max} K_Q^2}, \frac{Q_0}{B_{max} K_Q^2}, \frac{\frac{K_W}{\gamma_G} + \frac{K_A}{A_{max}} + \frac{K_B}{B_{max}}}{K_Q^2} \right)^2.$$

We note that $\delta f_Q^2 = \mu_Q^2 (Q - \tilde{Q})^2$. Combining the above expressions, we obtain

$$\|\delta f\|_{\mathbb{R}^6}^2 \leq K_1 \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|_{\mathbb{R}^6}^2 + K_2 \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2}^2 + K_3 \sum_{i=1}^3 \|\mathbf{x}(t - \tau_i) - \tilde{\mathbf{x}}(t - \tau_i)\|_{\mathbb{R}^6}^2,$$

for delays $\tau_1 = 22, \tau_2 = 26, \tau_3 = 30$ and constants

$$K_1 = \max [3(b_{AW} - \mu_{AW})^2, 3\mu_{AQ}^2, 3(b_{BW} - \mu_{BW})^2, 3\mu_{BQ}^2, 2\mu_W^2, 2\mu_M^2, 2\mu_G^2],$$

$$K_2 = \max [3K_l^2, K_{S_W}, K_{S_M}, K_{S_G}],$$

$$K_3 = \max [4b_{W,max}^2, 4b_{M,max}^2, 4b_{G,max}^2].$$

Then, taking $K = \max(\sqrt{K_1}, \sqrt{K_2}, \sqrt{K_3})$, we have the Lipschitz condition

$$\|\delta f\|_{\mathbb{R}^6} \leq K \left(\|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|_{\mathbb{R}^6} + \|\mathbf{x}_t - \tilde{\mathbf{x}}_t\|_{L_2} + \sum_{i=1}^3 \|\mathbf{x}(t - \tau_i) - \tilde{\mathbf{x}}(t - \tau_i)\|_{\mathbb{R}^6} \right).$$

3. THEORETICAL DEVELOPMENTS OF SOLUTION APPROXIMATIONS

3.1. NOTATION AND PRELIMINARY BACKGROUND

In [4] (see also a summary in [5]), the author presents the development of solutions for a general class of spline approximations to nonlinear functional differential equations. As stated in Section 2.1, we consider problems of the form (2) satisfying conditions (H1) and (H2). We next carry out discussions of underlying proofs of Theorems 2.1 and 2.2 and the supporting Lemmas 2.2-2.6 cited in [4]. We first introduce the necessary notation and some preliminary results.

We define the function $F : [0, T] \times Z \rightarrow \mathbb{R}^n$ by

$$F(t, \eta, \psi) = f(t, \eta, \psi, \psi(-\tau_1), \dots, \psi(-\tau_\nu)), \tag{3}$$

and we define the nonlinear operator $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}) \subset Z \rightarrow Z$ by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= W \equiv \{(\psi(0), \psi) \mid \psi \in H^1(-r, 0)\}, \\ \mathcal{A}(t)(\psi(0), \psi) &= (F(t, \psi(0), \psi), \psi'). \end{aligned}$$

We let Z^N be the approximating linear spline subspace (see [8, 4, 5]) of Z such that

$$\begin{aligned} Z^N &= \{(\phi(0), \phi) \mid \phi \text{ a continuous, linear spline with nodes at} \\ &\qquad t_j^N = -jr/N, \quad j = 0, 1, \dots, N\}, \end{aligned}$$

and $P^N = P_g^N$ be the orthogonal projection in $\langle \cdot, \cdot \rangle_g$ of Z onto Z^N , where

$$\langle (\psi(0), \psi), (\tilde{\psi}(0), \tilde{\psi}) \rangle_g = \langle \psi(0), \tilde{\psi}(0) \rangle_{\mathbb{R}^n} + \int_{-r}^0 \langle \psi(\sigma), \tilde{\psi}(\sigma) \rangle_{\mathbb{R}^n} g(\sigma) d\sigma$$

for g the linear weighting function defined on $[-r, 0]$ given by

$$g(\xi) = j \quad \text{for } \xi \in [-\tau_{\nu-j+1}, -\tau_{\nu-j}], \quad j = 1, 2, \dots, \nu.$$

We recall [8, 5] that the associated norm $\|\cdot\|_g$ generates a topology on $Z = \mathbb{R}^n \times L_2(-r, 0)$ that is equivalent to the Z topology. Furthermore, all of the usual density results involving the spaces $H^j(-r, 0)$ hold in the equivalent topology on Z as well as in the standard $\mathbb{R}^n \times L_2$ topology.

We restate Lemma 2.1 of [4], which will be of use in Section 3.2 below.

Lemma 2.1. If X is a Hilbert space and $x : [a, b] \rightarrow X$ is given by $x(t) = x(a) + \int_a^t y(\sigma) d\sigma$, then

$$\|x(t)\|^2 = \|x(a)\|^2 + 2 \int_0^t \langle x(\sigma), y(\sigma) \rangle d\sigma. \tag{4}$$

Lemma 2.1 is a restatement of a well-established equality (see [11, p. 100]) which follows immediately from

$$\frac{d}{dt} \frac{1}{2} \|x(t)\|^2 = \langle \dot{x}(t), x(t) \rangle.$$

We will also utilize the integral form of Gronwall’s Inequality for continuous functions [21], as stated below.

Gronwall’s Inequality. Let I be an interval in \mathbb{R} ($[a, \infty)$, $[a, b]$ or $[a, b]$ with $a \leq b$). Let α, β , and u be real-valued functions on I with the following conditions: β and u are continuous, the negative part of α is integrable on every closed, bounded subinterval of I , β is non-negative, α is non-decreasing, and u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(\sigma) u(\sigma) d\sigma \quad \forall t \in I.$$

Then

$$u(t) \leq \alpha(t) e^{\int_a^t \beta(\sigma) d\sigma}, \quad \forall t \in I.$$

Finally, we will use the following inequality for inner products on Banach spaces. If x, y are elements of a Banach space, then

$$\langle x, y \rangle \leq \frac{1}{2}(\|x\|^2 + \|y\|^2) \tag{5}$$

where $\| \cdot \|$ is the norm induced by the inner product.

3.2. THEORETICAL DEVELOPMENTS

We next discuss arguments underlying proofs of a series of theorems and lemmas.

Theorem 2.1. *Assume that (H1) holds and let $z(t; \phi, f_2) = (x(t; \phi, f_2), x_t(\phi, f_2))$, where x is the solution of (2) corresponding to $\phi \in H^1, f_2 \in L_2$. Then for $\zeta = (\phi(0), \phi)$, $z(t; \phi, f_2)$ is the unique solution on $[0, T]$ of*

$$z(t) = \zeta + \int_0^t [\mathcal{A}(\sigma)z(\sigma) + (f_2(\sigma), 0)]d\sigma. \tag{6}$$

Furthermore, $f_2 \rightarrow z(t; \phi, f_2)$ is weakly sequentially continuous from L_2 (weak) to Z (strong).

This is precisely Theorem 2.1 of [4]. The existence part of the proof is based on a standard uniform contraction principle [22, p. 7] well known to investigators of dynamical systems in the 1970's and 1980's. Earlier versions of Theorem 2.1 for linear delay systems are given in [3, 33, 34]. The theorem is stated as Theorem 2.1 in [29]. A detailed proof (which we will not repeat here) is given in [28]. To prove solution uniqueness, we make use of the dissipative inequality stated and proved below.

DDE Dissipative Inequality: For $z, \bar{z} \in \mathcal{D}(\mathcal{A})$,

$$\langle \mathcal{A}(t)z - \mathcal{A}(t)\bar{z}, z - \bar{z} \rangle_g \leq w(t) \|z - \bar{z}\|_g^2. \tag{7}$$

To establish this inequality, we let $z = (\phi(0), \phi)$ and $\bar{z} = (\bar{\phi}(0), \bar{\phi})$, $z, \bar{z} \in$

$\mathcal{D}(\mathcal{A})$. Then by definition of \mathcal{A} , we have

$$\begin{aligned}
 \langle \mathcal{A}(t)z - \mathcal{A}(t)\bar{z}, z - \bar{z} \rangle_g &= \langle (F(t, \phi(0), \phi), \phi') - (F(t, \bar{\phi}(0), \bar{\phi}), \bar{\phi}'), \\
 &\quad (\phi(0), \phi) - (\bar{\phi}(0), \bar{\phi}) \rangle_g \\
 &= \langle (F(t, \phi(0), \phi) - F(t, \bar{\phi}(0), \bar{\phi}), \phi' - \bar{\phi}'), \\
 &\quad (\phi(0) - \bar{\phi}(0), \phi - \bar{\phi}) \rangle_g \\
 &= \langle F(t, \phi(0), \phi) - F(t, \bar{\phi}(0), \bar{\phi}), \phi(0) - \bar{\phi}(0) \rangle_{\mathbb{R}^n} \\
 &\quad + \int_{-r}^0 \langle \phi'(\sigma) - \bar{\phi}'(\sigma), \phi(\sigma) - \bar{\phi}(\sigma) \rangle_{\mathbb{R}^n} g(\sigma) d\sigma.
 \end{aligned} \tag{8}$$

By (H1), we have

$$\begin{aligned}
 &\langle F(t, \phi(0), \phi) - F(t, \bar{\phi}(0), \bar{\phi}), \phi(0) - \bar{\phi}(0) \rangle_{\mathbb{R}^n} \\
 &\quad \leq \|F(t, \phi(0), \phi) - F(t, \bar{\phi}(0), \bar{\phi})\|_{\mathbb{R}^n} \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n} \\
 &\quad \leq K \left(\|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n} + \|\phi - \bar{\phi}\|_{L_2(-r,0)} \right. \\
 &\quad \quad \left. + \sum_{i=1}^{\nu} \|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n} \right) \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}.
 \end{aligned}$$

By definition of g , the second term in the inner product in (8) is

$$\begin{aligned}
 &\int_{-r}^0 \langle \phi'(\sigma) - \bar{\phi}'(\sigma), \phi(\sigma) - \bar{\phi}(\sigma) \rangle_{\mathbb{R}^n} g(\sigma) d\sigma \\
 &= \int_{-r}^0 \frac{d}{d\sigma} \left(\frac{\langle \phi(\sigma) - \bar{\phi}(\sigma), \phi(\sigma) - \bar{\phi}(\sigma) \rangle_{\mathbb{R}^n}}{2} \right) g(\sigma) d\sigma \\
 &= \sum_{i=1}^{\nu} \int_{-\tau_i}^{-\tau_{i-1}} \frac{d}{d\sigma} \left(\frac{\|\phi(\sigma) - \bar{\phi}(\sigma)\|_{\mathbb{R}^n}^2}{2} \right) (\nu + 1 - i) d\sigma \\
 &= \sum_{i=1}^{\nu} (\nu + 1 - i) \left(\frac{\|\phi(-\tau_{i-1}) - \bar{\phi}(-\tau_{i-1})\|_{\mathbb{R}^n}^2}{2} - \frac{\|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n}^2}{2} \right) \\
 &= \frac{\nu}{2} \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 - \sum_{i=1}^{\nu} \frac{\|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n}^2}{2}.
 \end{aligned}$$

Substitution yields

$$\begin{aligned}
 \langle \mathcal{A}(t)z - \mathcal{A}(t)\bar{z}, z - \bar{z} \rangle_g &\leq (K + \frac{\nu}{2}) \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 \\
 &+ K \|\phi - \bar{\phi}\|_{L_2(-r,0)} \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n} \\
 &+ \sum_{i=1}^{\nu} K \|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n} \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n} - \frac{\|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n}^2}{2} \\
 &\leq (K + \frac{\nu}{2}) \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 + K \left(\frac{\|\phi - \bar{\phi}\|_{L_2(-r,0)}^2}{2} + \frac{\|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2}{2} \right) \\
 &+ \sum_{i=1}^{\nu} \left(\frac{\|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n}^2}{2} + \frac{\|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 K^2}{2} \right) \\
 &- \frac{\|\phi(-\tau_i) - \bar{\phi}(-\tau_i)\|_{\mathbb{R}^n}^2}{2} \\
 &= (K + \frac{\nu}{2} + \frac{K}{2} + \frac{K^2\nu}{2}) \|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 + \frac{K}{2} \|\phi - \bar{\phi}\|_{L_2(-r,0)}^2 \\
 &\leq \left(\frac{\nu}{2} + \frac{3K}{2} + \frac{K^2\nu}{2} \right) \left(\|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 + \|\phi - \bar{\phi}\|_{L_2(-r,0)}^2 \right).
 \end{aligned}$$

We define $w(t) = \left(\frac{\nu}{2} + \frac{3K}{2} + \frac{K^2\nu}{2} \right)$ and note that because $g(\sigma) \geq 0$ for all $\sigma \in [-r, 0]$ we have

$$\|\phi(0) - \bar{\phi}(0)\|_{\mathbb{R}^n}^2 + \|\phi - \bar{\phi}\|_{L_2(-r,0)}^2 \leq \|z - \bar{z}\|_g^2.$$

The dissipative inequality $\langle \mathcal{A}(t)z - \mathcal{A}(t)\bar{z}, z - \bar{z} \rangle_g \leq w(t) \|z - \bar{z}\|_g^2$ therefore holds.

To establish the desired uniqueness of solutions to equation (6), we let $\phi, \bar{\phi} \in H^1(-r, 0)$ and define $\zeta = (\phi(0), \phi)$ and $\bar{\zeta} = (\bar{\phi}(0), \bar{\phi})$. Then the corresponding solutions to (6) are given by

$$\begin{aligned}
 z(t) &= \zeta + \int_0^t [\mathcal{A}(\sigma)z(\sigma) + (f_2(\sigma), 0)]d\sigma, \\
 \bar{z}(t) &= \bar{\zeta} + \int_0^t [\mathcal{A}(\sigma)\bar{z}(\sigma) + (f_2(\sigma), 0)]d\sigma.
 \end{aligned}$$

Suppose $\zeta = \bar{\zeta}$. Then we have

$$z(t) - \bar{z}(t) = \int_0^t [\mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma)]d\sigma.$$

Application of Lemma 2.1, the dissipative inequality (7), and Gronwall’s Inequality yields

$$\begin{aligned} \|z(t) - \bar{z}(t)\|_g^2 &\leq 2 \int_0^t \langle \mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma), z(\sigma) - \bar{z}(\sigma) \rangle_g d\sigma \\ &\leq 2 \int_0^t w(\sigma) \|z(\sigma) - \bar{z}(\sigma)\|_g^2 d\sigma \leq 0 \cdot \exp\left(\int_0^t 2w(\sigma)d\sigma\right) = 0. \end{aligned}$$

We therefore have $z(t) = \bar{z}(t)$ and the solution to (6) is unique.

Let Z^N and $P^N = P_g^N$ be the linear spline subspaces and projections as defined above. For the approximating operator $\mathcal{A}^N(t) = P^N \mathcal{A}(t) P^N$, we define the approximating solutions in Z^N

$$z^N(t) = P^N \zeta + \int_0^t [\mathcal{A}^N(\sigma)z^N(\sigma) + P^N(f_2(\sigma), 0)]d\sigma, \tag{9}$$

which are the unique solutions of the differential equation

$$\begin{aligned} \frac{dz^N}{dt} &= \mathcal{A}^N(t)z^N(t) + P^N(f_2(t), 0), \\ z^N(0) &= P^N \zeta. \end{aligned} \tag{10}$$

We note that these equations are finite dimensional ordinary differential equations and hence existence of solutions are readily guaranteed. To prove uniqueness, we will make use of the following dissipative inequality.

Approximation Dissipative Inequality: For $z, \bar{z} \in \mathcal{D}(\mathcal{A})$,

$$\langle \mathcal{A}^N(t)z - \mathcal{A}^N(t)\bar{z}, z - \bar{z} \rangle_g \leq w(t) \|z - \bar{z}\|_g^2. \tag{11}$$

By definition of \mathcal{A}^N and (7), we have

$$\begin{aligned} \langle \mathcal{A}^N(t)z - \mathcal{A}^N(t)\bar{z}, z - \bar{z} \rangle_g &= \langle P^N \mathcal{A}(t) P^N z - P^N \mathcal{A}(t) P^N \bar{z}, z - \bar{z} \rangle_g \\ &\leq \langle \mathcal{A}(t) P^N z - \mathcal{A}(t) P^N \bar{z}, P^N z - P^N \bar{z} \rangle_g \\ &\leq w(t) \|P^N z - P^N \bar{z}\|_g^2 \\ &\leq w(t) \|z - \bar{z}\|_g^2. \end{aligned}$$

Now, let $\phi, \bar{\phi} \in H^1(-r, 0)$ and define $\zeta = (\phi(0), \phi)$ and $\bar{\zeta} = (\bar{\phi}(0), \bar{\phi})$. Then the corresponding solutions to (9) are given by

$$\begin{aligned} z^N(t) &= P^N \zeta + \int_0^t [\mathcal{A}^N(\sigma)z^N(\sigma) + P^N(f_2(\sigma), 0)]d\sigma, \\ \bar{z}^N(t) &= P^N \bar{\zeta} + \int_0^t [\mathcal{A}^N(\sigma)\bar{z}^N(\sigma) + P^N(f_2(\sigma), 0)]d\sigma. \end{aligned}$$

Suppose $\zeta = \bar{\zeta}$. Then we have

$$z^N(t) - \bar{z}^N(t) = \int_0^t [\mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}^N(\sigma)\bar{z}^N(\sigma)]d\sigma.$$

Application of Lemma 2.1, and (11) (noting that $Z^N \subset \mathcal{D}(\mathcal{A})$), and Gronwall’s Inequality yields

$$\begin{aligned} \|z^N(t) - \bar{z}^N(t)\|_g^2 &\leq 2 \int_0^t \langle A^N(\sigma)z^N(\sigma) - A^N(\sigma)\bar{z}^N(\sigma), z^N(\sigma) - \bar{z}^N(\sigma) \rangle_g d\sigma \\ &\leq 2 \int_0^t w(\sigma) \|z^N(\sigma) - \bar{z}^N(\sigma)\|_g^2 d\sigma \\ &\leq 0 \cdot \exp\left(\int_0^t 2w(\sigma)d\sigma\right) = 0. \end{aligned}$$

Theorem 2.3. *Assume (H1), (H2). Let $\zeta = (\phi(0), \phi)$, $\phi \in H^1$ and $f_2 \in H^0(0, T)$ be given, with z^N and z the corresponding solutions on $[0, T]$ of (10) and (6) respectively. Then $z^N(t) \rightarrow z(t) = (x(t; \phi, f_2), x_t(\phi, f_2))$ as $N \rightarrow \infty$, uniformly in t on $[0, T]$.*

The proof of this theorem follows from the Lemmas below.

Lemma 2.2. Assume (H1) and let $\mathcal{Z} = \{z = (\phi(0), \phi) | \phi \in H^2\}$. Then $\mathcal{A}^N(t)z \rightarrow \mathcal{A}(t)z$ as $N \rightarrow \infty$ for each $z \in \mathcal{Z}$.

Proof. For $z = (\phi(0), \phi)$ and corresponding approximation $P^N z = (P^N \phi(0), P^N \phi)$, we consider the quantity $\|\mathcal{A}^N(t)z - \mathcal{A}(t)z\|_g$. Applying the definition of \mathcal{A}^N (and suppressing the dependence on t), we have

$$\begin{aligned} \|\mathcal{A}^N z - \mathcal{A}z\|_g &= \|P^N \mathcal{A}P^N z - \mathcal{A}z\|_g \\ &= \|P^N \mathcal{A}P^N z - P^N \mathcal{A}z + P^N \mathcal{A}z - \mathcal{A}z\|_g \tag{12} \\ &\leq \|P^N \mathcal{A}P^N z - P^N \mathcal{A}z\|_g + \|P^N \mathcal{A}z - \mathcal{A}z\|_g. \end{aligned}$$

However, $P^N \mathcal{A}z \rightarrow \mathcal{A}z$ as $N \rightarrow \infty$. We additionally note that

$$\|P^N \mathcal{A}P^N z - P^N \mathcal{A}z\|_g \leq \|\mathcal{A}P^N z - \mathcal{A}z\|_g.$$

By definition of the operator \mathcal{A} and the g -norm, we have

$$\begin{aligned} \|\mathcal{A}P^N z - \mathcal{A}z\|_g^2 &= \|(f(t, P^N \phi(0), P^N \phi), (P^N \phi)') - (f(t, \phi(0), \phi), \phi')\|_g^2 \\ &= \|f(t, P^N \phi(0), P^N \phi) - f(t, \phi(0), \phi)\|_{\mathbb{R}^n}^2 \tag{13} \\ &\quad + \int_{-r}^0 \|(P^N \phi)'(\sigma) - \phi'(\sigma)\|_{\mathbb{R}^n}^2 g(\sigma)d\sigma. \end{aligned}$$

By (H1),

$$\begin{aligned} & \|f(t, P^N \phi(0), P^N \phi) - f(t, \phi(0), \phi)\|_{\mathbb{R}^n} \\ & \leq K \left(\|P^N \phi(0) - \phi(0)\|_{\mathbb{R}^n} + \|P^N \phi - \phi\|_{L_2(-r,0)} \right. \\ & \quad \left. + \sum_{i=1}^{\nu} \|P^N \phi(-\tau_i) - \phi(-\tau_i)\|_{\mathbb{R}^n} \right). \end{aligned} \tag{14}$$

But from Theorems 2.5 and 2.6 of [36], we have that for $\phi \in H^2(-r, 0)$, $(P^N \phi)' - \phi' \rightarrow 0$ in L^2_g and $P^N \phi - \phi \rightarrow 0$ pointwise on $[-r, 0]$ as well as in L^2_g .

Lemma 2.3. Let $\mathcal{T} = \{(\zeta, f_2) \in W \times H^0(0, T) = \mathcal{D}(\mathcal{A}) \times L_2(0, T) \mid \phi \in H^2(-r, 0), f_2 \in H^1(0, T), \text{ with } \dot{\phi}(0) = F(0, \zeta) + f_2(0) \text{ where } \zeta = (\phi(0), \phi)\}$. Assume that (H1), (H2) hold. Then for $(\zeta, f_2) \in \mathcal{T}$, the corresponding solution $\sigma \rightarrow z(\sigma) = (x(\sigma), x_\sigma)$ of (6) (where x is the solution of (2)) satisfies $z(\sigma) \in \mathcal{Z}$ for each $\sigma \in [0, T]$.

Proof. Recall that $x_\sigma(\tau) = x(\sigma + \tau)$ for $\tau \in [-r, T - \tau]$. Assume (H1), (H2). Let $(\zeta, f_2) \in \mathcal{T}$ and $\sigma \in [0, T]$. We want to show that $z(\sigma) \in \mathcal{Z} = \{(x(\sigma), x_\sigma) \mid x_\sigma \in H^2(-r, T - \tau)\}$. It is enough to show that $x(\sigma) \in \mathbb{R}^n$, $x_\sigma, \dot{x}_\sigma, \ddot{x}_\sigma \in L_2(-r, T - \tau)$, where x is the solution of (2) (see Theorem 2.1). Clearly, $x(\sigma) \in \mathbb{R}^n$ because $f : [0, T] \times Z \times \mathbb{R}^{n\nu} \rightarrow \mathbb{R}^n$. Furthermore, $x_\sigma \in L_2(-r, T - \tau) \iff x_\sigma$ is integrable and real-valued on $[-r, T - \tau]$ and

$$\|x_\sigma\|_2 = \left(\int_{-r}^{T-\tau} |x_\sigma(t)|^2 dt \right)^{1/2} < \infty.$$

By definition of F , x_σ is continuous and real-valued on $[-r, T - \tau]$. Because $[-r, T - \tau]$ is a finite interval, we have $\|x_\sigma\|_2 < \infty$. Therefore $x_\sigma \in L_2[-r, T - \tau]$. The first derivative $\dot{x}_\sigma(t)$ has history $\dot{\zeta} = (\dot{\phi}(0), \dot{\phi})$ with $\phi \in H^2(-r, 0)$ and

$$\dot{\phi}(0) = F(0, \zeta) + f_2(0) = f(0, \phi(0), \phi, \phi(-\tau_1), \dots, \phi(-\tau_\nu)) + f_2(0).$$

With $\phi \in H^2(-r, 0)$, we have $\dot{\phi} \in L_2(-r, 0)$, and $\dot{\phi}$ is continuous and real valued on $[-r, 0]$. By definition of F , we have continuity of \dot{x} across $t = 0$. Therefore \dot{x}_σ is continuous on $[-r, T - \tau]$ and thus we find $\|\dot{x}_\sigma\|_2 < \infty$.

Finally, $\phi \in H^2(-r, 0) \Rightarrow \ddot{\phi} \in L_2$, so $\ddot{\phi}$ is integrable and real-valued on $[-r, 0]$. By definition of $\dot{\phi}$ for $(\zeta, f_2) \in \mathcal{T}$, we have

$$\ddot{\phi}(0^-) = \dot{F}(0^+, \zeta) + \dot{f}_2(0^+) = \dot{f}(0^+, \phi(0^+), \phi, \phi(-\tau_1), \dots, \phi(-\tau_\nu)) + \dot{f}_2(0^+).$$

Since f and f_2 are differentiable with dominated partial derivatives for f and derivative for f_2 on $[0, T]$, we have integrability of $\ddot{\phi}$ across $t = 0$. Therefore \ddot{x}_σ is real-valued and integrable on $[-r, T - \tau]$ and thus $\|\ddot{x}_\sigma\|_2 < \infty$.

Lemma 2.4. Assume (H1), (H2) and let $(\zeta, f_2) \in \mathcal{T}$ with z^N and z the respective solutions to (9) and (6). Then $z^N(t) \rightarrow z(t)$ uniformly in t on $[0, T]$.

Proof. Let $\phi \in H^1(-r, 0)$, $f_2 \in L_2(0, T)$, and define $\zeta = (\phi(0), \phi)$. Denote the solution to (6) for arguments $\{\zeta, f_2\}$ as $z(t)$ and the corresponding approximating solution given by (9) as $z^N(t)$. We consider the quantity $\Delta^N(t) = z^N(t) - z(t)$, or

$$\Delta^N(t) = (P^N - I)\zeta + \int_0^t [\mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}(\sigma)z(\sigma) + (P^N - I)(f_2(\sigma), 0)]d\sigma.$$

By Lemma 2.1, we have

$$\begin{aligned} \|\Delta^N(t)\|_g^2 &= \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2 + 2 \int_0^t \langle \mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}(\sigma)z(\sigma) \\ &\quad + (P^N - I)(f_2(\sigma), 0), \Delta^N(\sigma) \rangle_g d\sigma \\ &= \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2 + 2 \int_0^t \langle \mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}^N(\sigma)z(\sigma) + \mathcal{A}^N(\sigma)z(\sigma) \\ &\quad - \mathcal{A}(\sigma)z(\sigma), \Delta^N(\sigma) \rangle_g d\sigma \\ &\quad + 2 \int_0^t \langle (P^N - I)(f_2(\sigma), 0), \Delta^N(\sigma) \rangle_g d\sigma \\ &= \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2 + 2 \int_0^t \langle \mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}^N(\sigma)z(\sigma), \Delta^N(\sigma) \rangle_g d\sigma \\ &\quad + 2 \int_0^t \langle \mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma), \Delta^N(\sigma) \rangle_g d\sigma \\ &\quad + 2 \int_0^t \langle (P^N - I)(f_2(\sigma), 0), \Delta^N(\sigma) \rangle_g d\sigma. \end{aligned}$$

By (11), we obtain

$$2 \int_0^t \langle \mathcal{A}^N(\sigma)z^N(\sigma) - \mathcal{A}^N(\sigma)z(\sigma), \Delta^N(\sigma) \rangle_g d\sigma \leq 2 \int_0^t w(\sigma) \|\Delta^N(\sigma)\|_g^2 d\sigma$$

and by (5), we find

$$\begin{aligned}
 & 2 \int_0^t \langle \mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma), \Delta^N(\sigma) \rangle_g d\sigma \\
 & \leq \int_0^t \|\mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma)\|_g^2 + \|\Delta^N(\sigma)\|_g^2 d\sigma
 \end{aligned}$$

$$\begin{aligned}
 & 2 \int_0^t \langle (P^N - I)(f_2(\sigma), 0), \Delta^N(\sigma) \rangle_g d\sigma \\
 & \leq \int_0^t \|(P^N - I)(f_2(\sigma), 0)\|_g^2 + \|\Delta^N(\sigma)\|_g^2 d\sigma.
 \end{aligned}$$

Substitution yields

$$\begin{aligned}
 \|\Delta^N(t)\|_g^2 & \leq \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2 + \int_0^t \|\mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma)\|_g^2 d\sigma \\
 & \quad + \int_0^t \|(P^N - I)(f_2(\sigma), 0)\|_g^2 d\sigma + 2 \int_0^t (\omega + 1) \|\Delta^N(\sigma)\|_g^2 d\sigma \\
 & \leq \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2 + \int_0^T \|\mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma)\|_g^2 d\sigma \\
 & \quad + \int_0^T \|(P^N - I)(f_2(\sigma), 0)\|_g^2 d\sigma + 2 \int_0^t (\omega + 1) \|\Delta^N(\sigma)\|_g^2 d\sigma \\
 & = \epsilon_1(N) + \epsilon_2(N) + \epsilon_3(N) + 2 \int_0^t (\omega + 1) \|\Delta^N(\sigma)\|_g^2 d\sigma
 \end{aligned}$$

for

$$\begin{aligned}
 \epsilon_1(N) & = \|(P^N - I)\zeta\|_{\mathbb{R}^n}^2, \\
 \epsilon_2(N) & = \int_0^T \|\mathcal{A}^N(\sigma)z(\sigma) - \mathcal{A}(\sigma)z(\sigma)\|_g^2 d\sigma, \\
 \epsilon_3(N) & = \int_0^T \|(P^N - I)(f_2(\sigma), 0)\|_g^2 d\sigma.
 \end{aligned}$$

Then, by Gronwall’s Inequality, we have

$$\|\Delta^N(t)\|_g^2 \leq [\epsilon_1(N) + \epsilon_2(N) + \epsilon_3(N)] \exp \left[2 \int_0^t (\omega(\sigma) + 1) d\sigma \right].$$

Observing that $\exp \left[2 \int_0^t (\omega(\sigma) + 1) d\sigma \right]$ is bounded for $t \in [0, T]$, we find that it suffices to prove that $\epsilon_i(N) \rightarrow 0, i=1,2,3$. We have that the convergence of ϵ_1 and ϵ_3 to 0 is immediately argued while a consideration of Lemma 2.2 along with reference to Theorems 2.5 and 2.6 of [36] reveals that the convergence of $\mathcal{A}^N(\sigma)z(\sigma) \rightarrow \mathcal{A}(\sigma)z(\sigma)$ is dominated on the interval $[0, T]$ and hence we also have $\epsilon_2 \rightarrow 0$.

Lemma 2.5. Assume (H1). Then the solutions of (6) and (9) depend continuously (in the $Z \times H^0$ topology) on (ζ, f_2) in $W \times H^0$, uniformly in t on $[0, T]$.

Proof. Let $\phi, \bar{\phi} \in H^1(-r, 0)$ and $f_2, \bar{f}_2 \in L_2(0, T)$ and define $\zeta = (\phi(0), \phi)$ and $\bar{\zeta} = (\bar{\phi}(0), \bar{\phi})$. Then for arguments $\{\zeta, f_2\}$ and $\{\bar{\zeta}, \bar{f}_2\}$ denote solutions to (6) as $z(t)$ and $\bar{z}(t)$ respectively. We define the quantity

$$\Delta(t) = z(t; \zeta, f_2) - \bar{z}(t; \bar{\zeta}, \bar{f}_2)$$

and by definition of solutions $z(t), \bar{z}(t)$ we obtain

$$\begin{aligned} \Delta(t) &= \zeta + \int_0^t [\mathcal{A}(\sigma)z(\sigma) + (f_2(\sigma), 0)]d\sigma - \left(\bar{\zeta} + \int_0^t [\mathcal{A}(\sigma)\bar{z}(\sigma) + (\bar{f}_2(\sigma), 0)]d\sigma \right) \\ &= \zeta - \bar{\zeta} + \int_0^t [\mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma) + (f_2(\sigma) - \bar{f}_2(\sigma), 0)]d\sigma \\ &= \delta_1(0) + \int_0^t \delta_2(\sigma)d\sigma \end{aligned}$$

for $\delta_1(0) = \zeta - \bar{\zeta} = (\phi(0) - \bar{\phi}(0), \phi - \bar{\phi})$ and $\delta_2(\sigma) = \mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma) + (f_2(\sigma) - \bar{f}_2(\sigma), 0)$.

By Lemma 2.1, we therefore have

$$\begin{aligned} \langle \Delta(t), \Delta(t) \rangle_g &= \langle \delta_1(0), \delta_1(0) \rangle_g + 2 \int_0^t \langle \Delta(\sigma), \delta_2(\sigma) \rangle_g d\sigma \\ &= \langle \delta_1(0), \delta_1(0) \rangle_g + 2 \int_0^t \langle \delta_2(\sigma), \Delta(\sigma) \rangle_g d\sigma \\ &= \|\zeta - \bar{\zeta}\|_g^2 + 2 \int_0^t \langle \mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma), \Delta(\sigma) \rangle_g \\ &\quad + \langle (f_2(\sigma) - \bar{f}_2(\sigma), 0), \Delta(\sigma) \rangle_g d\sigma. \end{aligned}$$

We note that because $\Delta(t) = z(t) - \bar{z}(t)$ we have by (7)

$$\begin{aligned} \int_0^t \langle \mathcal{A}(\sigma)z(\sigma) - \mathcal{A}(\sigma)\bar{z}(\sigma), \Delta(\sigma) \rangle_g d\sigma &\leq \int_0^t w(\sigma) \langle \Delta(\sigma), \Delta(\sigma) \rangle_g \\ &= \int_0^t w(\sigma) \|\Delta(\sigma)\|_g^2 d\sigma. \end{aligned}$$

Additionally, by (5) and the definition of the g -inner product,

$$\int_0^t \langle (f_2(\sigma) - \bar{f}_2(\sigma), 0), \Delta(\sigma) \rangle_g \leq \int_0^t \frac{1}{2} \left(\|(f_2(\sigma) - \bar{f}_2(\sigma), 0)\|_g^2 + \|\Delta(\sigma)\|_g^2 \right) d\sigma$$

$$\begin{aligned}
 &= \int_0^t \frac{1}{2} \left\{ (\|f_2(\sigma) - \bar{f}_2(\sigma)\|_{\mathbb{R}^n} + 0)^2 + \|\Delta(\sigma)\|_g^2 \right\} d\sigma \\
 &= \frac{1}{2} \|f_2 - \bar{f}_2\|_{L_2(0,t)}^2 + \int_0^t \frac{1}{2} \|\Delta(\sigma)\|_g^2 d\sigma.
 \end{aligned}$$

Then substitution yields

$$\begin{aligned}
 \|\Delta(t)\|_g^2 = \langle \Delta(t), \Delta(t) \rangle_g &\leq \|\zeta - \bar{\zeta}\|_g^2 + \frac{1}{2} \|f_2 - \bar{f}_2\|_{L_2(0,t)}^2 \\
 &\quad + \int_0^t [2w(\sigma) + 1] \|\Delta(\sigma)\|_g^2 d\sigma.
 \end{aligned}$$

By Gronwall’s Inequality, we therefore have

$$\begin{aligned}
 \|\Delta(t)\|_g^2 &\leq \left(\|\zeta - \bar{\zeta}\|_g^2 + \frac{1}{2} \|f_2 - \bar{f}_2\|_{L_2(0,t)}^2 \right) \exp \left[\int_0^t 2w(\sigma) d\sigma + t \right] \\
 &= \left(\|\zeta - \bar{\zeta}\|_g^2 + \frac{1}{2} \|f_2 - \bar{f}_2\|_{L_2(0,t)}^2 \right) \exp \left[2\|w\|_{L_1(0,t)} + t \right] \\
 &\leq \left(\|\zeta - \bar{\zeta}\|_g^2 + \frac{1}{2} \|f_2 - \bar{f}_2\|_{L_2(0,T)}^2 \right) \exp \left[2\|w\|_{L_1(0,T)} + T \right].
 \end{aligned}$$

For all $\epsilon > 0$, we can choose $\zeta, \bar{\zeta}$ and f_2, \bar{f}_2 such that $\|\Delta(t)\|_g^2 < \epsilon$ for all $t \in [0, T]$ and therefore solutions $x(t; \zeta, f_2)$ depend uniformly continuously on arguments ζ, f_2 .

Lemma 2.6. The set \mathcal{T} defined in Lemma 2.3 is dense in $Z \times L_2(0, T)$.

Proof. We recall

$$\mathcal{Z} = \{ \zeta = (\phi(0), \phi) \in Z \mid \phi \in H^2(-r, 0) \}, \tag{15}$$

and that $H^2(a, b) \subset H^1(a, b) \subset L_2(a, b)$ where each of these spaces is dense in the $H^0 = L_2$ topology. For any $\epsilon > 0$, and $f_2 \in L_2(0, T)$ arbitrary and $\zeta \in \mathcal{Z}$, one can readily construct \bar{f}_2 with $\|f_2 - \bar{f}_2\|_{L_2(0,T)} < \epsilon$ with \bar{f}_2 piecewise $C^1(0, T)$ and $\bar{f}_2(0) = \dot{\phi}(0) - F(0, \zeta)$. Thus $(\zeta, \bar{f}_2) \in \mathcal{T}$. This gives rise to ordered pairs $(\zeta, \mathcal{I}(\zeta))$ where

$$\mathcal{I}(\zeta) \equiv \{ f_2 \in L_2(0, T) \mid f_2 \in H^1(0, T), f_2(0) = \dot{\phi}(0) - F(0, \zeta) \}$$

with $\mathcal{I}(\zeta)$ dense in $L_2(0, T)$. Hence $\bigcup_{\zeta \in \mathcal{Z}} \{(\zeta, \mathcal{I}(\zeta))\}$ is dense in $W \times L_2(0, T)$ where we recall $W = \mathcal{D}(\mathcal{A}) = \{(\phi(0), \phi) \mid \phi \in H^1(0, T)\}$. The desired density results follow immediately from

$$\bigcup_{\zeta \in \mathcal{Z}} \{(\zeta, \mathcal{I}(\zeta))\} \subset \mathcal{T} \subset W \times L_2(0, T) \subset \mathbb{R}^n \times L_2(-r, 0) \times L_2(0, T)$$

since $\bigcup_{\zeta \in \mathcal{Z}} \{(\zeta, \mathcal{I}(\zeta))\}$ is dense in $\mathbb{R}^n \times L_2(-r, 0) \times L_2(0, T)$.

4. CONCLUSION

We have shown that the bumblebee models satisfy the conditions outlined in the theory first presented in [4]. Thus the corresponding methods based on spline approximations may be applied to this problem. In addition, we have here finally completed the theory first outlined in [4]. Because this numerical method allows one to approximate solutions for such a general non-linear delay equation, this method can be used extensively for approximations to ecological models requiring delays as well as to our subsequently developed bumblebee models.

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