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# QUADRATURE RULES AND ITERATIVE NUMERICAL METHOD FOR TWO-DIMENSIONAL NONLINEAR FREDHOLM FUZZY INTEGRAL EQUATIONS

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**ABSTRACT:** In this paper, we introduce some generalized quadrature rules to approximate two-dimensional Henstock integral of fuzzy-number-valued functions by giving error bounds for Henstock integrable, bounded mappings in terms of uniform modulus of continuity. We also consider generalizations of classical quadrature rules, such as midpoint-type, trapezoidal and three-point-type quadrature. Moreover, we propose an iterative procedure based on trapezoidal quadrature to solve two-dimensional nonlinear Fredholm fuzzy integral equations. The error estimation of the proposed method is given in terms of uniform and partial modulus of continuity. Finally , an illustrative numerical experiment confirms the theoretical results and demonstrates the accuracy of the method.

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**Key Words:** numerical integration, quadrature formula, two-dimensional Fredholm fuzzy integral equation, iterative method

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## 1. INTRODUCTION

The concepts of fuzzy integral and differential equations have been studied by many mathematicians and authors. The study of fuzzy integral equations begins with the investigations performed by of Kaleva [7], Seikkala [10], Goetshel and Voxman [6] and others. The Henstock integral for fuzzy-valued functions was introduced and studied in [11]. Since many problems in engineering and applied sciences can be put in the form of two-dimensional fuzzy integral equations, it is important to develop numerical methods for solving such integral equations.The interest in fuzzy Fredholm integral equations is based primarily on its applications in fuzzy financial and economic systems [4]. In this paper , we introduce two-dimensional fuzzy integrals and propose some generalized quadrature rules and their depended theorems for Henstock integrable, bounded mappings. Also, we present the conditions for existence of unique solution for two-dimensional nonlinear Fredholm fuzzy integral equation. Finaly we introduce iterative method for solving of this equation. The error estimation of the proposed method is given in terms of uniform modulus of continuity. Some numerical example is uncluded in order to confirm the theoretical results test.

## 2. PRELIMINARIES

In this section, we review some necessary basic notions and results about fuzzy numbers, fuzzy-number-valued functions and fuzzy integrals.

**Definition 1.** [6] A fuzzy number is a function $u : \mathbb{R} \to [0, 1]$ satisfying the following properties:

1. $u$ is upper semicontinuous on $\mathbb{R}$,
2. $u(x) = 0$ outside of some interval $[c, d]$,
3. there are the real numbers $a$ and $b$ with $c \leq a \leq b \leq d$ , such that $u$ is increasing on $[c, a]$, decreasing on $[b, d]$, and $u(x) = 1$ for each $x \in [a, b]$,
4. $u$ is fuzzy convex set ( that is $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$).

The set of all fuzzy numbers is denoted by $E^1$. The neutral element with respect to $\oplus$ in $E^1$ is denoted by $\tilde{0} = \chi_{\{0\}}$. According to [11], any real number $\alpha \in \mathbb{R}$ can be interpreted as a fuzzy number $\alpha = \chi_{\{\alpha\}}$ and therefore $\mathbb{R} \subset E^1$.

**Definition 2.** [6, 11] For any $0 < r \leq 1$ an arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, which satisfies the following properties:

1. $\underline{u}(r)$ is bounded left continuous nondecreasing function over $[0, 1]$,
2. $\bar{u}(r)$ is bounded left continuous nonincreasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \bar{u}(r)$.

For $u,\ v \in E^1$, $k \in \mathbb{R}$, the linear operations of addition and scalar multiplication are defined by $(u \oplus v)(r)(\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$ and

$$(k \odot u)(r) = \begin{cases} (k\underline{u}(r), k\bar{u}(r)), & \text{if } k \geq 0, \\ (k\underline{u}(r), k\bar{u}(r)), & \text{if } k < 0, \end{cases}$$

for all $r \in [0, 1]$.

According , [2, 11] the following algebraic properties for any $u,\ v,\ w \in E^1$:

1. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ and $u \oplus v = v \oplus u$,
2. $u \oplus \tilde{0} = \tilde{0} \oplus u$,
3. with respect to $\tilde{0}$, none $u \in E^1 \setminus \mathbb{R}$, $u \neq \tilde{0}$ has opposite in $(E^1, \oplus)$,
4. $(a \oplus b) \odot u = a \odot u \oplus b \odot u$ for all $a, b \in \mathbb{R}$ with $ab \geq 0$ or $ab \leq 0$,
5. $a \odot (b \odot u) = (a.b) \odot u$, for all $a \in \mathbb{R}$ and $1 \odot u = u$.

**Definition 3.** (see [5, 11]) For arbitrary fuzzy numbers $u = (\underline{u}(r), \bar{u}(r))$, $v = (\underline{v}(r), \bar{v}(r))$ the quantity

$$D(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}$$

is the distance between $u$ and $v$ and also the following properties hold:

1. $(E^1, D)$ is a complete metric space,

2. $D(u \oplus w, v \oplus w) = D(u, v)$, for all $u, v, w \in E^1$,

3. $D(k \odot u, k \odot v) = |k| D(u, v)$, for all $u, v \in E^1$, for all $k \in \mathbb{R}$,

4. $D(u \oplus v, w \oplus e) = D(u, w) + D(v, e)$, for all $u, v, w, e \in E^1$,

5. $D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2| D(u, \tilde{0})$, for all $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 \geq 0$ and $u \in E^1$.

For any fuzzy-number-valued function $f : A = [a, b] \times [c, d] \to E^1$ we can define the functions $\underline{f}(.,.,r)$, $\overline{f}(.,.,r) : A \to \mathbb{R}$, by $\underline{f}(s,t,r) = (\underline{f(s,t)})(r)$ and $\overline{f}(s,t,r) = (\overline{f(s,t)})(r)$ for each $(s,t) \in A$ and $r \in [0,1]$. These functions are called the left and right $r-$level functions of $f$.

**Definition 4.** (see [7, 9]) A fuzzy-number-valued function $f : A \to E^1$ is said to be continuous at $(s_0, t_0) \in A$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(f(s,t), f(s_0, t_0)) < \varepsilon$ whenever $|s - s_0| + |t - t_0| \leq \delta$. If $f$ be continuous for each $(s,t) \in A$, then we say that $f$ is continuous on $A$. A fuzzy number $u \in E^1$ is upper bound for a fuzzy-number-valued function $f : A \to E^1$ if $\underline{f}(s,t,r) \leq \underline{u}(r)$ and $\overline{f}(s,t,r) \leq \overline{u}(r)$ for all $(s,t) \in A$ and $r \in [0,1]$. Similarly, a fuzzy number $u \in E^1$ is lower bound for a fuzzy-number-valued function $f : A \to E^1$ if $\underline{f}(s,t,r) \geq \underline{u}(r)$ and $\overline{f}(s,t,r) \geq \overline{u}(r)$ for all $(s,t) \in A$ and $r \in [0,1]$. A fuzzy-number-valued function $f : A \to E^1$ is said to be bounded if it has a lower bound and upper bound.

On the set

$$C(A, E^1) = \{f : A \to E^1 : f \text{ is continuous}\},$$

we define

$$D^*(f, g) = \sup_{(s,t) \in A} D(f(s,t), g(s,t)),$$

for all $f, g \in C(A, E^1)$ and $D^*(., \tilde{0})$ is denoted by $\|.\|_{\mathcal{F}}$ . It is obvious that $(C(A, E^1), D^*)$ is a complete metric space.

**Definition 5.** (see [1]) Let $f : A \to E^1$, be a bounded mapping, then the function

$$\omega^1_{[a,b]}(f, .),\ \omega^2_{[c,d]}(f, .) : \mathbb{R}_+ \cup 0 \to \mathbb{R}_+$$

defined by

$$\omega^1_{[a,b]}(f,\delta) = \sup\{|f(s_1,t) - f(s_2,t)| : |s_1 - s_2| \le \delta; t \in [c,d]\},$$
$$\omega^2_{[c,d]}(f,\delta) = \sup\{|f(s,t_1) - f(s,t_2)| : |t_1 - t_2| \le \delta; s \in [a,b]\},$$

is called the modulus of oscillation of $f$ on $[a,b]$ , $[c,d]$. In addition if $f \in C(A, E^1)$, then $\omega^1_{[a,b]}(f,\delta)$ and $\omega^2_{[c,d]}(f,\delta)$ is called uniform modulus of continuity of $f$.

According to [3], the following properties hold true:

1. $D(f(x,y), f(s,t)) \le \omega^1_{[a,b]}(f,|x-s|)+\omega^2_{[c,d]}(f,|y-t|)$ for any $(x,y),(s,t) \in A$,
2. $\omega^1_{[a,b]}(f,\delta)$ and $\omega^2_{[c,d]}(f,\delta)$ are a non-decreasing mapping in $\delta$,
3. $\omega^1_{[a,b]}(f,0) = 0$, $\omega^2_{[c,d]}(f,0) = 0$,
4. $\omega^1_{[a,b]}(f,\delta_1+\delta_2) \le \omega^1_{[a,b]}(f,\delta_1) + \omega^1_{[a,b]}(f,\delta_2)$ and $\omega^2_{[c,d]}(f,\delta_1+\delta_2) \le \omega^2_{[c,d]}(f,\delta_1) + \omega^2_{[c,d]}(f,\delta_2)$ for any $\delta_1, \delta_2 \ge 0$,
5. $\omega^1_{[a,b]}(f,n\delta) \le n\omega^1_{[a,b]}(f,\delta)$ and $\omega^2_{[c,d]}(f,n\delta) \le n\omega^2_{[c,d]}(f,\delta)$, for any $\delta \ge 0$ and $n \in \mathbb{N}$,
6. $\omega^1_{[a,b]}(f,\lambda\delta) \le (\lambda+1)\omega^1_{[a,b]}(f,\delta)$ and $\omega^2_{[c,d]}(f,\lambda\delta) \le (\lambda+1)\omega^2_{[c,d]}(f,\delta)$, for any $\delta, \lambda \ge 0$,
7. If $[a_1,b_1] \subseteq [a,b]$ and $[c_1,d_1] \subseteq [c,d]$, then $\omega^1_{[a_1,b_1]}(f,\delta) \le \omega^1_{[a,b]}(f,\delta)$ and $\omega^2_{[c_1,d_1]}(f,\delta) \le \omega^2_{[c,d]}(f,\delta)$ for all $\delta \ge 0$.

**Definition 6.** [11] Let $f : [a,b] \times [c,d] \to E^1$, for $\Delta^n_x : a = x_0 < x_1 < ... < x_n = b$ and $\Delta^n_y : c = y_0 < y_1 < ... < y_n = d$, be two partitions of the intervals $[a,b]$ and $[c,d]$, respectively. Let one consider the intermidiate points $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$, $i = 1,...,n$; $j = 1,...,n$, and $\delta : [a,b] \to \mathbb{R}_+$ and $\sigma : [c,d] \to \mathbb{R}_+$. The divisions $P_x = ([x_{i-1}, x_i]; \xi_i)$, $i = 1,...,n$, and $P_y = ([y_{j-1}, y_i]; \eta_j)$, $j = 1,...,n$, denoted shortly by $P_x = (\Delta^n, \xi)$ and $P_y = (\Delta^n, \eta)$ are said to be $\delta$-fine and $\sigma$-fine, respectively, if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $[y_{j-1}, y_j] \subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))$.

The function $f$ is said to be two-dimensional Henstock integrable to $I \in E^1$ if for every $\varepsilon > 0$ there are functions $\delta : [a,b] \to E^1$ and $\sigma : [c,d] \to E^1$ such that for any $\delta$-fine and $\sigma$-fine divisions we have $D(\sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j), I) < \varepsilon$, where $\sum$ denotes the fuzzy summation. Then, $I$ is called the two-dimensional Henstock integral of $f$ and is denoted by $I(f) = (FH)\int_c^d \int_a^b f(s,t)dsdt$.

If the above $\delta$ and $\sigma$ are constant functions, then one recaptures the concept of Riemann integral. In this case, $I \in E^1$ will be called two-dimensional integral of $f$ on $[a,b] \times [c,d]$ and will be denoted by $(FR)\int_c^d \int_a^b f(s,t)dsdt$.

In [11], the authors introduced and concept of the Henstock integral for a fuzzy number-valued function.

## 3. QUADRATURE RULES FOR TWO-DIMENSIONAL HENSTOCK INTEGRALS

In this section, we present some quadrature rules for two-dimensional Henstock integral.

**Theorem 1.** *Let $f : A \to E^1$ be Henstock integrable, bounded mappings. Then, for any divisions $a = x_0 < x_1 < ... < x_n = b$ and $b = y_0 < y_1 < ... < y'_n = d$ and any points $\xi_i \in [x_{i-1}, x_i], \eta_j \in [y_{j-1}, y_j]$, one has*

$$D((FH)\int_c^d (FH)\int_a^b f(s,t)dsdt, \sum_{j=1}^{n'}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j))$$
$$\leq \sum_{j=1}^{n'}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1})(\omega^1_{[x_{i-1},x_i]}(f, |x_i - x_{i-1}|)$$
$$+ \omega^2_{[y_{j-1},y_j]}(f, |y_j - y_{j-1}|)).$$

**Proof.** It is known that the Henstock integrals are additive related to interval. This leads us to

$$D((FH)\int_c^d (FH)\int_a^b f(s,t)dsdt, \sum_{j=1}^{n'}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j))$$

$$=D\Big(\sum_{j=1}^{n'}\sum_{i=1}^{n}(FH)\int_{y_{j-1}}^{y_j}(FH)\int_{x_{i-1}}^{x_i} f(s,t)dsdt, \sum_{j=1}^{n'}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})$$
$$\odot f(\xi_i,\eta_j)\Big)$$
$$\leq \sum_{j=1}^{n'}\sum_{i=1}^{n}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} D(f(s,t), f(\xi_i,\eta_j))dsdt$$
$$\leq \sum_{j=1}^{n'}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})(\omega^1_{[x_{i-1},x_i]}(f,|x_i-x_{i-1}|)$$
$$+\,\omega^2_{[y_{j-1},y_j]}(f,|y_j-y_{j-1}|)),$$

which completes the proof. □

From the above inequality, we infer some generalization of well-known midpoint-type, trapezoidal-type and tree-point-type inequalities with error estimations.

**Corollary 1.** *Assume that $f : A \to E^1$ is a Henstock integrable, bounded mapping. Then, with the notation $\omega^1_{[a,b]} = \omega^1_{[a,b]}(f, b-a)$ and $\omega^2_{[c,d]} = \omega^2_{[c,d]}(f, d-c)$ one has:*

$$a)D\Big((FH)\int_c^d(FH)\int_a^b f(s,t)dsdt, (b-a)(d-c)\odot f(x,y)\Big) \leq$$
$$\leq (x-a)(y-c)(\omega^1_{[a,x]}+\omega^2_{[c,y]}) + (x-a)(d-y)(\omega^1_{[a,x]}+\omega^2_{[y,d]})+$$
$$+(b-x)(y-c)(\omega^1_{[x,b]}+\omega^2_{[c,y]}) + (b-x)(d-y)(\omega^1_{[x,b]}+\omega^2_{[y,d]}),$$
*for any $(x,y)\in A$,*

$$b)D\Big((FH)\int_c^d(FH)\int_a^b f(s,t)dsdt, (x-a)(y-c)\odot f(u,\alpha)\oplus$$
$$\oplus(x-a)(d-y)\odot f(u,\beta)\oplus(b-x)(y-c)\odot f(v,\alpha)\oplus(b-x)(d-y)\odot f(v,\beta)\Big) \leq$$
$$\leq (x-a)(y-c)(\omega^1_{[a,x]}+\omega^2_{[c,y]}) + (x-a)(d-y)(\omega^1_{[a,x]}+\omega^2_{[y,d]})+$$
$$+(b-x)(y-c)(\omega^1_{[x,b]}+\omega^2_{[c,y]}) + (b-x)(d-y)(\omega^1_{[x,b]}+\omega^2_{[y,d]}),$$
*for any $(x,y)\in A, u\in[a,x], v\in[x,b], \alpha\in[c,y], \beta\in[y,d]$,*

$$c)D\Big((FH)\int_c^d(FH)\int_a^b f(s,t)dsdt, (\alpha-a)(\theta-c)\odot f(u,r)\oplus$$
$$\oplus(\alpha-a)(\gamma-\theta)\odot f(u,p)\oplus(\alpha-a)(d-\gamma)\odot f(u,z)\oplus(\beta-\alpha)(\theta-c)\odot f(v,r)\oplus$$
$$\oplus(\beta-\alpha)(\gamma-\theta)\odot f(v,p)\oplus(\beta-\alpha)(d-\gamma)\odot f(u,z)\oplus$$
$$\oplus(b-\beta)(\theta-c)\odot f(w,r)\oplus(b-\beta)(\gamma-\theta)\odot f(w,p)\oplus(b-\beta)(d-\gamma)\odot f(w,z)\Big) \leq$$

$\leq (\alpha-a)(\theta-c)(\omega^1_{[a,\alpha]}+\omega^2_{[c,\theta]})+(\alpha-a)(\gamma-\theta)(\omega^1_{[a,\alpha]}+\omega^2_{[\theta,\gamma]})+$
$+(\alpha-a)(d-\gamma)(\omega^1_{[a,\alpha]}+\omega^2_{[\gamma,d]})+(\beta-\alpha)(\theta-c)(\omega^1_{[\alpha,\beta]}+\omega^2_{[c,\theta]})+$
$+(\beta-\alpha)(\gamma-\theta)(\omega^1_{[\alpha,\beta]}+\omega^2_{[\theta,\gamma]})+(\beta-\alpha)(d-\gamma)(\omega^1_{[\alpha,\beta]}+\omega^2_{[\gamma,d]})+$
$+(b-\beta)(\theta-c)(\omega^1_{[\beta,b]}+\omega^2_{[c,\theta]})+(b-\beta)(\gamma-\theta)(\omega^1_{[\beta,b]}+\omega^2_{[\theta,\gamma]})+$
$+(b-\beta)(d-\gamma)(\omega^1_{[\beta,b]}+\omega^2_{[\gamma,d]})$
*for any* $u,\alpha,v,\beta,w,r,\theta,p,\gamma$ *and* $z$ *with* $a\leq u<\alpha<v<\beta<w\leq b$ *and* $c\leq r<\theta<p<\gamma<z\leq d$.

Proof. Analogous to the proof of Corollary 18 in [8].

**Corollary 2.** *Let* $f:A\to E^1$ *be a a two-dimensional Henstock integrable, bounded mapping. Then, the following inequalities hold:*

*a)*$D((FH)\int\limits_c^d(FH)\int\limits_a^b f(s,t)dsdt,(d-c)(b-a)\odot f(\frac{a+b}{2},\frac{c+d}{2}))\leq$
$\leq (d-c)(b-a)(\omega^1_{[a,b]}(f,\frac{b-a}{2})+\omega^2_{[c,d]}(f,\frac{d-c}{2}))$,

*b)* $D((FH)\int\limits_c^d(FH)\int\limits_a^b f(s,t)dsdt,\frac{(d-c)(b-a)}{4}\odot(f(a,c)\oplus f(a,d)\oplus f(b,c)\oplus f(b,d)))\leq$
$\leq (d-c)(b-a)(\omega^1_{[a,b]}(f,\frac{b-a}{2})+\omega^2_{[c,d]}(f,\frac{d-c}{2}))$,

*c)*$D((FH)\int\limits_c^d(FH)\int\limits_a^b f(s,t)dsdt,\frac{(d-c)(b-a)}{36}\odot(f(a,c)\oplus f(a,d)\oplus 4\odot f(a,\frac{c+d}{2})\oplus$
$\oplus 4\odot f(\frac{a+b}{2},c))\oplus 16\odot f(\frac{a+b}{2},\frac{c+d}{2})\oplus 4\odot f(\frac{a+b}{2},d)\oplus 4\odot f(b,\frac{c+d}{2})\oplus f(b,c)\oplus f(b,d))\leq$

$\leq (d-c)(b-a)(\omega^1_{[a,b]}(f,\frac{b-a}{6})+\omega^2_{[c,d]}(f,\frac{d-c}{6}))$.

**Proof.** a) If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in the assertion a) of Corollary 1 and Theorem 1 we obtain the the required inequality a).

b)Taking $x=\frac{a+b}{2},y=\frac{c+d}{2},u=a,v=b,\alpha=c,\beta=d$ in the assertion a) of Corollary 1 and Theorem 1 we obtain the required inequality b).

c) Taking $n=4$,$u=a$, $\alpha=\frac{5a+b}{6}$, $v=\frac{a+b}{2}$, $\beta=\frac{a+5b}{6}$, $w=b$, $r=c$, $\theta=\frac{5c+d}{6}$, $p=\frac{c+d}{2}$, $\gamma=\frac{c+5d}{6}$, $z=d$ in the assertion c) of Corollary 1 and Theorem 1 we obtain the required inequality c). □

**Remark 1.** (see also [3]) Since any Riemann integrable fuzzy-number-valued function is also Henstock integrable, it follows that the above quadrature rules hold for Riemann integrable too.

## 4. THE SEQUENCE OF SUCCESSIVE APPROXIMATIONS

In this section, we present some quadrature rule for two-dimensional Henstock integral and sequence of successive approximations for numerical solution of

$$\begin{aligned} F(s,t) =& g(s,t) \oplus \lambda (FR)\int_c^d (FR)\int_a^b H(s,t,x,y) \odot G(F(x,y))dxdy, \\ & s \in [a,b],\ t \in [c,d], \end{aligned} \tag{1}$$

where $H(s,t,x,y)$ is an arbitrary positive kernel on $A \times A$, $g : A \to E^1$, $G : E^1 \to E^1$ are continuous fuzzy-number valued functions and $\lambda = 1$.

We introduce the following conditions:

(i) $g \in C(A, E^1)$, $H \in C(A \times A, \mathbb{R}_+)$, $G \in C(E^1, E^1)$,

(ii) there exists $\alpha$, such that $D(G(u), G(v)) \leq \alpha D(u,v)$ for all $u, v \in E^1$,

(iii) $B = N_H \alpha \Delta < 1$, where $N_H = max\{|H(s,t,x,y)| : (s,t),\ (x,y) \in A\}$, according to the continuity of $H$ and $\Delta = (b-a)(d-c)$.

**Theorem 2.** *(see [9]) Let the conditions (i)-(iii) are fulfilled. Then the integral equation (1) has unique solution $F \in C(A, E^1)$ and the sequence of successive approximations $\{F_m\}_{m\in\mathbb{N}} \subset C(A, E^1)$*

$$F_m(s,t) = g(s,t) \oplus (FR)\int_c^d (FR)\int_a^b H(s,t,x,y) \odot G(F_{m-1}(x,y))dxdy, \tag{2}$$

*$(s,t) \in A,\ m \in \mathbb{N}$ converges to $F$ in $C(A, E^1)$ for any choice of $F_0 \in C(A, E^1)$. In addition, the following error estimates hold:*

$$D(F(s,t), F_m(s,t)) \leq \frac{B^m}{1-B} D(F_1(s,t), F_0(s,t)), \tag{3}$$

*for all $(s,t) \in A,\ m \in \mathbb{N}$.*

*Choosing $F_0 \in C(A, E^1)$, $F_0 = g$ the inequality (3) becomes*

$$D(F(s,t), F_m(s,t)) \leq \frac{B^m}{1-B} N_H \Delta \|G(g)\|_{\mathcal{F}} \ \text{for all}\ (s,t) \in A,\ m \in \mathbb{N}. \tag{4}$$

## 5. THE ERROR ESTIMATION

We define uniform partitions $a = x_0 < x_1 < ... < x_n = b$ and $c = y_0 < y_1 < ... < y_{n'}$, $i = \overline{1, n}$, $j = \overline{1, n'}$, $\sigma = \frac{b-a}{n}$ and $\sigma' = \frac{d-c}{n'}$.

Let $f(.,.,.,.) : A \times A \to E^1$, and $f \in C(A \times A, E^1)$ , then we denote

$$\omega^3_{[a,b]}(f, \delta) = \sup_{s\in[a,b],t,y\in[c,d]} \{D(f(s, t, x_1, y), f(s, t, x_2, y)) : |x_1 - x_2| \le \delta\}$$

and

$$\omega^4_{[c,d]}(f, \delta) = \sup_{s,t\in[a,b],x\in[c,d]} \{D(f(s, t, x, y_1), f(s, t, x, y_2)) : |y_1 - y_2| \le \delta\}.$$

Now, we apply the trapezoidal quadrature rule b) of Corollary 2 of the sequence of successive approximations (2). Then, the following iterative procedure gives the approximate solution of (1) in point $(s, t) \in A$.

Let $\tilde{f}_m(s, t, x, y) = H(s, t, x, y) \odot G(\tilde{F}_m(x, y))$ for $m = 1, 2, ...$

$$\begin{aligned} \tilde{F}_0(s,t) =& g(s,t) \\ \tilde{F}_m(s,t) =& g(s,t) \oplus \frac{\sigma'\sigma}{4} \sum_{j=1}^{n'} \sum_{i=1}^{n} (\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1}) \\ & \oplus \tilde{f}_{m-1}(s,t,x_{i-1},y_j) \oplus \tilde{f}_{m-1}(s,t,x_i,y_{j-1}) \\ & \oplus \tilde{f}_{m-1}(s,t,x_i,y_j)). \end{aligned} \tag{5}$$

We denote $f_m(s, t, x, y) = H(s, t, x, y) \odot G(F_m(x, y))$ for $m = 1, 2, ....$

**Lemma 1.** *Let the conditions (i)-(iii) are fulfilled. Then the inequalities are hold*

*a)* $\omega^3_{[a,b]}(f_m, \frac{\sigma}{2}) \le N_H \alpha \omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \frac{\|G(g)\|_{\mathcal{F}}}{1-B}(B\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + \omega^3_{[a,b]}(H, \frac{\sigma}{2}))$,

*b)* $\omega^4_{[c,d]}(f_m, \frac{\sigma'}{2}) \le N_H \alpha \omega^2_{[c,d]}(g, \frac{\sigma'}{2}) + \frac{\|G(g)\|_{\mathcal{F}}}{1-B}(B\omega^2_{[c,d]}(H, \frac{\sigma'}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2}))$.

**Proof.** a) First we calculate $\omega^1_{[a,b]}(F_m, \frac{\sigma}{2})$. Let $s_1, s_2 \in [a, b]$ and $|s_1 - s_2| < \frac{\sigma}{2}$.

$$\begin{aligned} & D(F_m(s_1,t), F_m(s_2,t)) \le D(g(s_1,t), g(s_2,t)) \\ & + D((FR)\int_c^d (FR)\int_a^b H(s_1,t,x,y) \odot G(F_{m-1}(x,y))dxdy, \end{aligned}$$

$$(FR)\int_c^d (FR)\int_a^b H(s_2,t,x,y)\odot G(F_{m-1}(x,y))dxdy)$$
$$\leq \omega^1_{[a,b]}(g,\frac{\sigma}{2})+\int_c^d\int_a^b D(H(s_1,t,x,y)\odot G(F_{m-1}(x,y)),H(s_2,t,x,y)$$
$$\odot\, G(F_{m-1}(x,y)))dxdy$$
$$\leq \omega^1_{[a,b]}(g,\frac{\sigma}{2})+\omega^1_{[a,b]}(H,\frac{\sigma}{2})\int_c^d\int_a^b D(G(F_{m-1}(x,y)),\tilde{0})dxdy.$$

We calculate

$$D(G(F_m(x,y)),\tilde{0})\leq D(G(F_m(x,y)),G(F_{m-1}(x,y)))+D(G(F_{m-1}(x,y)),\tilde{0})$$
$$\leq \alpha D(F_m(x,y),F_{m-1}(x,y))+D(G(F_{m-1}(x,y)),\tilde{0}).$$

So we have

$$D(G(F_m(x,y)),\tilde{0})\leq \alpha D(F_m(x,y),F_{m-1}(x,y))+D(G(F_{m-1}(x,y)),\tilde{0}),$$
$$D(G(F_{m-1}(x,y)),\tilde{0})\leq \alpha D(F_{m-1}(x,y),F_{m-2}(x,y))+D(G(F_{m-2}(x,y)),\tilde{0}),$$
$$\vdots$$
$$D(G(F_1(x,y)),\tilde{0})\leq \alpha D(F_1(x,y),g(x,y))+D(G(g(x,y)),\tilde{0}).$$

Summing these inequalities and using Theorem 2 then we have

$$D(G(F_m(x,y)),\tilde{0})\leq \alpha(D(F_m(x,y),F_{m-1}(x,y))+D(F_{m-1}(x,y),F_{m-2}(x,y))$$
$$+\ldots+D(F_1(x,y),g(x,y)))+\|G(g)\|_{\mathcal{F}}$$
$$\leq \alpha(B^{m-1}D(F_1(x,y),g(x,y))+B^{m-2}D(F_1(x,y),g(x,y))$$
$$+\ldots+D(F_1(x,y),g(x,y)))$$
$$+\|G(g)\|_{\mathcal{F}}\leq \alpha\frac{1-B^m}{1-B}N_H\Delta\|G(g)\|_{\mathcal{F}}+\|G(g)\|_{\mathcal{F}}$$
$$\leq \frac{1}{1-B}\|G(g)\|_{\mathcal{F}}.$$

Hence

$$\omega^1_{[a,b]}(F_m,\frac{\sigma}{2})\leq \omega^1_{[a,b]}(g,\frac{\sigma}{2})+\frac{\Delta\|G(g)\|_{\mathcal{F}}}{1-B}\omega^1_{[a,b]}(H,\frac{\sigma}{2}).$$

Let $x_1, x_2 \in [a,b]$ and $|x_1 - x_2| < \frac{\sigma}{2}$.

$$\begin{aligned}
&D(f_m(s,t,x_1,y), f_m(s,t,x_2,y)) \\
=&D(H(s,t,x_1,y) \odot G(F_m(x_1,y)), H(s,t,x_2,y) \odot G(F_m(x_2,y))) \\
\leq&D(H(s,t,x_1,y) \odot G(F_m(x_1,y)), H(s,t,x_2,y) \odot G(F_m(x_1,y))) \\
&+ D(H(s,t,x_2,y) \odot G(F_m(x_1,y)), H(s,t,x_2,y) \odot G(F_m(x_2,y))) \\
\leq&\omega^3_{[a,b]}(H, \frac{\sigma}{2}) D(F_m(x_1,y), \tilde{0}) + N_H \alpha D(F_m(x_1,y), F_m(x_2,y)) \\
\leq&\omega^3_{[a,b]}(H, \frac{\sigma}{2}) \frac{1}{1-B} \|G(g)\|_{\mathcal{F}} + N_H \alpha \omega^1_{[a,b]}(F_m, \frac{\sigma}{2}) \\
\leq&N_H \alpha \omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \frac{\|G(g)\|_{\mathcal{F}}}{1-B}(B\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + \omega^3_{[a,b]}(H, \frac{\sigma}{2})).
\end{aligned}$$

b) Analogous to a) we have

$$\omega^2_{[c,d]}(F_m, \frac{\sigma'}{2}) \leq \omega^2_{[c,d]}(g, \frac{\sigma'}{2}) + \frac{\Delta \|G(g)\|_{\mathcal{F}}}{1-B} \omega^2_{[c,d]}(H, \frac{\sigma'}{2})$$

and then hence b). □

**Theorem 3.** *Under the conditions (i)-(iii) the iterative method (5) converges to the unique solution $F$ of the equation (1) and for any $(s,t) \in A$ its error estimate is as follows*

$$\begin{aligned}
D^*(F, \tilde{F}_m) \leq& \frac{B^m}{1-B} N_H \Delta \|G(g)\|_{\mathcal{F}} \\
&+ \frac{B+B^m}{1-B}(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2})) \\
&+ \frac{\Delta \|G(g)\|_{\mathcal{F}} B}{(1-B)^2}(\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + \omega^2_{[c,d]}(H, \frac{\sigma'}{2})) \qquad (6) \\
&+ \frac{\Delta \|G(g)\|_{\mathcal{F}}(1 + B^{m-1} + B^{m+1})}{(1-B)^2}(\omega^3_{[a,b]}(H, \frac{\sigma}{2}) \\
&+ \omega^4_{[c,d]}(H, \frac{\sigma'}{2})).
\end{aligned}$$

**Proof.** Considering iterative procedure (5), for all $(s,t) \in A$ we have

$$\begin{aligned}
D(F_m(s,t), &\tilde{F}_m(s,t)) = D(g(s,t), g(s,t)) \\
&+ D((FR)\int_c^d (FR)\int_a^b f_{m-1}(s,t,x,y)dxdy,
\end{aligned}$$

$$
\begin{aligned}
&\frac{\sigma'\sigma}{4}\sum_{j=1}^{n'}\sum_{i=1}^{n}(\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus \tilde{f}_{m-1}(s,t,x_{i-1},y_j)\oplus \tilde{f}_{m-1}(s,t,x_i,y_{j-1})\oplus \tilde{f}_{m-1}(s,t,x_i,y_j))\\
\leq& D(\sum_{j=1}^{n'}\sum_{i=1}^{n}(FR)\int_{y_{j-1}}^{y_j}(FR)\int_{x_{i-1}}^{x_i} f_{m-1}(s,t,x,y)dxdy,\\
&\frac{\sigma'\sigma}{4}\sum_{j=1}^{n'}\sum_{i=1}^{n}(\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus \tilde{f}_{m-1}(s,t,x_{i-1},y_j)\oplus \tilde{f}_{m-1}(s,t,x_i,y_{j-1})\oplus \tilde{f}_{m-1}(s,t,x_i,y_j))\\
\leq& \sum_{j=1}^{n'}\sum_{i=1}^{n}D((FR)\int_{y_{j-1}}^{y_j}(FR)\int_{x_{i-1}}^{x_i} f_{m-1}(s,t,x,y)dxdy,\\
&\frac{\sigma'\sigma}{4}(\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus \tilde{f}_{m-1}(s,t,x_{i-1},y_j)\oplus \tilde{f}_{m-1}(s,t,x_i,y_{j-1})\oplus \tilde{f}_{m-1}(s,t,x_i,y_j))\\
\leq& \sum_{j=1}^{n'}\sum_{i=1}^{n}D((FR)\int_{y_{j-1}}^{y_j}(FR)\int_{x_{i-1}}^{x_i} f_{m-1}(s,t,x,y)dxdy,\\
&\frac{\sigma'\sigma}{4}(f_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus f_{m-1}(s,t,x_{i-1},y_j)\oplus f_{m-1}(s,t,x_i,y_{j-1})\oplus f_{m-1}(s,t,x_i,y_j))\\
&+\sum_{j=1}^{n'}\sum_{i=1}^{n}D(\frac{\sigma'\sigma}{4}(f_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus f_{m-1}(s,t,x_{i-1},y_j)\oplus f_{m-1}(s,t,x_i,y_{j-1})\\
&\oplus f_{m-1}(s,t,x_i,y_j)),\frac{\sigma'\sigma}{4}(\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1})\\
&\oplus \tilde{f}_{m-1}(s,t,x_{i-1},y_j)\oplus \tilde{f}_{m-1}(s,t,x_i,y_{j-1})\oplus \tilde{f}_{m-1}(s,t,x_i,y_j)))\\
\leq& \sum_{j=1}^{n'}\sum_{i=1}^{n}\sigma'\sigma(\omega^3_{[x_{i-1},x_i]}(f_{m-1},\frac{\sigma}{2})+\omega^4_{[y_{j-1},y_j]}(f_{m-1},\frac{\sigma'}{2}))\\
&+\sum_{j=1}^{n'}\sum_{i=1}^{n}\frac{\sigma'\sigma}{4}(D(f_{m-1}(s,t,x_{i-1},y_{j-1}),\tilde{f}_{m-1}(s,t,x_{i-1},y_{j-1}))\\
&+D(f_{m-1}(s,t,x_i,y_{j-1}),\tilde{f}_{m-1}(s,t,x_i,y_{j-1}))
\end{aligned}
$$

$$
\begin{aligned}
&+ D(f_{m-1}(s,t,x_{i-1},y_j), \tilde{f}_{m-1}(s,t,x_{i-1},y_j)) \\
&+ D(f_{m-1}(s,t,x_i,y_j), \tilde{f}_{m-1}(s,t,x_i,y_j))) \\
\leq & \sum_{j=1}^{n'} \sum_{i=1}^{n} \sigma'\sigma(\omega^3_{[a,b]}(f_{m-1}, \frac{\sigma}{2}) + \omega^4_{[c,d]}(f_{m-1}, \frac{\sigma'}{2})) \\
&+ \sum_{j=1}^{n'} \sum_{i=1}^{n} \frac{\sigma'\sigma}{4} N_H \alpha(D(F_{m-1}(x_{i-1},y_{j-1}), \tilde{F}_{m-1}(x_{i-1},y_{j-1})) \\
&+ D(F_{m-1}(x_{i-1},y_j), \tilde{F}_{m-1}(x_{i-1},y_j)) \\
&+ D(F_{m-1}(x_i,y_{j-1}), \tilde{F}_{m-1}(x_i,y_{j-1})) \\
&+ D(F_{m-1}(x_i,y_j), \tilde{F}_{m-1}(x_i,y_j))) \\
\leq & \Delta(\omega^3_{[a,b]}(f_{m-1}, \frac{\sigma}{2}) + \omega^4_{[c,d}(f_{m-1}, \frac{\sigma'}{2})) + BD^*(F_{m-1}, \tilde{F}_{m-1}).
\end{aligned}
$$

Hence from Lemma 1

$$
\begin{aligned}
D^*(F_m, \tilde{F}_m) \leq & \Delta(\omega^3_{[a,b]}(f_{m-1}, \frac{\sigma}{2}) + \omega^4_{[c,d]}(f_{m-1}, \frac{\sigma'}{2})) + BD^*(F_{m-1}, \tilde{F}_{m-1}) \\
\leq & B(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2})) \\
&+ \frac{\Delta \|G(g)\|_{\mathcal{F}}}{1-B}(B\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + B\omega^2_{[c,d]}(H, \frac{\sigma'}{2}) \\
&+ \omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})) \\
&+ BD^*(F_{m-1}, \tilde{F}_{m-1}).
\end{aligned}
$$

We denote

$$
\begin{aligned}
P = B(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2})) + \frac{\Delta \|G(g)\|_{\mathcal{F}}}{1-B}(B\omega^1_{[a,b]}(H, \frac{\sigma}{2}) \\
+ B\omega^2_{[c,d]}(H, \frac{\sigma'}{2}) + \omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})).
\end{aligned}
$$

So we have

$$
\begin{aligned}
D^*(F_m, \tilde{F}_m) \leq & P + BD^*(F_{m-1}, \tilde{F}_{m-1}), \\
D^*(F_{m-1}, \tilde{F}_{m-1}) \leq & P + BD^*(F_{m-2}, \tilde{F}_{m-2}), \vdots \\
D^*(F_2, \tilde{F}_2) \leq & P + BD^*(F_1, \tilde{F}_1), \\
D^*(F_1, \tilde{F}_1) \leq & B(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2}))
\end{aligned}
$$

$$+ \Delta\|G(g)\|_{\mathcal{F}}(\omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})).$$

Multiplying these inequalities by 1, $B$, ... , $B^{m-1}$ , respectively, and summing then we have

$$\begin{aligned} D^*(F_m, \tilde{F}_m) \leq & \frac{1}{1-B}P + B^m(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2})) \\ & + B^{m-1}\Delta\|G(g)\|_{\mathcal{F}}(\omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})), \end{aligned}$$

i.e.

$$\begin{aligned} D^*(F_m, \tilde{F}_m) \leq & \frac{B+B^m}{1-B}(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2})) \\ & + \frac{\Delta\|G(g)\|_{\mathcal{F}}B}{(1-B)^2}(\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + \omega^2_{[c,d]}(H, \frac{\sigma'}{2})) \\ & + \frac{\Delta\|G(g)\|_{\mathcal{F}}(1 + B^{m-1} + B^{m+1})}{(1-B)^2} \\ & (\omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})). \end{aligned} \tag{7}$$

Now

$$D^*(F, \tilde{F}_m) \leq D^*(F, F_m) + D^*(F_m, \tilde{F}_m).$$

From (7) and (4) of Theorem 2 we obtained (5). □

## 6. NUMERICAL STABILITY ANALYSIS

We study the numerical stability of the iterative algorithm (5) with respect to small changes in the starting approximation. We consider $F_0 = g$ and another starting approximation $Q_0 = g_1 \in C(A, E^1)$ such that exists $\varepsilon > 0$ for which $D(F_0(s,t), Q_0(s,t)) < \varepsilon$ , for all $(s,t) \in A$ . The obtained sequence of successive approximations is:

$$\begin{aligned} Q_0(s,t) =& g_1(s,t), \\ Q_m(s,t) =& g(s,t) \oplus (FR)\int_c^d (FR)\int_a^b H(s,t,x,y) \odot G(Q_{m-1}(x,y))dxdy, \\ & m = 1, 2, \ldots, \end{aligned}$$

and applying the quadrature rule b) of Corollary 2 we obtain

$$\begin{aligned}\tilde{Q}_0(s,t) =& g_1(s,t),\\ \tilde{Q}_m(s,t) =& g(s,t) \oplus \frac{\sigma'\sigma}{4}\sum_{j=1}^{n'}\sum_{i=1}^{n}(H(s,t,x_{i-1},y_{j-1}) \odot G(\tilde{Q}_{m-1}(x_{i-1},y_{j-1})\\ &\oplus H(s,t,x_{i-1},y_j) \odot G(\tilde{Q}_{m-1}(x_{i-1},y_j)\\ &\oplus H(s,t,x_i,y_{j-1}) \odot G(\tilde{Q}_{m-1}(x_i,y_{j-1})\\ &\oplus H(s,t,x_i,y_j) \odot G(\tilde{Q}_{m-1}(x_i,y_j)), \quad m = 1,2,...\end{aligned}$$

**Definition 7.** The method of successive approximations applied to the integral equation (1) is said to be numerically stable with respect to the choice of the first iteration if for all $(s,t) \in A$ there exist constants $k_1$, $k_2$, $k_3$, $k_4 > 0$ which are independent by $\sigma = \frac{b-a}{n}$ and $\sigma' = \frac{d-c}{n'}$ such that

$$\begin{aligned}D(\tilde{F}_m(s,t), \tilde{Q}_m(s,t)) < k_1\varepsilon + k_2(\omega^1_{[a,b]}(g,\frac{\sigma}{2}) + \omega^2_{[c,d]}(g,\frac{\sigma'}{2}))\\ + k_3(\omega^1_{[a,b]}(H,\frac{\sigma}{2}) + \omega^2_{[c,d]}(H,\frac{\sigma'}{2})) + k_4(\omega^3_{[a,b]}(H,\frac{\sigma}{2}) + \omega^4_{[c,d]}(H,\frac{\sigma'}{2})).\end{aligned}$$

**Theorem 4.** *Under the conditions (i)-(iii) the iterative method is numerically stable with respect to the choice of the first iteration.*

Proof. First, we observe that

$$D^*(\tilde{F}_m, \tilde{Q}_m) \leq D^*(\tilde{F}_m, F_m) + D^*(F_m, Q_m) + D^*(Q_m, \tilde{Q}_m).$$

From inequality (7) of Theorem 3 we have

$$\begin{aligned}D^*(Q_m, \tilde{Q}_m) \leq& \frac{B+B^m}{1-B}(\omega^1_{[a,b]}(g,\frac{\sigma}{2}) + \omega^2_{[c,d]}(g,\frac{\sigma'}{2}))\\ &+ \frac{\Delta\|G(g)\|_{\mathcal{F}}B}{(1-B)^2}(\omega^1_{[a,b]}(H,\frac{\sigma}{2}) + \omega^2_{[c,d]}(H,\frac{\sigma'}{2}))\\ &+ \frac{\Delta\|G(g)\|_{\mathcal{F}}(1+B^{m-1}+B^{m+1})}{(1-B)^2}(\omega^3_{[a,b]}(H,\frac{\sigma}{2}) + \omega^4_{[c,d]}(H,\frac{\sigma'}{2})).\end{aligned}$$

By hipothesis, $D(F_0(s,t), Q_0(s,t)) < \varepsilon$, for all $(s,t) \in A$ and thus

$$D(F_m(s,t), Q_m(s,t)) \leq D(g(s,t), g(s,t))$$

$$+ D(\int_c^d \int_a^b H(s,t,x,y) \odot G(F_{m-1}(x,y))dxdy,$$

$$\int_c^d \int_a^b H(s,t,x,y) \odot G(Q_{m-1}(x,y))dxdy)$$

$$\leq \int_c^d \int_a^b |H(s,t,x,y)| D(G(F_{m-1}(x,y)), G(Q_{m-1}(x,y)))dxdy)$$

$$\leq \Delta N_H \alpha D^*(F_{m-1}, Q_{m-1}) = BD^*(F_{m-1}, Q_{m-1}).$$

Then

$$D^*(F_m, Q_m) \leq B^m D^*(F_0, Q_0) \leq B^m \varepsilon,$$

for all $(s,t) \in A, m \geq 1$ and

$$D^*(\tilde{F}_m, \tilde{Q}_m) \leq k_1 \varepsilon + k_2(\omega^1_{[a,b]}(g, \frac{\sigma}{2}) + \omega^2_{[c,d]}(g, \frac{\sigma'}{2}))$$
$$+ k_3(\omega^1_{[a,b]}(H, \frac{\sigma}{2}) + \omega^2_{[c,d]}(H, \frac{\sigma'}{2})) + k_4(\omega^3_{[a,b]}(H, \frac{\sigma}{2}) + \omega^4_{[c,d]}(H, \frac{\sigma'}{2})),$$

where

$$k_1 = B^m, \qquad k_2 = \frac{2(B + B^m)}{1 - B},$$
$$k_3 = \frac{2\Delta \|G(g)\|_{\mathcal{F}} B}{(1-B)^2}, \qquad k_4 = \frac{2\Delta \|G(g)\|_{\mathcal{F}} (1 + B^{m-1} + B^{m+1})}{(1-B)^2}.$$

## 7. NUMERICAL EXPERIMENT

In this section, we intent to illustrate the obtained theoretical results on some numerical example testing the convergence of the method and the numerical stability with respect to the choice of the first iteration. The algorithm was implemented using Javascript. The program can be found on the following web address: `http://enkov.com/solver2DFErr`.

**Example.** Let $A = [0,1] \times [0,1]$. For the integral equation

$$F(s,t) = g(s,t) \oplus (FR) \int_c^d (FR) \int_a^b s^2 t^2 (x+y) \odot \sqrt{F(x,y)} dxdy, \quad (s,t) \in A$$

the exact solution is

$$\underline{F}(s,t,r) = (4+r)e^{2(s+t)} \text{ and } \bar{F}(s,t,r) = (6-r)e^{2(s+t)}.$$

Here

$$\underline{g}(s,t,r) = (4+r)e^{2(s+t)} - 2s^2t^2(e-1)\sqrt{4+r},$$

$$\bar{g}(s,t,r) = (6-r)e^{2(s+t)} - 2s^2t^2(e-1)\sqrt{6-r}.$$

Applying the iterative procedure for various $m, n, n'$ and $r$. We obtain the computational errors

$$\underline{E}_m(r) = \underline{E}_m(s_0,t_0,r) = |\tilde{\underline{F}}_m(s_0,t_0,r) - \underline{F}(s_0,t_0,r)|,$$

$$\overline{E}_m(r) = \overline{E}_m(s_0,t_0,r) = |\overline{\tilde{F}}_m(s_0,t_0,r) - \overline{F}(s_0,t_0,r)|$$

in the point $(s_0,t_0) = (0.5, 0.5)$.

The numerical stability is tested by considering $\varepsilon = 0.1$, for various $m, n, n'$ and $r$. The results are expressed by

$$\underline{D}_m(r) = \underline{D}_m(s_0,t_0,r) = |\tilde{\underline{F}}_m(s_0,t_0,r) - \tilde{\underline{Q}}_m(s_0,t_0,r)|$$

and

$$\overline{D}_m(r) = \overline{D}_m(s_0,t_0,r) = |\overline{\tilde{F}}_m(s_0,t_0,r) - \overline{\tilde{Q}}_m(s_0,t_0,r)|$$

in the point $(s_0,t_0) = (0.5, 0.5)$.

We get that for any $0 \le r \le 1$, the norm of the errors tend to zero as $m, n, n' \to \infty$. We present these results in Tables 1, 2, 3.

For $n$=$n'$=15 after $m$=6 and 7 iteration we obtain that the order of the errors $\underline{D}_m(r)$ and $\overline{D}_m(r)$ is $e-15$. In Table 3 for $m$=9 we write 0.

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| $r$ | $\underline{E}_m(r)$ | $\overline{E}_m(r)$ | $\underline{D}_m(r)$ | $\overline{D}_m(r)$ |
|---|---|---|---|---|
| 0 | 7.87141e-3 | 9.62370e-3 | 4.33758e-10 | 1.92927e-10 |
| 0,2 | 8.06396e-3 | 9.46319e-3 | 3.93399e-10 | 2.06484e-10 |
| 0,4 | 8.25196e-3 | 9.29990e-3 | 3.58419e-10 | 2.21519e-10 |
| 0,6 | 8.43575e-3 | 9.13367e-3 | 3.27908e-10 | 2.38259e-10 |
| 0,8 | 8.61557e-3 | 8.96433e-3 | 3.01128e-10 | 2.56975e-10 |
| 1 | 8.79169e-3 | 8.79169e-3 | 2.77502e-10 | 2.77971e-10 |

Table 1: Table 1: $m$=4, $n$=$n'$=5

| $r_i$ | $\underline{E}_m(r)$ | $\overline{E}_m(r)$ | $\underline{D}_m(r)$ | $\overline{D}_m(r)$ |
|---|---|---|---|---|
| 0.0 | 1.96506e-3 | 2.40259e-3 | 3.19744e-14 | 7.10543e-15 |
| 0.2 | 2.01313e-3 | 2.36252e-3 | 3.19744e-14 | 1.42109e-14 |
| 0.4 | 2.06008e-3 | 2.32174e-3 | 2.84217e-14 | 1.42109e-14 |
| 0.6 | 2.10597e-3 | 2.28024e-3 | 2.13163e-14 | 1.42109e-14 |
| 0.8 | 2.15087e-3 | 2.23795e-3 | 2.13163e-14 | 1.42109e-14 |
| 1.0 | 2.19485e-3 | 2.19485e-3 | 2.13163e-14 | 1.42109e-14 |

Table 2: Table 2: $m$=6, $n$=$n'$=10

| $r_i$ | $\underline{E}_m(r)$ | $\overline{E}_m(r)$ | $\underline{D}_m(r)$ | $\overline{D}_m(r)$ |
|---|---|---|---|---|
| 0.0 | 8.73129e-4 | 1.06755e-3 | 0 | 0 |
| 0.2 | 8.94491e-4 | 1.04974e-3 | 0 | 0 |
| 0.4 | 9.15351e-4 | 1.03162e-3 | 0 | 0 |
| 0.6 | 9.35741e-4 | 1.01318e-3 | 0 | 0 |
| 0.8 | 9.55693e-4 | 0.99439e-3 | 0 | 0 |
| 1.0 | 9.75234e-4 | 0.97523e-3 | 0 | 0 |

Table 3: Table 3: $m$=9, $n$=$n'$=15

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