Communications in Applied Analysis, **22**, No. 1 (2018), 1-17 ISSN: 1083-2564

NEW INTEGRAL INEQUALITIES ON TIME SCALES WITH APPLICATIONS TO THE CONTINUOUS AND DISCRETE CALCULUS

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ABSTRACT: In this paper, we present some new weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales. Our results generalize many other results in the literature. Besides generalization, we also obtain some interesting inequalities by considering special cases of time scales.

AMS Subject Classification: 26D15, 54C30, 26D10

Key Words: Montgomery's identity, Ostrowski's inequality, Ostrowski-Grüss inequality, parameter function, time scales

Received: August 9, 2017; **Accepted:** August 31, 2017;
Published: September 10, 2018. **doi:** 10.12732/caa.v22i1.1
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

The classical Ostrowski inequality is stated in the following theorem.

Theorem 1 ([5]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\ dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In 1997, Dragomir and Wang [4] obtained the following result, which is today known as the Ostrowski-Grüss type inequality.

Theorem 2. *If $f : [a, b] \to \mathbb{R}$ is differentiable with bounded derivative, then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds - \frac{f(b) - f(a)}{b-a} \left(t - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma), \quad (1)$$

where $\gamma := \inf_{t \in [a,b]} f'(t)$ and $\Gamma := \sup_{t \in [a,b]} f'(t)$.

In order to unify the continuous and discrete calculus, Hilger [6] in 1988 introduced the concept of time scales (see Section 2 for a brief introduction). Since the introduction of this theory, loads of work have been be done in extending Theorems 1 and 2 to time scales. See, for example, the following papers and the related references therein [3, 8, 9, 10, 12, 16, 13, 14, 7].

In [3], Bohner and Matthews extended Theorem 1 to time scales by proving the following foundational result.

Theorem 3. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \triangle s \right| \leq \frac{M}{b-a} \left(h_2(t, a) + h_2(t, b) \right), \quad (2)$$

where $h_2(\cdot, \cdot)$ is given in Definition 12 and $M = \sup_{a<t<b} \left| f^{\triangle}(t) \right| < \infty$. This inequality is sharp in the sense that the right-hand side of (2) cannot be replaced by a smaller one.

By introducing a parameter, Liu and Ngô [8] generalized Theorem 3. Motivated by their work, Xu and Fang [16] recently proved the following broader generalization via a parameter function technique.

Theorem 4. *Suppose that $a, b, s, t \in \mathbb{T}$, $a < b$, $f : [a, b] \to \mathbb{R}$ are differentiable, and ψ is a function of $[0, 1]$ into $[0, 1]$. Then the following inequality holds*

$$\left| \frac{1 + \psi(1-\lambda) - \psi(\lambda)}{2} f(t) + \frac{\psi(\lambda) f(a) + (1 - \psi(1-\lambda)) f(b)}{2} \right.$$

$$
\begin{aligned}
&- \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\Bigg| \\
&\leq \frac{M}{b-a}\Bigg[h_2\left(a, a+\psi(\lambda)\frac{b-a}{2}\right) \\
&+ h_2\left(t, a+\psi(\lambda)\frac{b-a}{2}\right) \\
&+ h_2\left(t, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) \\
&+ h_2\left(b, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\Bigg]
\end{aligned}
$$

for all $\lambda \in [0,1]$ *such that* $a+\psi(\lambda)\frac{b-a}{2}$ *and* $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ *are in* $\mathbb{T}$, *and*

$$t \in \left[a+\psi(\lambda)\frac{b-a}{2}, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right],$$

where $M = \sup_{a<t<b} |f^\Delta(t)| < \infty$.

In 2010, Tuna and Daghan [15] proved the following time scale version of the Ostrowski-Grüsss type inequality (see also [11] for yet another version).

Theorem 5. *Let* $a, b, t, x \in \mathbb{T},\ a < b$ *and if* $f : [a, b] \to \mathbb{R}$ *be differentiable. If* f^Δ *is rd-continuous and* $\gamma \leq f^\Delta(t) \leq \Gamma$ *for all* $t \in [a, b]$, *then for all* $x \in [a, b]$, *we have*

$$
\begin{aligned}
&\Bigg|\left(1-\frac{\lambda}{2}\right)f(x) + \lambda\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t \\
&\quad - \frac{\Gamma+\gamma}{2}\frac{1}{b-a}\left[h_2(x,a) - h_2(x,b) - \lambda\left(\frac{(x-a)^2-(b-x)^2}{2}\right)\right]\Bigg| \\
&\quad \leq \frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(a, a+\lambda\frac{x-a}{2}\right) + h_2\left(x, a+\lambda\frac{x-a}{2}\right)\right. \\
&\quad \left. + h_2\left(x, b-\lambda\frac{b-x}{2}\right) + h_2\left(b, b-\lambda\frac{b-x}{2}\right)\right]
\end{aligned}
$$

for all $\lambda \in [0,1]$ *such that* $a+\lambda\frac{x-a}{2}$ *and* $b-\lambda\frac{b-x}{2}$ *are in* $\mathbb{T}$.

Inspired by the above works, we prove a generalization of Theorem 4 by introducing a weight function, which in turn generalizes other Ostrowski type results in the literature. In addition, we give a new version of the weighted Ostrowski-Grüss type inequality and obtain from them other interesting inequalities. To do this, we first prove a weighted Montgomery identity and then use it to obtain the desired results.

This paper is arranged as follows: In Section 2, we give a brief overview of the time scale theory. Thereafter, we state and prove a weighted Montgomery identity, Ostrowski and Ostrowski-Grüss type results in Section 3.

2. TIME SCALE ESSENTIALS

In this section, we lay a brief foundation of the theory of time scales by collecting concepts that will aid in better understanding of our results. For an indept study, we refer the interested reader to the books of Bohner and Peterson [1, 2].

Definition 6. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward *jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and backward *jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \in \mathbb{T}$, respectively. Clearly, we see that $\sigma(t) \geq t$ and $\rho(t) \leq t$ for all $t \in \mathbb{T}$. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. If $\sigma(t) = t$, then t is called right dense, and if $\rho(t) = t$ then t is called left dense. Points that are both right dense and left dense are called dense. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$; otherwise, $\mathbb{T}^k = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$, we define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals and half-open intervals are defined in the same manner.

Definition 7. The function $f : \mathbb{T} \to \mathbb{R}$, is called differentiable at $t \in \mathbb{T}^k$, with delta derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\epsilon > 0$ there exist a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \qquad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \dfrac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

Definition 8. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be $rd-$continuous if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left dense points $t \in \mathbb{T}$.

Definition 9. Let f be a $rd-$continuous function. Then $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we have

$$\int_a^b f(s)\Delta s = g(b) - g(a).$$

Definition 10. The function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined as

$$f^{\sigma}(t) := f(\sigma(t))$$

for any $t \in \mathbb{T}$.

Theorem 11. *If $a, b, c \in \mathbb{T}$ with $a < c < b$, $\alpha \in \mathbb{R}$ and f, g are $rd-$continuous, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.$

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

(v) $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\Delta t$ *for all* $t \in [a, b]$.

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g^{\sigma}(t))\Delta t.$

Definition 12. Let $h_k : \mathbb{T}^2 \to \mathbb{T}$, $k \in \mathbb{N}$ be functions that are recursively defined as

$$h_0(t, s) = 1$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad for\ all \quad s, t \in \mathbb{T}.$$

In view of the above definition, we make the following remarks.

Remark 13. Using the fact that for all $s, t \in \mathbb{T}$, $h_1(t, s) = t - s$, we get that $h_2(t, s) = \int_s^t (\tau - s)\Delta\tau$. When $\mathbb{T} = \mathbb{R}$, then for all $s, t \in \mathbb{T}$, $h_k(t, s) = \frac{(t-s)^k}{k!}$.

When $\mathbb{T} = \mathbb{Z}$, then for all $s, t \in \mathbb{T}$, $h_k(t,s) = \binom{t-s}{k} = \prod_{i=1}^{k} \dfrac{t-s+1-i}{i}$. When $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then for all $s, t \in \mathbb{T}$, $h_k(t,s) = \dfrac{(t-s)_q^k}{[k]!}$ for $k \in \mathbb{N}_0$, where $[k]_q := \dfrac{q^k - 1}{q-1}$ for $q \in \mathbb{R}\backslash\{1\}$ and $k \in \mathbb{N}_0$, $[k]! := \prod_{j=1}^{k} [j]_q$ for $k \in \mathbb{N}_0$,

$$(t-s)_q^k := \prod_{j=0}^{k-1} (t - q^j s) \quad \text{for} \quad k \in \mathbb{N}_0.$$

3. MAIN RESULTS

For the proof of our main results, we will need the following lemmas.

Lemma 14. *Let $a, b, s, t \in \mathbb{T}$, $a < b$, $f, h : [a,b] \to \mathbb{R}$ be differentiable functions, and ψ a function of $[0,1]$ into $[0,1]$. Then for all $t \in [a,b]$, we have the following Montgomery identity*

$$\begin{aligned} &\frac{h(b)-h(a)}{2}\Big[1 + \psi(1-\lambda) - \psi(\lambda)\Big] f(t) \\ &\qquad = -\frac{h(b)-h(a)}{2}\Big[\psi(\lambda) f(a) + (1 - \psi(1-\lambda))\, f(b)\Big] \\ &\qquad\quad + \int_a^b K(s,t) f^{\Delta}(s)\Delta s + \int_a^b h^{\Delta}(s) f^{\sigma}(s) \Delta s, \end{aligned} \tag{3}$$

where

$$K(s,t) = \begin{cases} h(s) - \left(h(a) + \psi(\lambda)\dfrac{h(b)-h(a)}{2}\right), & s \in [a,t), \\ h(s) - \left(h(a) + (1 + \psi(1-\lambda))\dfrac{h(b)-h(a)}{2}\right), & s \in [t,b]. \end{cases} \tag{4}$$

Proof. Applying the integration by part formula as given in item (vi) of Theorem 11, we get

$$\begin{aligned} &\int_a^t \left[h(s) - \left(h(a) + \psi(\lambda)\frac{h(b)-h(a)}{2}\right)\right] f^{\Delta}(s)\Delta s + \int_a^t h^{\Delta}(s) f^{\sigma}(s)\Delta s \\ &\quad = \left[h(t) - \left(h(a) + \psi(\lambda)\frac{h(b)-h(a)}{2}\right)\right] f(t) + \psi(\lambda)\frac{h(b)-h(a)}{2} f(a) \end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\int_t^b \left[h(s) - \left(h(a) + (1+\psi(1-\lambda))\frac{h(b)-h(a)}{2}\right)\right] f^\Delta(s)\Delta s \\
&\quad + \int_t^b h^\Delta(s) f^\sigma(s)\Delta s \\
&= \left[h(b) - \left(h(a) + (1+\psi(1-\lambda))\frac{h(b)-h(a)}{2}\right)\right] f(b) \\
&\quad - \left[h(t) - \left(h(a) + (1+\psi(1-\lambda))\frac{h(b)-h(a)}{2}\right)\right] f(t). \qquad (6)
\end{aligned}$$

Adding Equations (5) and (6), and using item (iv) of Theorem 11, gives

$$\begin{aligned}
&\int_a^b K(s,t) f^\Delta(s)\Delta s + \int_a^b h^\Delta(s) f^\sigma(s)\Delta s \\
&\qquad = \frac{h(b)-h(a)}{2}\Big[1 + \psi(1-\lambda) - \psi(\lambda)\Big] f(t) \\
&\qquad + \frac{h(b)-h(a)}{2}\Big[\psi(\lambda) f(a) + (1-\psi(1-\lambda))\, f(b)\Big]. \qquad (7)
\end{aligned}$$

Hence, Equation (13) follows. □

Lemma 15. *Let $K(\cdot,\cdot)$ be given as in Lemma 14 above with $h(t) = t$. Then for all $\lambda \in [0,1]$ such that $a + \psi(\lambda)\frac{b-a}{2}$ and $a + (1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and*

$$t \in \left[a + \psi(\lambda)\frac{b-a}{2}, a + (1+\psi(1-\lambda))\frac{b-a}{2}\right],$$

we have

(1).

$$\begin{aligned}
\int_a^b |K(s,t)|\Delta s &= h_2\left(a, a + \psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t, a + \psi(\lambda)\frac{b-a}{2}\right) \\
&+ h_2\left(t, a + (1+\psi(1-\lambda))\frac{b-a}{2}\right) + h_2\left(b, a + (1+\psi(1-\lambda))\frac{b-a}{2}\right).
\end{aligned}$$

(2).

$$\begin{aligned}
\int_a^b K(s,t)\Delta s &= h_2\left(t, a + \psi(\lambda)\frac{b-a}{2}\right) - h_2\left(a, a + \psi(\lambda)\frac{b-a}{2}\right) \\
&+ h_2\left(b, a + (1+\psi(1-\lambda))\frac{b-a}{2}\right) - h_2\left(t, a + (1+\psi(1-\lambda))\frac{b-a}{2}\right).
\end{aligned}$$

Proof. The proof of item 1 is somewhat embedded in the proof of Theorem 1 in [16]. For the sake of completeness, we present the proof. Using Theorem 11, gives

$$\begin{aligned}
\int_a^b |K(s,t)|\Delta s = & \int_a^t |K(s,t)|\Delta s + \int_t^b |K(s,t)|\Delta s \\
= & \int_a^t \left| s - \left(a + \psi(\lambda)\frac{b-a}{2} \right) \right| \Delta s \\
& + \int_t^b \left| s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \right| \Delta s \\
= & \int_a^{a+\psi(\lambda)\frac{b-a}{2}} \left| s - \left(a + \psi(\lambda)\frac{b-a}{2} \right) \right| \Delta s \\
& + \int_{a+\psi(\lambda)\frac{b-a}{2}}^t \left| s - \left(a + \psi(\lambda)\frac{b-a}{2} \right) \right| \Delta s \\
& + \int_t^{a+(1+\psi(1-\lambda))\frac{b-a}{2}} \left| s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \right| \Delta s \\
& + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^b \left| s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \right| \Delta s \\
= & \int_{a+\psi(\lambda)\frac{b-a}{2}}^a \left[s - \left(a + \psi(\lambda)\frac{b-a}{2} \right) \right] \Delta s \\
& + \int_{a+\psi(\lambda)\frac{b-a}{2}}^t \left[s - \left(a + \psi(\lambda)\frac{b-a}{2} \right) \right] \Delta s \\
& + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^t \left[s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \right] \Delta s \\
& + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^b \left[s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \right] \Delta s \\
= & h_2\left(a, a + \psi(\lambda)\frac{b-a}{2} \right) + h_2\left(t, a + \psi(\lambda)\frac{b-a}{2} \right) \\
& + h_2\left(t, a + (1+\psi(1-\lambda))\frac{b-a}{2} \right) \\
& + h_2\left(b, a + (1+\psi(1-\lambda))\frac{b-a}{2} \right).
\end{aligned}$$

The proof of item 2 follows from the same line of reasoning. □

Corollary 16. *If we take* $\psi(\lambda)=\lambda$*, then Lemma 14 amounts to*

$$(h(b)-h(a))(1-\lambda)f(t) = -\lambda\frac{h(b)-h(a)}{2}(f(a)+f(b)) + \int_a^b K(s,t)f^{\Delta}(s)\Delta s + \int_a^b h^{\Delta}(s)f^{\sigma}(s)\Delta s, \quad (8)$$

where

$$K(s,t) = \begin{cases} h(s) - \left(h(a)+\lambda\frac{h(b)-h(a)}{2}\right), & s\in[a,t), \\ h(s) - \left(h(a)+(2-\lambda)\frac{h(b)-h(a)}{2}\right), & s\in[t,b]. \end{cases} \quad (9)$$

Remark 17. Lemma 14 boils down to Lemma 1 in [16] if $h(t)=t$. With $h(t)=t$ in Corollary 16, we get Lemma 3.2 of [8] which in turn reduces to Lemma 3.1 of the paper [3] if $\lambda=0$.

Corollary 18. *If* $\mathbb{T}=\mathbb{R}$*, then the equation in Lemma 14 becomes*

$$\begin{aligned}\frac{h(b)-h(a)}{2}&\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)\\ &= -\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]\\ &\quad+\int_a^b K(s,t)f'(s)ds+\int_a^b h'(s)f(s)ds. \end{aligned} \quad (10)$$

Corollary 19. *If we take* $\mathbb{T}=\mathbb{Z}$ *in Lemma 14, then we get*

$$\begin{aligned}\frac{h(b)-h(a)}{2}&\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)\\ &= -\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]\\ &\quad+\sum_{s=a}^{b-1}K(s,t)\Big(f(s+1)-f(s)\Big)\\ &\quad+\sum_{s=a}^{b-1}f(s+1)\Big(h(s+1)-h(s)\Big). \end{aligned} \quad (11)$$

Corollary 20. *Let* $\mathbb{T}=q^{\mathbb{N}_0}$*, with* $q>1$*. Then Lemma 14 becomes*

$$\frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)$$

$$= -\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a) + (1-\psi(1-\lambda))\,f(b)\Big]$$
$$+\int_a^b K(s,t)\frac{f(qs)-f(s)}{(q-1)s}d_qs + \int_a^b \frac{h(qs)-h(s)}{(q-1)s}f(qs)d_qs. \quad (12)$$

3.1. A WEIGHTED OSTROWSKI TYPE INEQUALITY

Theorem 21. *Let $a,b,s,t \in \mathbb{T}$, $a < b$, $f,h : [a,b] \to \mathbb{R}$ be differentiable functions, and ψ a function of $[0,1]$ into $[0,1]$. Then for all $t \in [a,b]$, we have the following inequality*

$$\Bigg|\frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t) + \frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big] - \int_a^b h^{\Delta}(s)f^{\sigma}(s)\Delta s\Bigg| \le M\int_a^b |K(s,t)|\Delta s, \quad (13)$$

where $K(\cdot,\cdot)$ is as defined in (9) and $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$.

Proof. By rearranging the equation in Lemma 14, taking absolute value of both sides and then applying item (v) of Theorem 11, we get the desired inequality. □

Remark 22. If we take $h(t) = t$ in Theorem 21, then we get Theorem 4.

Corollary 23. *If we take $h(t) = t^2$ and $\psi(\lambda) = \lambda^2$, then Theorem 21 boils down to*

$$\left|(1-\lambda)f(t) + \frac{\lambda^2 f(a) + (2\lambda-\lambda^2)f(b)}{2} - \frac{1}{b^2-a^2}\int_a^b (\sigma(s)+s)f^{\sigma}(s)\Delta s\right| \le \frac{M}{b^2-a^2}\int_a^b |K(s,t)|\Delta s, \quad (14)$$

where

$$K(s,t) = \begin{cases} s^2 - \left(a^2 + \lambda^2\dfrac{b^2-a^2}{2}\right), & s \in [a,t), \\ s^2 - \left(a^2 + (\lambda^2-2\lambda+2)\dfrac{b^2-a^2}{2}\right), & s \in [t,b]. \end{cases}$$

Corollary 23 reduces to the following inequalities for different choices of λ.

1. For $\lambda = 0$, we get

$$\left| f(t) - \frac{1}{b^2 - a^2} \int_a^b (\sigma(s) + s) f^\sigma(s) \Delta s \right| \le \frac{M}{b^2 - a^2} \int_a^b |K(s,t)| \Delta s,$$

where

$$K(s,t) = \begin{cases} (s-a)(s+a), & s \in [a,t), \\ (s-b)(s+b), & s \in [t,b]. \end{cases}$$

2. For $\lambda = 1$, we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b^2 - a^2} \int_a^b (\sigma(s) + s) f^\sigma(s) \Delta s \right| \le \frac{M}{b^2 - a^2} \int_a^b |K(s,t)| \Delta s, \tag{15}$$

where

$$K(s,t) = s^2 - \frac{a^2 + b^2}{2}, \qquad s \in [a,b].$$

3. For $\lambda = \frac{1}{2}$, we have

$$\left| f(t) + \frac{f(a) + 3f(b)}{4} - \frac{2}{b^2 - a^2} \int_a^b (\sigma(s) + s) f^\sigma(s) \Delta s \right| \le \frac{2M}{b^2 - a^2} \int_a^b |K(s,t)| \Delta s, \tag{16}$$

where

$$K(s,t) = \begin{cases} s^2 - \frac{7a^2 + b^2}{8}, & s \in [a,t), \\ s^2 - \frac{3a^2 + 5b^2}{8}, & s \in [t,b]. \end{cases}$$

Corollary 24. *If $\mathbb{T} = \mathbb{R}$, then the inequality in Theorem 21 becomes*

$$\left| \frac{h(b) - h(a)}{2} \Big[1 + \psi(1-\lambda) - \psi(\lambda) \Big] f(t) + \frac{h(b) - h(a)}{2} \Big[\psi(\lambda) f(a) + (1 - \psi(1-\lambda))\, f(b) \Big] \right.$$

$$- \int_a^b h'(s)f(s)\, ds \Bigg| \le M \int_a^b |K(s,t)|\, ds, \tag{17}$$

where $M = \sup_{a<t<b} |f'(t)| < \infty$.

Corollary 25. *If* $\mathbb{T} = \mathbb{Z}$, *then the inequality in Theorem 21 becomes*

$$\begin{aligned}\Bigg| \frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big] f(t) \\ + \frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a) + (1-\psi(1-\lambda))\, f(b)\Big] \\ - \sum_{s=a}^{b-1} f(s+1)\Big(h(s+1)-h(s)\Big)\Bigg| \le M \sum_{s=a}^{b-1} |K(s,t)|,\end{aligned}$$

where $M = \sup_{a<t<b} |f(t+1)-f(t)| < \infty$.

Corollary 26. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, *with* $q > 1$, $a = q^m$ *and* $b = q^n$ *with* $m < n$. *Then Theorem 21 becomes*

$$\begin{aligned}\Bigg| \frac{h(q^n)-h(q^m)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big] f(t) - \int_{q^m}^{q^n} \frac{h(qs)-h(s)}{(q-1)s} f(qs) d_q s \\ + \frac{h(q^n)-h(q^m)}{2}\Big[\psi(\lambda)f(q^m) + (1-\psi(1-\lambda))\, f(q^n)\Big]\Bigg| \le M \sum_{j=m}^{n-1} |K(q^j,t)|,\end{aligned}$$

where $M = \sup_{q^m<q^j<q^n} \left| \frac{f(q^{j+1})-f(q^j)}{(q-1)q^j} \right| < \infty$, *and*

$$K(q^j,t) = \begin{cases} h(q^j) - \left(h(q^m) + \psi(\lambda)\frac{h(q^n)-h(q^m)}{2}\right), & q^j \in [q^m, t), \\ h(q^j) - \left(h(q^m) + (1+\psi(1-\lambda))\frac{h(q^n)-h(q^m)}{2}\right), & q^j \in [t, q^n], \end{cases}$$

3.2. A WEIGHTED OSTROWSKI-GRÜSS TYPE INEQUALITY

Theorem 27. *Let* $a, b, s, t \in \mathbb{T}$, $a < b$, $f, h : [a,b] \to \mathbb{R}$ *be differentiable functions such that there exist constants* γ, $\Gamma \in \mathbb{R}$, *with* $\gamma \le f^{\Delta}(t) \le \Gamma$, $t \in [a,b]$, *and* ψ *a function of* $[0,1]$ *into* $[0,1]$. *Then for all* $t \in [a,b]$, *we have the following inequality*

$$\Bigg| \frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big] f(t)$$

$$+\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]$$
$$-\int_a^b h^{\Delta}(s)f^{\sigma}(s)\Delta s-\frac{\Gamma+\gamma}{2}\int_a^b K(s,t)\Delta s\Bigg|\leq\frac{\Gamma-\gamma}{2}\int_a^b |K(s,t)|\Delta s, \quad (18)$$

where $K(\cdot,\cdot)$ is as defined in (9) above.

Proof. From Lemma 14, we get

$$\int_a^b K(s,t)f^{\Delta}(s)\Delta s=\frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)$$
$$+\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]$$
$$-\int_a^b h^{\Delta}(s)f^{\sigma}(s)\Delta s. \quad (19)$$

Setting $\Phi=\frac{\Gamma+\gamma}{2}$, and using the assumption that $\gamma\leq f^{\Delta}(t)\leq\Gamma$, for $t\in[a,b]$, we get

$$\sup_{a<t<b}|f^{\Delta}(t)-\Phi|\leq\frac{\Gamma-\gamma}{2}. \quad (20)$$

Also, using (19) we obtain

$$\int_a^b K(s,t)\Big(f^{\Delta}(s)-\Phi\Big)\Delta s=\frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)$$
$$+\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]$$
$$-\int_a^b h^{\Delta}(s)f^{\sigma}(s)\Delta s-\frac{\Gamma+\gamma}{2}\int_a^b K(s,t)\Delta s. \quad (21)$$

Furthermore, applying item (v) of Theorem 11 and (20) to the left hand side of (21) amounts to

$$\left|\int_a^b K(s,t)\Big(f^{\Delta}(s)-\Phi\Big)\Delta s\right|\leq\sup_{a<t<b}|f^{\Delta}(t)-\Phi|\int_a^b|K(s,t)|\Delta s$$
$$\leq\frac{\Gamma-\gamma}{2}\int_a^b|K(s,t)|\Delta s. \quad (22)$$

The desired inequality follows from using (21) in (22). □

Corollary 28. *If we take $h(t) = t$ in Theorem 27, then we obtain by using Lemma 15 the following inequality*

$$\begin{aligned}
&\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)}{2}\\
&-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\\
&-\frac{\Gamma+\gamma}{2(b-a)}\Bigg[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\\
&+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\Bigg]\Bigg|\\
\leq&\frac{\Gamma-\gamma}{2(b-a)}\Bigg[h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)\\
&+h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\Bigg],
\end{aligned}$$

for all $\lambda \in [0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $t \in \left[a+\psi(\lambda)\frac{b-a}{2}, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

Now applying Theorem 27 to the continuous, discrete, and quantum cases, we get

Corollary 29. *For $\mathbb{T} = \mathbb{R}$, the inequality in Theorem 27 becomes*

$$\begin{aligned}
&\Bigg|\frac{h(b)-h(a)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)\\
&+\frac{h(b)-h(a)}{2}\Big[\psi(\lambda)f(a)+(1-\psi(1-\lambda))\,f(b)\Big]\\
&-\int_a^b h'(s)f(s)\;ds-\frac{\Gamma+\gamma}{2}\int_a^b K(s,t)\;ds\Bigg|\\
\leq&\frac{\Gamma-\gamma}{2}\int_a^b |K(s,t)|\;ds.
\end{aligned}$$

Corollary 30. *For $\mathbb{T} = \mathbb{Z}, a = 0, b = n, f(k) = x_k, s = j$ and $t = i$, the inequality in Theorem 27 becomes*

$$\Bigg|\frac{h(n)-h(0)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]x_i$$

$$
\begin{aligned}
&+\frac{h(n)-h(0)}{2}\Big[\psi(\lambda)x_0+(1-\psi(1-\lambda))\,x_n\Big]\\
&-\sum_{j=0}^{n-1}x_{j+1}\Big(h(j+1)-h(j)\Big)-\frac{\Gamma+\gamma}{2}\sum_{j=0}^{n-1}K(j,i)\Bigg|\\
\leq&\frac{\Gamma-\gamma}{2}\sum_{j=0}^{n-1}|K(j,i)|,
\end{aligned}
$$

where

$$
K(j,i)=\begin{cases} h(j)-\left(h(0)+\psi(\lambda)\dfrac{h(n)-h(0)}{2}\right), & j\in[0,i),\\ h(j)-\left(h(0)+(1+\psi(1-\lambda))\dfrac{h(n)-h(0)}{2}\right), & j\in[i,n-1]. \end{cases}
$$

Corollary 31. *Let $\mathbb{T}=q^{\mathbb{N}_0}$, with $q>1$, $a=q^m$ and $b=q^n$ with $m<n$. Then Theorem 27 becomes*

$$
\begin{aligned}
&\Bigg|\frac{h(q^n)-h(q^m)}{2}\Big[1+\psi(1-\lambda)-\psi(\lambda)\Big]f(t)\\
&+\frac{h(q^n)-h(q^m)}{2}\Big[\psi(\lambda)f(q^m)+(1-\psi(1-\lambda))\,f(q^n)\Big]\\
&-\int_{q^m}^{q^n}\frac{h(qs)-h(s)}{(q-1)s}f(qs)d_qs-\frac{\Gamma+\gamma}{2}\sum_{j=m}^{n-1}K(q^j,t)\Bigg|\\
\leq&\frac{\Gamma-\gamma}{2}\sum_{j=m}^{n-1}|K(q^j,t)|,
\end{aligned}
$$

where

$$
K(q^j,t)=\begin{cases} h(q^j)-\left(h(q^m)+\psi(\lambda)\dfrac{h(q^n)-h(q^m)}{2}\right), & q^j\in[q^m,t),\\ h(q^j)-\left(h(q^m)+(1+\psi(1-\lambda))\dfrac{h(q^n)-h(q^m)}{2}\right), & q^j\in[t,q^n]. \end{cases}
$$

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