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LINEAR COMBINATIONS OF 2-ORTHOGONAL POLYNOMIALS: GENERATION AND DECOMPOSITION PROBLEMS

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ABSTRACT: In this work we are interested in the study of the 2-orthogonality of sequences of monic 2-orthogonal polynomials $\{P_n\}_{n\geq\ 0}$ and $\{Q_n\}_{n\geq 0}$ satisfying the relation $Q_{n+1}(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x)$, $n \geq 0$,where α_n, $n \geq 1$, are nonzero complex numbers. The sequence $\{Q_n\}_{n\geq 0}$ is said to be generated with 2 terms of the sequence $\{P_n\}_{n\geq 0}$ and the sequence $\{P_n\}_{n\geq 0}$ is said to be a decomposition of the sequence $\{Q_n\}_{n\geq 0}$ with 2 terms. First, we give necessary and sufficient conditions for the 2-orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ assuming the 2-orthogonality of the sequence $\{P_n\}_{n\geq 0}$. Second, assuming the sequence $\{Q_n\}_{n\geq 0}$ is 2-orthogonal we get necessary and sufficient conditions for the existence of a sequence $\{P_n\}_{n\geq 0}$ satisfying the above relation and such that it is 2 orthogonal. Indeed, we characterize the 2-orthogonality of these sequences in terms of the coefficients of the corresponding four term récurrence relations. Next, we study our problem as an inverse problem for 2-monic orthogonal polynomials. Furthermore, the relation between the banded Hessenberg matrices associated with the multiplication operator in terms of the bases $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ is analyzed. Finally, we give many examples of such related 2-orthogonal polynomial sequences.

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1. INTRODUCTION

Recently, in [1] and [3] the authors have studied the standard orthogonality of a sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ given in terms of a linear combination of a sequence $\{P_n\}_{n\geq 0}$ of monic polynomials orthogonal with respect to a linear functional u in the standard sense, i.e, $\langle u, P_n(x)P_m(x)\rangle = 0$, $m \neq n$. Their approach is based on the three term recurrence relations that such polynomial sequences satisfy. Furthermore, they deduce the connection between the corresponding linear functionals. Finally, in [2] the algebraic relation between the Jacobi matrices associated with such polynomial sequences is stated by using the so called Geronimus spectral transformation (see [17]). This kind of questions are known in the literature of orthogonal polynomials as inverse problems.

If the sequence $\{Q_n\}_{n\geq 0}$ of monic polynomials can be represented in terms of a sequence of standard monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ as $Q_{n+1}(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x)$, $n \geq 0$, where α_n, $n \geq 1$, are nonzero complex numbers, we say that $\{Q_n\}_{n\geq 0}$ is generated by $\{P_n\}_{n\geq 0}$ when it satisfies the standard orthogonality in terms of a linear functional v. Necessary and sufficient conditions for such an orthogonality have been studied in [12]. On the other hand, $v = \lambda(x-c)^{-1}u + \delta(x-c)$ (see [12]), where λ, c depend on α_1, α_2, and the coefficients of the polynomials $P_1(x)$ and $P_2(x)$. Conversely, given a sequence $\{Q_n\}_{n\geq 0}$ of standard monic orthogonal polynomials, if there exists a sequence of standard monic polynomials $\{P_n\}_{n\geq 0}$ such that the above relation holds, we say that $\{Q_n\}_{n\geq 0}$ is decomposed in terms of $\{P_n\}_{n\geq 0}$. Notice that you have such a decomposition when $Q_n(c) \neq 0$ for every $n \geq 0$ and some real number c. Indeed, the polynomials P_n are the monic kernel polynomials which are orthogonal with respect to the linear functional $(x-c)v$ (see [8])

The aim of our contribution is to analyze a similar problem in the framework of 2-orthogonality. Let $\{P_n\}_{n\geq 0}$ be a sequence of 2-orthogonal monic polynomials (2-MOPS, in short) with respect to a bi-dimensional vector linear functional U. Let consider a sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ defined by

$$Q_{n+1}(x) = P_{n+1}(x) + \alpha_{n+1} P_n(x), \; n \geq 0, \tag{1.1}$$

where α_n, $n \geq 1$, are nonzero complex numbers. We are interested to find necessary and sufficient conditions in order to $\{Q_n\}_{n\geq 0}$ is a 2-MOPS with respect to a bi-dimensional vector functional V. As a next step, to find the relation between the vector functionals U and V.

Our approach is based on the fact that sequences of monic 2-orthogonal polynomials are characterized by a four term recurrence relation (see [11]). Thus, from (1.1) we can deduce when the polynomial sequence $\{Q_n\}_{n\geq 0}$ satisfies a four term recurrence relation. As a straightforward consequence, we can obtain the corresponding coefficients in such a recurrence relation. Conversely, given a 2-MOPS $\{Q_n\}_{n\geq 0}$, we analyze its decomposition as a linear combination of another 2-MOPS $\{P_n\}_{n\geq 0}$ according to (1.1). Then, $\{Q_n\}_{n\geq 0}$ is said to be generated by 2 terms of the sequence $\{P_n\}_{n\geq 0}$. The sequence $\{P_n\}_{n\geq 0}$ is said to be the decomposition of the sequence $\{Q_n\}_{n\geq 0}$ with 2 terms. As a special case, you have the so-called 2-kernel polynomials (see [5]) .

By using an algebraic approach, in Proposition 2.4 of [14], finite type relations between sequences of orthogonal polynomials $\{Q_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ as $\Phi(x)Q_n(x) = \sum_{k=n-s}^{n+t} \lambda_{n,k} P_k(x)$, $n \geq s$, where $\Phi(x)$ is a polynomial of degree t and there exists $r \geq s$ such that $\lambda_{r,r-s}$, are studied. The connection between the corresponding linear functionals not only for the standard orthogonality but also for 2-orthogonal polynomials is stated. Furthermore, in [16] some analog questions are considered for special finite type relations between sequences of polynomials in the framework of d-orthogonality. Notice that the problem we are dealing in our contribution has not been studied therein since the terms involved in the finite type relation are $dl+1$. Indeed, in Section 4 of [16], the authors consider a finite type relation involving $2l+1$ terms, with

$l \geq 1$.

The structure of the paper is as follows. In Section 2 we recall some algebraic notations on the dual space of polynomials with complex coefficients. In Section 3, we state and prove our main result concerning necessary and sufficient conditions for the 2-orthogonality of the sequence $\{Q_n\}_{n\geq 0}$. In such a case, we find the coefficients of its four term recurrence relation in terms of those of the four term recurrence relation satisfied by the sequence $\{P_n\}_{n\geq 0}$ and conversely. The previous result allows us to introduce the concept of generation and decomposition of 2-MOPS. Moreover, we give the relation between the banded Hessenberg matrices associated with the multiplication operator with respect to the bases $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ respectively. Finally, in Section 4, we show some illustrative examples of 2-MOPS $\{Q_n\}_{n\geq 0}$ which can be generated by fixed 2-MOPS as well as fixed 2-MOPS $\{Q_n\}_{n\geq 0}$ which can be decomposed as 2-MOPS, furthermore such sequences can be repeat the process of (generation/decomposition) for once, finitely or infinitely time.

2. PRELIMINARIES AND BASIC BACKGROUND

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u with respect to the canonical sequence $\{x^n\}_{n\geq 0}$. When $(u)_0 = 1$, the linear functional u is said to be normalized. In $\mathcal{P}'$ we introduce the following algebraic operations that will be useful in the sequel. The left multiplication of a linear functional $u \in \mathcal{P}'$ by a polynomial h and the division by a polynomial $x - c$, where $c \in \mathbf{C}$, are defined as follows

$$\langle hu, f\rangle := \langle u, hf\rangle$$
$$\left\langle (x-c)^{-1} u, f \right\rangle := \langle u, \theta_c f\rangle \text{ where } (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials with $\deg P_n = n$, $n \geq 0$, (PS, in short) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m\rangle := \delta_{n,m}$, n, $m \geq 0$.

We recall some definitions and results on the $d-$orthogonality[11] (see, for instance [9]).

Definition 1. ([11]) Let $U = (u_0, u_1, ..., u_{d-1})$ be a d-dimensional vector of linear functionals and let $\{P_n\}_{n\geq 0}$ be a PS. $\{P_n\}_{n\geq 0}$ is said to be a d-orthogonal polynomial sequence with respect to the vector functional $U = (u_0, u_1, ..., u_{d-1})$ if for each positive integer number α, with $0 \leq \alpha \leq d-1$, we get

$$\begin{cases} \langle u_\alpha, P_m P_n \rangle = 0, & n \geq md + \alpha + 1, \ m \geq \ 0, \\ \langle u_\alpha, P_m P_{md+\alpha} \rangle \neq 0, & m \geq \ 0. \end{cases} \tag{2.1}$$

In this case, the vector functional U is said to be regular. If the d-orthogonal polynomials are monic, then the sequence is said to be a d-orthogonal polynomial sequence (d-MOPS, in short)

For $d = 1$, we get the standard orthogonality.

Theorem 1. *([11]) Let $\{P_n\}_{n\geq 0}$ be a PS. Then the following statements are equivalent.*

1. *$\{P_n\}_{n\geq 0}$ is a d-MOPS with respect to $U = (u_0, u_1, ..., u_{d-1})$.*
2. *$\{P_n\}_{n\geq 0}$ satisfies a difference equation of order $d+1$, $d \geq 1$,*

$$P_{m+d+1}(x) = (x - \beta_{m+d})\, P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \ m \geq \ 0,$$

with initial data

$$P_0(x) = 1, P_1(x) = x - \beta_0,$$
$$P_m(x) = (x - \beta_{m-1})\, P_{m-1}(x) - \textstyle\sum_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1-\nu} P_{m-2-\nu}(x), \ 0 \ \leq m \leq \ 2,$$

where $\gamma_{m+1}^0 \neq 0$, $m \geq 0$. (Regularity conditions)

Theorem 2. *([11]) For every d-MOPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

1. *$\{P_n\}_{n\geq 0}$ is $d-$symmetric.*
2. *$\{P_n\}_{n\geq 0}$ satisfies the recurrence relation*

$$P_n(x) = x^n, \ 0 \leq n \leq d,$$
$$P_{n+d+1}(x) = x P_{n+d}(x) - \gamma_{n+1}^0 P_n(x), \ \gamma_{n+1}^0 \neq 0, \ n \geq 0.$$

Now we will focus our attention on the case $d = 2$. In such a situation, $\{P_n\}_{n\geq 0}$ satisfies a four term recurrence relation

$$\begin{cases} P_{n+3}(x) = (x-\beta_{n+2})P_{n+2} - \gamma_{n+2}P_{n+1}(x) - \delta_{n+1}P_n(x), \ n \geq 0, \\ P_0(x) = 1, \ P_1(x) = x - \beta_0, \ P_2(x) = (x-\beta_1)P_1(x) - \gamma_1, \end{cases} \tag{2.2}$$

where $\delta_{n+1} \neq 0$, $n \geq 0$, (regularity conditions).

The following definitions and lemma will be useful in the sequel

Definition 2. ([13]) For any 2-MOPS $\{P_n\}_{n\geq 0}$ with respect to $U = (u_0, u_1)$, let $\{P_n^{(r)}\}_{n\geq 0}$ be the associated 2-MOPS of order r satisfying the shifted recurrence relation

$$P_{n+3}^{(r)}(x) = (x-\beta_{n+r+2})P_{n+2}^{(r)} - \gamma_{n+r+2}P_{n+1}^{(r)}(x) - \delta_{n+r+1}P_n^{(r)}(x), \ n \geq 0,$$

$$P_0^{(r)}(x) = 1, \ P_1^{(r)}(x) = x - \beta_r, \ P_2^{(r)}(x) = (x-\beta_{r+1})P_1^{(r)}(x) - \gamma_{r+1}.$$

Also, let $\left\{P_n^*\left(x, \mu_0, \mu_1, \mu_1^1\right)\right\}_{n\geq 0} := \{P_n^*(x;\mu)\}_{n\geq 0}$ be the co-recursive 2-MOPS defined by

$$\begin{aligned} P_{n+3}^*(x;\mu) = (x-\beta_{n+2})P_{n+2}^*(x;\mu) - \gamma_{n+2}P_{n+1}^*(x;\mu) \\ - \delta_{n+1}P_n^*(x;\mu), \quad n \geq 0, \end{aligned}$$

$$\begin{aligned} P_0^*(x) =& 1, \\ P_1^*(x) =& x - \beta_0 - \mu_0, \\ P_2^*(x) =& (x-\beta_1-\mu_1)P_1^*(x) - \left(\gamma_1 + \mu_1^1\right). \end{aligned}$$

It is very well-known, see [13], that the sequence $\{P_n^*\}_{n\geq 0}$ can be given in terms of the 2-MOPS and their associated 2-MOPS of order 1 and 2. Indeed,

Lemma 1.

$$P_n^*(x) = P_n(x) - \mu_0 P_{n-1}^{(1)}(x) - \left\{\mu_1 x + \mu_1^1 - \mu_1(\beta_0+\mu_0)\right\} P_{n-2}^{(2)}(x), \ n \geq 0.$$

On the other hand, we will remind a basic result concerning vector functionals such that their corresponding 2-MOPS are related in a linear way (see [14]).

Proposition 1. Let $\{P_n\}_{n\geq 0}$ be a 2-MOPS with respect to $U = (u_0, u_1)$ and $\{Q_n\}_{n\geq 0}$ a 2-MOPS with respect to $V = (v_0, v_1)$. Let Φ be a monic polynomial with $\deg \Phi = t$. Then the following statements are equivalent.

1. There is a positive integer s such that

$$\Phi(x) Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} P_\nu(x), \ n \geq s, \quad \exists r \geq s: \ \lambda_{r,r-s} \neq 0.$$

2. There exists a matrix X_0 with polynomial entries

$$X_0 = \begin{pmatrix} A_0 & B_0 \\ A_1 & B_1 \end{pmatrix}$$

such that

$$\Phi U = X_0 V.$$

Notice that the polynomials A_0, B_0 are not simultaneously identically zero.

3. MAIN RESULTS

Let $\{P_n\}_{n\geq 0}$ be a 2-MOPS with respect to a regular bi-dimensional vector functional U, with recurrence coefficients denoted by $\{\beta_n, \gamma_{n+1}, \delta_{n+1}\}_{n\geq 0}$ and let $\{Q_n\}_{n\geq 0}$ be the sequence of monic polynomials defined by

$$Q_{n+1}(x) = P_{n+1}(x) + \alpha_{n+1} P_n(x), \ \alpha_{n+1} \neq 0, \ \forall n \geq 0.$$

First, we will give necessary and sufficient conditions such that the sequence $\{Q_n\}_{n\geq 0}$ be a 2- MOPS with respect to a regular bi-dimensional vector functional V. As a straightforward consequence, we get the relation between the vector functionals U and V. On the other hand, we will obtain a set of equations relating the coefficients of the recurrence relations satisfied by $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

Proposition 2. Let $\{Q_n\}_{n\geq 0}$ be the monic sequence of polynomials defined by (1.1), where $\{\alpha_n\}_{n\geq 0}$ are complex numbers with $\alpha_n \neq 0$, $\forall n \geq 0$. Then

$\{Q_n\}_{n\geq 0}$ is a 2-MOPS such that the coefficients of the recurrence relation are $\left\{\tilde{\beta}_n, \tilde{\gamma}_{n+1}, \tilde{\delta}_{n+1}\right\}_{n\geq 0}$ if and only if

$$\begin{cases} \tilde{\beta}_{n+1} = \alpha_{n+1} - \alpha_{n+2} + \beta_{n+1}, \qquad n \geq 0, & (3.1) \\ \tilde{\gamma}_{n+1} = \gamma_{n+1} + \alpha_{n+1}\left(\beta_n - \tilde{\beta}_{n+1}\right), \qquad n \geq 0, & (3.2) \\ \tilde{\delta}_{n+1} = \delta_{n+1} + \alpha_{n+2}\gamma_{n+1} - \alpha_{n+1}\tilde{\gamma}_{n+2}, \qquad n \geq 0, & (3.3) \\ \dfrac{\tilde{\delta}_{n+2}}{\alpha_{n+3}} = \dfrac{\delta_{n+1}}{\alpha_{n+1}}, \qquad n \geq 0, & (3.4) \end{cases}$$

with initial condition $\tilde{\beta}_0 = \beta_0 - \alpha_1$.

Proof. Multiplying both sides in (1.1) by x, applying (1.3) to $xP_{n+1}(x)$ $xP_n(x)$, and using (1.1) we get

$$\begin{aligned} xQ_{n+1}(x) =& Q_{n+2}(x) + (\beta_{n+1} + \alpha_{n+1} - \alpha_{n+2}) P_{n+1}(x) \\ &+ (\gamma_{n+1} + \alpha_{n+1}\beta_n) P_n(x) \\ &+ (\delta_n + \alpha_{n+1}\gamma_n) P_{n-1}(x) + \alpha_{n+1}\delta_{n-1}P_{n-2}(x), \quad n \geq 2 \end{aligned}$$

But, according to (1.1) we get

$$\begin{aligned} xQ_{n+1}(x) =& Q_{n+2}(x) + (\beta_{n+1} + \alpha_{n+1} - \alpha_{n+2}) Q_{n+1}(x) \\ &+ (\gamma_{n+1} + \alpha_{n+1}\beta_n - \alpha_{n+1}(\beta_{n+1} + \alpha_{n+1} - \alpha_{n+2})) Q_n(x) \\ &+ (\delta_n + \alpha_{n+1}\gamma_n - \alpha_n(\gamma_{n+1} + \alpha_{n+1}\beta_n \\ &-\alpha_{n+1}(\beta_{n+1} + \alpha_{n+1} - \alpha_{n+2}))) Q_{n-1}(x) \\ &- (\delta_n + \alpha_{n+1}\gamma_n - \alpha_n(\gamma_{n+1} + \alpha_{n+1}\beta_n \\ &-\alpha_{n+1}(\beta_{n+1} + \alpha_{n+1} - \alpha_{n+2}))) \alpha_{n-1}P_{n-2}(x) \\ &+ (\alpha_{n+1}\delta_{n-1}) P_{n-2}(x) \\ =& Q_{n+2}(x) + \tilde{\beta}_{n+1}Q_{n+1}(x) + \tilde{\gamma}_{n+1}Q_n(x) + \tilde{\delta}_n Q_{n-1}(x), \quad n \geq 1. \end{aligned}$$

And, after the identification of coefficients, we get our result. □

Remark 1. The regularity condition $\tilde{\delta}_n \neq 0$, $\forall n \geq 2$, follows from (3.4).

Next, we show that the 2-orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ can be characterized by the fact that the parameters $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}, \delta_n$ and $\alpha_{n+1}, \tilde{\beta}_{n+1}, \tilde{\gamma}_{n+1}, \tilde{\delta}_n$ satisfy simultaneously the above expression.

Corollary 1. *$\{Q_n\}_{n\geq 0}$ is a 2-MOPS if and only*

$$\frac{\delta_{n+2}}{\alpha_{n+3}\alpha_{n+2}}-\frac{\delta_{n+1}}{\alpha_{n+1}\alpha_{n+2}}-\left(\frac{\gamma_{n+3}}{\alpha_{n+3}}-\frac{\gamma_{n+2}}{\alpha_{n+2}}\right)$$
$$+\left(\beta_{n+3}-\beta_{n+2}\right)-\left(\alpha_{n+4}-\alpha_{n+3}\right)=0,\ n\geq 0, \quad (3.5)$$

and

$$\begin{cases}\tilde{\beta}_{n+1}=\alpha_{n+1}-\alpha_{n+2}+\beta_{n+1}, & n\geq 0,\\ \tilde{\gamma}_{n+1}=\gamma_{n+1}+\alpha_{n+1}\left(\beta_n-\tilde{\beta}_{n+1}\right), & n\geq 0,\\ \tilde{\delta}_{n+1}=\delta_{n+1}+\alpha_{n+2}\gamma_{n+1}-\alpha_{n+1}\tilde{\gamma}_{n+2}, & n\geq 0.\end{cases}$$

Proof. If we replace (3.1) in (3.2), then we get

$$\tilde{\gamma}_{n+1}=\gamma_{n+1}+\alpha_{n+1}\left(\beta_n-\tilde{\beta}_{n+1}\right)=\gamma_{n+1}+\alpha_{n+1}\left(\beta_n-\alpha_{n+1}+\alpha_{n+2}-\beta_{n+1}\right),$$
$$n\geq 0,$$

or, equivalently,

$$\tilde{\gamma}_{n+1}=\gamma_{n+1}+\alpha_{n+1}\left(\alpha_{n+2}-\beta_{n+1}\right)-\alpha_{n+1}\left(\alpha_{n+1}-\beta_n\right),\ n\geq 0.$$

If we replace the above expression in (3.3), then

$$\tilde{\delta}_{n+1}=\delta_{n+1}+\alpha_{n+2}\gamma_{n+1}-\alpha_{n+1}\tilde{\gamma}_{n+2},\quad n\geq 0.$$

In other words, $\forall n\geq 0$

$$\tilde{\delta}_{n+1}=\delta_{n+1}+\alpha_{n+2}\gamma_{n+1}$$
$$-\alpha_{n+1}\left(\gamma_{n+2}+\alpha_{n+2}\left(\alpha_{n+3}-\beta_{n+2}\right)-\alpha_{n+2}\left(\alpha_{n+2}-\beta_{n+1}\right)\right),$$

or, equivalently, $\forall n\geq 0$

$$\tilde{\delta}_{n+1}=\delta_{n+1}+\alpha_{n+2}\gamma_{n+1}-\alpha_{n+1}\gamma_{n+2}$$
$$-\alpha_{n+1}\alpha_{n+2}\left(\alpha_{n+3}-\beta_{n+2}\right)+\alpha_{n+1}\alpha_{n+2}\left(\alpha_{n+2}-\beta_{n+1}\right).$$

From (3.4)

$$\tilde{\delta}_{n+1}=\frac{\alpha_{n+2}}{\alpha_n}\delta_n,\ n\geq 1,$$

and, as a consequence, $\forall n\geq 1$

$$\frac{\alpha_{n+2}}{\alpha_n}\delta_n = \delta_{n+1} + \alpha_{n+2}\gamma_{n+1} - \alpha_{n+1}\gamma_{n+2} - \alpha_{n+1}\alpha_{n+2}\left(\alpha_{n+3} - \beta_{n+2}\right) + \alpha_{n+1}\alpha_{n+2}\left(\alpha_{n+2} - \beta_{n+1}\right).$$

Dividing in both sides by $\alpha_{n+2}\alpha_{n+1}$, we get

$$\frac{\delta_{n+1}}{\alpha_{n+2}\alpha_{n+1}} - \frac{\delta_n}{\alpha_n\alpha_{n+1}} - \left(\frac{\gamma_{n+2}}{\alpha_{n+2}} - \frac{\gamma_{n+1}}{\alpha_{n+1}} -\right) + \left(\beta_{n+2} - \beta_{n+1}\right) - \left(\alpha_{n+3} - \alpha_{n+2}\right) = 0, \ n \geq 1. \quad \square$$

In a similar way as above,

Corollary 2. *$\{Q_n\}_{n\geq 0}$ is a 2-MOPS if and only if*

$$\frac{\tilde{\delta}_{n+2}}{\alpha_{n+3}\alpha_{n+2}} - \frac{\tilde{\delta}_{n+1}}{\alpha_{n+2}\alpha_{n+1}} - \left(\frac{\tilde{\gamma}_{n+2}}{\alpha_{n+2}} - \frac{\tilde{\gamma}_{n+1}}{\alpha_{n+1}}\right) + \left(\tilde{\beta}_{n+1} - \tilde{\beta}_n\right) - \left(\alpha_{n+1} - \alpha_n\right) = 0, \ n \geq 1, \quad (3.6)$$

and

$$\begin{cases} \beta_{n+1} = \tilde{\beta}_{n+1} + \alpha_{n+2} - \alpha_{n+1}, & n \geq 0, \\ \gamma_{n+1} = \tilde{\gamma}_{n+1} - \alpha_{n+1}\left(\beta_n - \tilde{\beta}_{n+1}\right), & n \geq 0, \\ \delta_{n+1} = \tilde{\delta}_{n+1} - \alpha_{n+2}\gamma_{n+1} + \alpha_{n+1}\tilde{\gamma}_{n+2}, & n \geq 0, \end{cases}$$

with

$$\frac{\tilde{\delta}_2}{\alpha_3\alpha_2} - \frac{\tilde{\delta}_1}{\alpha_2\alpha_1} - \left(\frac{\tilde{\gamma}_2}{\alpha_2} - \frac{\tilde{\gamma}_1}{\alpha_1}\right) + \left(\beta_0 - \tilde{\beta}_1\right) = 0,$$

where

$$\beta_0 = \tilde{\beta}_1 + \left(\frac{\tilde{\gamma}_2}{\alpha_2} - \frac{\tilde{\gamma}_1}{\alpha_1}\right) - \left(\frac{\tilde{\delta}_2}{\alpha_2\alpha_3} - \frac{\tilde{\delta}_1}{\alpha_1\alpha_2}\right).$$

In the next corollary, we show that the 2-orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ can be also characterized by the fact that there exist two constants $\mathbf{y}$, $\mathbf{z}$ depending only on the first parameters of the recurrence relations of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively.

Corollary 3. *Let* $\mathbf{y} = \frac{\delta_1}{\alpha_1\alpha_2} - \frac{\gamma_2}{\alpha_2} + \beta_2 - \alpha_3$, $\mathbf{z} = \frac{\tilde{\delta}_2}{\alpha_3\alpha_2} - \frac{\tilde{\gamma}_2}{\alpha_2} + \tilde{\beta}_1 - \alpha_1$. *Then*

$$\frac{\delta_{n+2}}{\alpha_{n+3}\alpha_{n+2}} - \frac{\gamma_{n+3}}{\alpha_{n+3}} + \beta_{n+3} - \alpha_{n+4} = \mathbf{y}, \ n \geq 0, \quad (3.7)$$

and

$$\frac{\tilde{\delta}_{n+3}}{\alpha_{n+4}\alpha_{n+3}} - \frac{\tilde{\gamma}_{n+3}}{\alpha_{n+3}} + \tilde{\beta}_{n+2} - \alpha_{n+2} = \mathbf{z}, \ n \geq 0. \quad (3.8)$$

Proof. From Corollary 1, summing over n, we obtain

$$\sum_{k=1}^{n}\left[\frac{\delta_{k+1}}{\alpha_{k+2}\alpha_{k+1}}-\frac{\delta_k}{\alpha_k\alpha_{k+1}} -\left(\frac{\gamma_{k+2}}{\alpha_{k+2}}-\frac{\gamma_{k+1}}{\alpha_{k+1}}\right)+(\beta_{k+2}-\beta_{k+1})-(\alpha_{k+3}-\alpha_{k+2})\right]=0,$$

$$\left[\left(\frac{\delta_{n+1}}{\alpha_{n+2}\alpha_{n+1}}-\frac{\delta_1}{\alpha_1\alpha_2}\right)-\left(\frac{\gamma_{n+2}}{\alpha_{n+2}}-\frac{\gamma_2}{\alpha_2}-\right)+(\beta_{n+2}-\beta_2)-(\alpha_{n+3}-\alpha_3)\right]=0,$$

$$\frac{\delta_{n+2}}{\alpha_{n+3}\alpha_{n+2}}-\frac{\gamma_{n+3}}{\alpha_{n+3}}+\beta_{n+3}-\alpha_{n+4}=\frac{\delta_1}{\alpha_1\alpha_2}-\frac{\gamma_2}{\alpha_2}+\beta_2-\alpha_3=\mathbf{y},\quad n\geq 0$$

and

$$\frac{\tilde{\delta}_{n+3}}{\alpha_{n+4}\alpha_{n+3}}-\frac{\tilde{\gamma}_{n+3}}{\alpha_{n+3}}+\tilde{\beta}_{n+2}-\alpha_{n+2}=\frac{\tilde{\delta}_2}{\alpha_3\alpha_2}-\frac{\tilde{\gamma}_2}{\alpha_2}+\tilde{\beta}_1-\alpha_1=\mathbf{z},\ n\geq 0,\quad \square$$

Remark 2. Noticed that expressions (3.5) and (3.6), which henceforth will be denoted ($\mathbf{R}.\mathbf{1}$) and ($\mathbf{R}.\mathbf{2}$) respectively are obtained from the same proposition, but the first one is given in terms of the recurrence coefficients of $\{P_n\}_{n\geq 0}$ and the second one in terms of the recurrence coefficients of $\{Q_n\}_{n\geq 0}$.

The previous result allows us to state the following

Definition 3. Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two MPS such that

$$Q_{n+1}(x)=P_{n+1}(x)+\alpha_{n+1}P_n(x), n\geq 0, \quad \text{where} \quad \alpha_n\neq 0.\mathbb{C}$$

If $\{P_n\}_{n\geq 0}$ is a 2-MOPS and satisfies ($\mathbf{R}.\mathbf{1}$), then $\{Q_n\}_{n\geq 0}$ is said to be generated by 2 terms of the sequence $\{P_n\}_{n\geq 0}$. On the other hand, if $\{Q_n\}_{n\geq 0}$ is a 2-MOPS and satisfies ($\mathbf{R}.\mathbf{2}$), then we say that $\{Q_n\}_{n\geq 0}$ is decomposed with 2 terms of the sequence $\{P_n\}_{n\geq 0}$.

Now we are going to specify $\{Q_n\}_{n\geq 0}$ to be 2-orthogonal. In addition, the relation between the vector functionals U and V will be deduced.

Theorem 3. *Let $\{P_n\}_{n\geq 0}$ be a 2-MOPS that satisfies the recurrence relation* (2.2) *and let $\{Q_n\}_{n\geq 0}$ be the sequence defined by (1.1). Let* $\mathbf{y}=\dfrac{\delta_1}{\alpha_2\alpha_1}-\dfrac{\gamma_2}{\alpha_2}-(\alpha_3-\beta_2)$. *If the sequence $\{\alpha_n\}_{n\geq 0}$ is defined by*

$$\alpha_{n+1}=-\frac{Y_{n+1}(\mathbf{y})}{Y_n(\mathbf{y})},\ n\geq\ 1,$$

then the 2-orthogonality of the monic polynomial sequence $\{Q_n\}_{n\geq 0}$ *depends, at most, of the choice of* $\alpha_1, \alpha_2,$ *and* α_3. *Indeed,*

i) *If*

$$\alpha_1 = \beta_0 - \mathbf{y} \text{ and } \alpha_2 = -\frac{P_2(\mathbf{y})}{P_1(\mathbf{y})}, \ n \geq 1,$$

then the sequence $\{Q_n\}_{n\geq 0}$ *is a 2-MOPS if and only for every* $n \geq 1$

$$Y_n(\mathbf{y}) = P_n(\mathbf{y}).$$

In this case,

$$Q_{n+1}(x) = P_{n+1}(x) - \frac{P_{n+1}(\mathbf{y})}{P_n(\mathbf{y})} P_n(x), \ n \geq 0.$$

ii) *If*

$$\mu_0 = 0, \ \mu_1 = \mathbf{y} - \beta_1 + \alpha_2, \ \mu_1^1 = -\gamma_1, \text{ and } \alpha_2 = -\frac{P_2^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)}{P_1^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)},$$

where $P_n^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)$ *is the co-recursive 2-MOPS of the sequence* $\{P_n\}_{n\geq 0}$, *then* $\{Q_n\}_{n\geq 0}$ *is a 2-MOPS if and only if for every* $n \geq 1$

$$Y_n(\mathbf{y}) = P_n^*(\mathbf{y}, \mu_0, \mu_1, \mu_1^1).$$

iii) *If*

$$\mu_0 = \mathbf{y} - \beta_0 + \alpha_1, \ \mu_1 = \mathbf{y} - \beta_1 + \alpha_2, \ \mu_1^1 = -\gamma_1 - \alpha_2\mu_0,$$

and

$$\alpha_2 = -\frac{P_2^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)}{P_1^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)},$$

then $\{Q_n\}_{n\geq 0}$ *is a 2-MOPS if and only if for every* $n \geq 1$

$$Y_n(\mathbf{y}) = P_n^*(\mathbf{y}, \mu_0, \mu_1, \mu_1^1).$$

iv) *If*

$$\mu_0 = \mathbf{y} - \beta_0 + \alpha_1, \ \mu_1 = \mathbf{y} - \beta_1 + \alpha_2, \ \mu_1^1 = -\gamma_1,$$

and

$$\alpha_2 = -\frac{P_2^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)}{P_1^*\left(\mathbf{y}, \mu_0, \mu_1, \mu_1^1\right)},$$

then $\{Q_n\}_{n\geq 0}$ *is a 2-MOPS if and only if for every* $n \geq 1$

$$Y_n(\mathbf{y}) = P_n^*(\mathbf{y}, \mu_0, \mu_1, \mu_1^1).$$

Proof. According to (3.7)

$$\frac{\delta_n}{\alpha_n} - \gamma_{n+1} - \alpha_{n+1}(\alpha_{n+2} - \beta_{n+1}) - \alpha_{n+1}\mathbf{y} = 0, \ n \geq 2,$$

or, equivalently,

$$\alpha_{n+1}\alpha_{n+2} + \alpha_{n+1}(\mathbf{y} - \beta_{n+1}) + \gamma_{n+1} - \frac{\delta_n}{\alpha_n} = 0, \ n \geq 1. \tag{3.9}$$

Now, let define the sequence $\{Y_n\}_{n\geq 0}$ in a recursive way as follows

$$\begin{cases} Y_{n+1} = -\alpha_{n+1}Y_n, \ n \geq 0, \\ Y_0 = 1. \end{cases}$$

Let replace it in (3.9). Then

$$\frac{Y_{n+2}}{Y_n} - (\mathbf{y} - \beta_{n+1})\frac{Y_{n+1}}{Y_n} + \gamma_{n+1} + \delta_n\frac{Y_{n-1}}{Y_n} = 0, \ n \geq 1,$$

i.e.,

$$\begin{cases} Y_{n+2} - (\mathbf{y} - \beta_{n+1})Y_{n+1} + \gamma_{n+1}Y_n + \delta_n Y_{n-1} = 0, \quad n \geq 1, \\ Y_0 = 1, \ Y_1 = -\alpha_1, \ Y_2 = -\alpha_2 Y_1. \end{cases}$$

The sequence $\{Y_n\}_{n\geq 0}$ satisfies a three term recurrence relation with arbitrary initial values Y_1 and Y_2. The sequence $\{Y_n\}_{n\geq 0}$ will be a sequence of polynomials if these initial values are polynomials of degrees 1 and 2, respectively. This yields four choices for Y_1 and Y_2 cited explicitly in Table 1.

$$\begin{array}{lll} A) & Y_1 = P_1(\mathbf{y}) & Y_2 = P_2(\mathbf{y}). \\ B) & Y_1 = P_1(\mathbf{y}) & Y_2 \neq P_2(\mathbf{y}). \\ C) & Y_1 \neq P_1(\mathbf{y}) & Y_2 = P_2(\mathbf{y}). \\ D) & Y_1 \neq P_1(\mathbf{y}) & Y_2 \neq P_2(\mathbf{y}). \end{array}$$

Now we are going to specify the sequence $\{Q_n\}_{n\geq 0}$ according to the above initial conditions □

Furthermore, if $U = (u_0, u_1)$ and $V = (v_0, v_1)$ are, respectively, the vector functionals associated with the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, then we have

Proposition 3. Let $\{Q_n\}_{n\geq 0}$ be a sequence of monic polynomials defined by (1.1) . Then

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ ax + b & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \tag{3.10}$$

Case	Y_1, Y_2	μ_0	μ_1	μ_1^1
A	$\begin{cases} Y_1 = -\alpha_1 \\ Y_2 = \alpha_1\alpha_2 \end{cases}$	0	0	0
B	$\begin{cases} Y_1 = \mathbf{y} - \beta_0 - \mu_0 \\ Y_2 = (\mathbf{y} - \beta_1 - \mu_1) Y_1 \\ \quad - (\gamma_1 + \mu_1^1) \end{cases}$	$\mathbf{y} - \beta_0 + \alpha_1$	$\mathbf{y} - \beta_1 + \alpha_2$	$-\gamma_1$
C	$\begin{cases} Y_1 = \mathbf{y} - \beta_0 - \mu_0 \\ Y_2 = (\mathbf{y} - \beta_1 - \mu_1) P_1(\mathbf{y}) \\ \quad - (\gamma_1 + \mu_1^1) \end{cases}$	$\mathbf{y} - \beta_0 + \alpha_1$	$\mathbf{y} - \beta_1 + \alpha_2$	$-\alpha_2\mu_0 - \gamma_1$
D	$\begin{cases} Y_1 = \mathbf{y} - \beta_0 - \mu_0 \\ Y_2 = (\mathbf{y} - \beta_1 - \mu_1) P_1(\mathbf{y}) \\ \quad - (\gamma_1 + \mu_1^1) \end{cases}$	$\mathbf{y} - \beta_0 + \alpha_1$	$\mathbf{y} - \beta_1 + \alpha_2$	$-\gamma_1$

Table 1: Initial values polynomials of degrees 1 and 2 respectively

where

$$\lambda = \alpha_1, \; a = \frac{\alpha_2}{\tilde{\delta}_1}, \; b = -\frac{\alpha_2\tilde{\beta}_0}{\tilde{\delta}_1}. \tag{3.11}$$

Proof. From Proposition 1,

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \frac{\alpha_2}{\tilde{\delta}_1} Q_1(x) & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix},$$

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \frac{\alpha_2}{\tilde{\delta}_1} \left(x - \tilde{\beta}_0\right) & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix},$$

and, therefore, (3.8) and (3.9) hold. □

3.1. A MATRIX INTERPRETATION

Now, we present a matrix interpretation of our problem in terms of monic banded matrices associated with 2-MOPS $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. Let consider two families of 2-MOPS $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with respect to the regular vector functionals U and V, and let denote

$$\mathbf{P}_n = (P_0, P_1, ..., P_n)^T,$$
$$\mathbf{Q}_n = (Q_0, Q_1, ..., Q_n)^T.$$

The corresponding recurrence relations read as

$$x\mathbf{P}_n = \mathbf{H}_n\mathbf{P}_n + P_{n+1}e_n, \tag{3.12}$$

$$x\mathbf{Q}_n = \tilde{\mathbf{H}}_n\mathbf{Q}_n + Q_{n+1}e_n. \tag{3.13}$$

Here $e_n = (0,0,...,0,1)^T \in \mathbb{R}^{n+1}$, $\mathbf{H}_n$ and $\tilde{\mathbf{H}}_n$ are the Hessenberg banded matrices associated with these recurrence relations, i.e.

$$\mathbf{H}_n = \begin{pmatrix} \beta_0 & 1 & 0 & . & . & 0 \\ \gamma_1 & \beta_1 & 1 & . & . & 0 \\ \delta_1 & \gamma_2 & \beta_2 & . & . & . \\ 0 & \delta_2 & \gamma_3 & . & . & 0 \\ . & . & . & . & . & 1 \\ 0 & 0 & . & . & & \beta_n \end{pmatrix}, \quad \tilde{\mathbf{H}}_n = \begin{pmatrix} \tilde{\beta}_0 & 1 & 0 & . & . & 0 \\ \tilde{\gamma}_1 & \tilde{\beta}_1 & 1 & . & . & 0 \\ \tilde{\delta}_1 & \tilde{\gamma}_2 & \tilde{\beta}_2 & . & . & . \\ 0 & \tilde{\delta}_2 & \tilde{\gamma}_3 & . & . & 0 \\ . & . & . & . & . & 1 \\ 0 & 0 & . & . & & \tilde{\beta}_n \end{pmatrix}.$$

On the other hand,

$$\mathbf{Q}_n = \wedge_n\mathbf{P}_n, \tag{3.14}$$

where

$$\wedge_n = \begin{pmatrix} 1 & 0 & 0 & . & 0 & 0 \\ \alpha_1 & 1 & 0 & . & 0 & 0 \\ 0 & \alpha_2 & 1 & . & 0 & 0 \\ 0 & . & \alpha_3 & . & 0 & 0 \\ . & . & . & . & 1 & 0 \\ 0 & 0 & . & . & \alpha_n & 1 \end{pmatrix}$$

is a lower bi-diagonal matrix.

Replacing the last expression in (3.12)

$$\begin{aligned} x \wedge_n \mathbf{P}_n &= \tilde{\mathbf{H}}_n \wedge_n \mathbf{P}_n + (P_{n+1} + \alpha_{n+1}P_n)\, e_n \\ &= \tilde{\mathbf{H}}_n \wedge_n \mathbf{P}_n + diag\,(0,0,...,\alpha_{n+1})\,\mathbf{P}_n + P_{n+1}e_n \\ &= \left[\tilde{\mathbf{H}}_n \wedge_n + diag\,(0,0,...,\alpha_{n+1})\right]\mathbf{P}_n + P_{n+1}e_n. \end{aligned}$$

The comparison with (3.13) yields

$$\mathbf{H}_n = \wedge_n^{-1}\left[\tilde{\mathbf{H}}_n \wedge_n + diag\,(0,0,...,\alpha_{n+1})\right].$$

In other words

$$\wedge_n\mathbf{H}_n = \tilde{\mathbf{H}}_n \wedge_n + diag\,(0,0,...,\alpha_{n+1})\,,$$

$$\wedge_n \mathbf{H}_n \wedge_n^{-1} = \tilde{\mathbf{H}}_n + diag\,(0, 0, ..., \alpha_{n+1})\wedge_n^{-1},$$

$$\begin{aligned} \tilde{\mathbf{H}}_n &= \wedge_n \left[\mathbf{H}_n - diag\,(0, 0, ..., \alpha_{n+1})\right]\wedge_n^{-1} \\ &= \wedge_n \left[\mathbf{H}_n - \alpha_{n+1} e_n^t e_n\right]\wedge_n^{-1}. \end{aligned}$$

Thus the matrix $\tilde{\mathbf{H}}_n$ is similar to a rank one perturbation of the matrix $\mathbf{H}_n$.

4. APPLICATIONS

In this section, we give several examples showing the generation and decomposition of such 2-orthogonal polynomials sequences where the sequence $\{\alpha_n\}_n$ is given by $\alpha_{n+1} = p_1(n)$, $\alpha_{n+1} = \dfrac{p_1(n)}{q_1(n)}$ or $\alpha_{n+1} = p_1(n)\,q_1(n)$, (where $\deg p_1(n) = \deg q_1(n) = 1$). Furthermore some of this sequences can be repeat the process (generation/decomposition) more than one time (Finitely or not) where we illustrate graphically this process for two cases of Sheffer-Meixner type.

4.1. 2-MOPS OF SHEFFER-MEIXNER TYPE

We recall that the sequences of Sheffer-Meixner type are defined by the generating function

$$G(x,\ t) = A(t)e^{xH(t)} = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!},$$

where

$$A(t) = \sum_{n\geq 0} a_n t^n \qquad \text{and} \qquad H(t) = \sum_{n\geq\,1} h_n t^n,$$

$$\text{with} \qquad A(0) = 1,\ H(0) = 0 \qquad \text{and} \qquad H'(0) = 1.$$

When $d = 2$, in [6] the authors proved that $A(t)$ and $H(\mathrm{t})$ satisfy

$$\begin{cases} H'(t) = \dfrac{1}{(1-\alpha t)(1-\beta t)(1-\gamma t)}, & \alpha,\ \beta,\ \gamma \in \mathbb{C}, \\ \dfrac{A'(t)}{A(t)} = \dfrac{\sigma_0 + \sigma_1 t + \sigma_2 t^2}{(1-\alpha t)(1-\beta t)(1-\gamma t)}, & \sigma_0,\ \sigma_1,\ \sigma_2 \in \mathbb{C},\ \sigma_2 \neq 0, \end{cases}$$

as well as this family consist of 9 sequences polynomials, moreover the sequence $\{S_n\}_{n\geq 0}$ satisfy a four term recurrence relation

$$\begin{aligned} S_{n+3}(x) = & [(x-\sigma_0+(n+2)(\alpha+\beta+\gamma)]S_{n+2}(x) \\ & -(n+2)[\sigma_1+(n+1)(\alpha\beta+\alpha\gamma+\beta\gamma)]S_{n+1}(x) \\ & -(n+1)(n+2)(\sigma_2-n\alpha\beta\gamma)S_n(x),\ n\geq 0. \end{aligned} \tag{4.1}$$

Also, we recall that the sequences of 2-MOPS $\{m_n(x)\}_{n\geq 0}$ and $\{M_n^{\alpha-\beta}(x)\}_{n\geq 0}$ defined by

$$m_n(x) = \frac{DS_{n+1}(x)}{n+1}, n\geq 0, \quad \text{and} \quad M_n^{\omega}(x) = \frac{\triangle_\omega S_{n+1}(x)}{n+1};\ n\geq 0,$$

with $\alpha_{n+1} = -(n+1)\,\alpha$ satisfy (see [5])

$$S_{n+1}(x) = m_{n+1}(x) - (n+1)\alpha m_n(x),\ n\geq 0, \tag{4.2}$$

$$S_{n+1}(x) = M_{n+1}^{\alpha-\beta}(x) - (n+1)\alpha M_n^{\alpha-\beta}(x),\ n\geq 0. \tag{4.3}$$

Here D and $\triangle_\omega$ denote the derivative and the difference operator, respectively.

Thus (**R**.**2**) holds for the sequence $\{S_n\}_{n\geq 0}$ in the special case B, (see Table 2 below). Now, we generalize this result for all cases in a more general context. In the both side of the sequences $\{P_n\}_{n\geq 0}$ or $\{Q_n\}_{n\geq 0}$ we can derive the sequence $\{\alpha_n\}_{n\geq 0}$ which satisfy (**R**.**1**) or (**R**.**2**)

Theorem 4. *Every 2-MOPS of Sheffer-Meixner type $\{S_n\}_{n\geq 0}$ is generated and decomposed for $\alpha_{n+1} = -k(n+1)$, where $k = \alpha$ or β or γ.*

Proof. Each case of 2-MOPS of Sheffer-Meixner type $\{S_n\}_{n\geq 0}$ (defined by 4.1) satisfies relations (**R**.**1**) and (**R**.**2**); where the sequence $\{\alpha_n\}_{n\geq 0}$, the generated and decomposed sequences are given in Table 2. □

Remark 3.

1. The decomposed/generated sequences are of the same type.
2. In both relations (**R**.**1**) and (**R**.**2**), for the case A, when $Q_{n+1}(x) = P_{n+1}(x)$, the sequence $\{Q_n\}_{n\geq 0}$ is a 2-MOPS of Hermite type, while the three sequences in B, C and D are the co-recursive 2-MOPS of Laguerre, Poisson-Charlier, and Euler type. Notice that the other sequences are new in the literature. We will call them Sheffer-Meixner type 2-MOPS with **perturbed recurrence coefficients.**

Case	$\alpha_{n+1} = -k\,(n+1)$	Denomination of $\{Q_n\}_{n>0}$	Denomination of $\{P_n\}_{n>0}$
A	$k = 0$	2-MOPS of Hermite type	2-MOPS of Hermite type
B	$k = \alpha = \beta$ and $\gamma = 0$	Corecursive 2-MOPS of Laguere type $\mu_0 = k,\ \mu_1 = -k,\ \ \mu_1^1 = -k^2$	Corecursive 2-MOPS of Laguere type $\mu_0 = -k,\ \mu_1 = k,\ \mu_1^1 = k^2$
C	$k = \alpha$ and $\beta = \gamma = 0$	Corecursive 2-MOPS of Poisson-charlier type $\mu_0 = k,\ \mu_1 = -k,\ \ \mu_1^1 = 0$	Corecursive 2-MOPS of Poisson-charlier type $\mu_0 = -k,\ \mu_1 = k,\ \ \mu_1^1 = 0$
D	$k = \alpha,$ or $k = \beta$ and $\gamma = 0$	Corecursive 2-MOPS of Euler type $\mu_0 = k,\ \mu_1 = -k$ $\mu_1^1 = -k\,(\alpha + \beta - k)$	Corecursive 2-MOPS of Euler type $\mu_0 = -k,\ \mu_1 = k$ $\mu_1^1 = k\,(\alpha + \beta - k)$
E	$k = \alpha = \beta = \gamma$	2-MOPS of Sheff er-Meixner type $\left\{S_n^{(k,k,k)}\right\}_{n\geq 0}$	2-MOPS of Sheffer-Meixner type $\left\{S_n^{(k,k,k)}\right\}_{n\geq 0}$
F	$k = \alpha = \beta,$ or $k = \alpha = \gamma,$ or $k = \beta = \gamma$	2-MOPS of Sheffer-Meixner type $\left\{S_n^{(k,k,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_{nn\geq 0}^{(k,\beta,k)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,k,k)}\right\}_{n>0}$	2-MOPS of Sheffer-Meixner type $\left\{S_n^{(k,k,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_{nn\geq 0}^{(k,\beta,k)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,k,k)}\right\}_{n>0}$
G	$k = \alpha,$ or $k = \beta,$ or $k = \gamma$	2-MOPS of Sheffer-Meixner type $\left\{S_n^{(k,\beta,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,k,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,\beta,k)}\right\}_{n\geq 0}$	2-MOPS of Sheffer-Meixner type $\left\{S_n^{(k,\beta,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,k,\gamma)}\right\}_{n\geq 0}$, or $\left\{S_n^{(\alpha,\beta,k)}\right\}_{n\geq 0}$.

Table 2: Generated/decomposed 2-MOPS of Sheffer-Meixner type.

3. For $\gamma = 0$, $\{S_n^{(\alpha,\beta,0)}\}_{n\geq 0}$ becomes the co-recursive sequence $\{S_n^*\}_{n\geq 0}$.

We can state also the relation between the recurrence coefficients for generated and decomposed sequence; thus according to proposition 2 we have

Corollary 4. *Let $\{S_n\}_{n\geq 0}$ be 2-MOPS of Sheffer-Meixner type defined as above, when $\alpha_n = -(n+1)\,k$ then (3.1), (3.2) and (3.3) are written respec-*

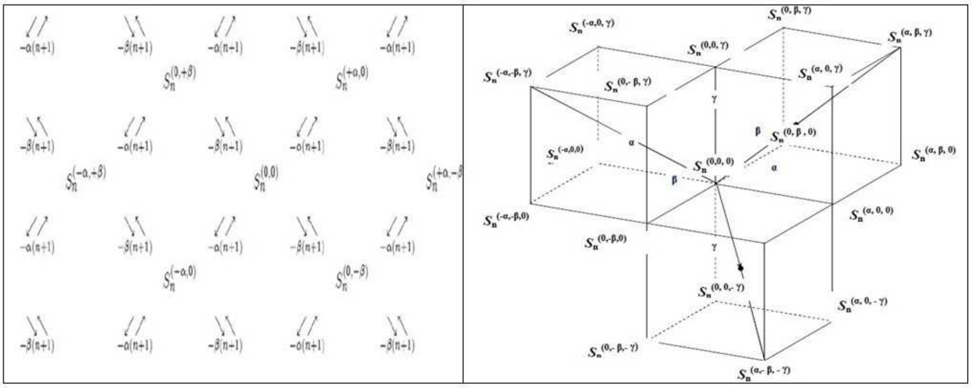

Figure 1: Infinitely generation and decomposition 2-MOPS of Sheffer-Meixner type for case D and G.

tively as

$$\begin{cases} \tilde{\beta}_{n+1} - \beta_n = k, \\ \tilde{\gamma}_{n+1} - \gamma_{n+1} = -(n+1)(\alpha+\beta+\gamma-k)\,k, \\ \tilde{\delta}_{n+1} - \delta_{n+1} = (n+1)(n+2)\left[\alpha\beta+\alpha\gamma+\beta\gamma-(\alpha+\beta+\gamma-k)\,k\right]k. \end{cases}$$

Now, let us express in both cases D and G the process of generation/decomposition repeated more than one time; where we illustrate with the following scheme the way in which this sequence $\{S_n\}_{n\geq 0}$ is generated or decomposed.

Here $\nearrow$ (respectively $\rightarrow$) denotes the generation process and $\searrow$ (respectively $\leftarrow$) denotes the decomposition process.

4.2. 2-MOPS OF GEGENBAUER TYPE

Theorem 5. *The* 2*-MOPS of Gegenbauer type introduced in [4] is generated (resp. decomposed) for* $\alpha_{n+1} = \dfrac{n+1}{n+\alpha}k$ *(resp.* $\alpha_{n+1} = \dfrac{n+1}{n+1+\alpha}k$*).*

Proof. Let $\{P_n\}_{n\geq 0}$ be the 2-MOPS of Gegenbauer type, with recurrence coefficients

$$\begin{cases} \beta_{n+1} = \beta_0, \\ \gamma_{n+2} = \dfrac{(n+2)(n+1+2\alpha)}{(n+1+\alpha)(n+2+\alpha)}\gamma, \ n \geq 0, \\ \delta_{n+1} = \dfrac{(n+1)(n+2)(n+3\alpha)}{(n+\alpha)(n+1+\alpha)(n+2+\alpha)}\delta_2, \ n \geq 0. \end{cases}$$

If we choose

$$\alpha_{n+1} = \frac{n+1}{n+\alpha}k,$$

Then we have

$$\left(\frac{\delta_{n+1}}{\alpha_{n+2}\alpha_{n+1}} - \frac{\delta_n}{\alpha_n\alpha_{n+1}}\right) - \left(\frac{\gamma_{n+2}}{\alpha_{n+2}} - \frac{\gamma_{n+1}}{\alpha_{n+1}}\right) - (\alpha_{n+3} - \alpha_{n+2})$$
$$= \frac{1}{(n+2+\alpha)(n+1+\alpha)}\left[\frac{\delta}{k^2}\left(-2\left(\alpha-1\right)\right) - \frac{\gamma}{k}\left(1-\alpha\right) - \left(\alpha-1\right)k\right] = 0,$$

i.e.,

$$-\frac{2\delta}{k^2}\left(\alpha-1\right) + \frac{\gamma}{k}\left(\alpha-1\right) - \left(\alpha-1\right)k = 0,$$

or

$$\begin{aligned} \frac{2\delta}{k^2} - \frac{\gamma}{k} + k &= 0 \Rightarrow 2\delta - \gamma k + k^3 = 0 \Rightarrow 2\delta - \gamma k + k^3 = 0 \\ &\Rightarrow \delta = \frac{k}{2}\left(\gamma - k^2\right). \end{aligned}$$

Thus relation (**R**.**1**) holds. If $\gamma \neq k^2$ you get the regularity condition. In a similar way we can prove that the same sequence satisfies (**R**.**2**) and we have for

$$\alpha_{n+1} = \frac{n+1}{n+1+\alpha}k,$$

$$\frac{\tilde{\delta}_{n+2}}{\alpha_{n+3}\alpha_{n+2}} - \frac{\tilde{\delta}_{n+1}}{\alpha_{n+2}\alpha_{n+1}} - \left(\frac{\tilde{\gamma}_{n+2}}{\alpha_{n+2}} - \frac{\tilde{\gamma}_{n+1}}{\alpha_{n+1}}\right) - (\alpha_{n+1} - \alpha_n)$$
$$= \frac{1}{(n+\alpha)(n+1+\alpha)}\left[-\frac{2\alpha\delta_2}{k^2} + \frac{\alpha\gamma}{k} - \alpha k\right] = 0.$$

This means that

$$-\frac{2\delta_2}{k^2} + \frac{\gamma}{k} - k = 0 \Leftrightarrow -2\delta_2 + \gamma k - k^3 = 0 \Leftrightarrow \delta_2 = \frac{k}{2}\left(\gamma - k^2\right).$$

Notice that the regularity condition holds if $\gamma \neq k^2$. □

Remark 4. **1** The 2-MOPS of Gegenbauer type satisfies $(R.1)$ (resp. $(R.2)$), whenever $\{Q_n\}_{n\geq 0}$ (resp. $\{P_n\}_{n\geq 0}$) is not of the same type $\tilde{\beta}_n \neq 0$, (resp. $\beta_n \neq 0$).

2 The sequence of Gegenbauer type repeat the process of generation/ decomposed for finite number of time.

Remark 5. For $\alpha = 1, \beta_0 = 0$, we get the 2-MOPS of Chebyshev polynomials of the second kind

$$\begin{cases} \beta_{n+1} = 0, \\ \gamma_{n+2} = \gamma,\ n \geq 0, \\ \delta_{n+1} = \delta_2,\ n \geq 0, \end{cases}$$

such that (**R.1**) and (**R.2**) hold for $\alpha_{n+1} = \alpha$.

Indeed, $\{P_n\}_{n\geq 0}$ is 2-symmetric and $\alpha_n = \alpha \neq 0$, but the sequence $\{Q_n\}_{n\geq 0}$ is not a 2-symmetric 2-MOPS. In fact, we have

$$\begin{aligned} xQ_n(x) &= Q_{n+1}(x) + \delta_1 Q_{n-2}(x), n \geq 2, \\ Q_0(x) &= 1, \\ Q_1(x) &= x + \alpha, \\ Q_2(x) &= x\,(x+\alpha)\,. \end{aligned}$$

Another particular case cited in [7] (Lemma 4.8) and before in [10] corresponds to the recurrence relation satisfied by the Bateman function $J_n^{u,v}$ [15]. Indeed,

$$\begin{cases} \beta_{n+1} = 3(n+1)^2 + (n+1)\,(2\alpha + 2\beta + 3) + (1+\alpha)\,(1+\beta)\,, n \geq 0, \\ \gamma_{n+1} = (n+1)\,(\alpha + \beta + 3\,(n+1))\,(n+1+\alpha)\,(n+1+\beta)\,,\ n \geq 0, \\ \delta_{n+1} = (n+1)(n+2)\,(n+1+\alpha)\,(n+2+\alpha)\,(n+1+\beta)\,(n+2+\beta)\,, n \geq 0. \end{cases}$$

Proposition 4. The 2-MOPS cited above is generated and decomposed for $\alpha_{n+1} = (n+1+\alpha)\,(n+1+\beta)$.

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REFERENCES

[1] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, On linearly related orthogonal polynomials and their functionals, *J. Math. Anal. Appl.*, **287** (2003), 307-319.

[2] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? *J. Comput. Appl. Math.*, **233** (2010), 1446-1452.

[3] M. Alfaro, A. Peña, M.L. Rezola, F. Marcellán, Orthogonal polynomials associated with an inverse quadratic spectral transform, *Comput. Math. Appl.*, **61** (2011), 888-900.

[4] A. Boukhemis, A study of a sequence of classical orthogonal polynomials of dimension 2, *J. Approx. Theory*, **90** (1997) 435-454.

[5] A. Boukhemis, On the classical 2-orthogonal polynomials sequences of Sheffer-Meixner type, *Math. J. Universidad de La Frontera*, **7**, No. 2 (2005), 39 pp.

[6] A. Boukhemis, P. Maroni, Une caractérisation des polynômes strictement $1/p$ orthogonaux de type Scheffer. Étude du cas $p = 2$, *J. Approx. Theory*, **54** (1988) 67-91.

[7] A. Boukhemis, E. Zerouki, Classical 2-orthogonal polynomials and differential equations, *Int. J. Math. and Math. Sci.* (2006), Art. ID 12640, 32 pp.

[8] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, New York, Gordon and Breach, 1978.

[9] K. Douak, P. Maroni, On d-orthogonal Tchebychev polynomials, *I Appl. Numer. Math.*, **24** (1997), 23-53.

[10] K. Douak, Y. Ben Cheikh, On two-orthogonal polynomials related to the Bateman's function, *Methods Appl. Anal.*, **7** (2000), 641-662.

[11] P. Maroni, L'orthogonalité et les récurrences de polynômes d'ordre supérieur à deux, *Ann. Fac. Sci. Toulouse*, **10** (1989), 105-139.

[12] P. Maroni, Sur la suite de polynômes orthogonaux associeé à la forme $u = \delta_c + \lambda(x-c)^{-1}L$, *Period Math Hungar*, **21** (1990), 223-248.

[13] P. Maroni, Two-dimensional orthogonal polynomials, their associated sets and the co-recursive sets, *Numer Algorithms*, **3** (1992), 299-311.

[14] P. Maroni, Semi-classical character and finite-type relations between polynomial sequences, *Appl. Numer. Math.*, **31** (1999), 295-330.

[15] E.D. Rainville, Special Functions, New York, The Macmillan Co 1, 1960.

[16] A. Saib, E. Zerouki, Some inverse problems for d-orthogonal polynomials, *Mediterr. J. Math.*, **10** (2013), 865-885.

[17] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, *J. Comput. Appl. Math.*, **85** (1997), 67-86.