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EXISTENCE AND UNIQUENESS OF SOLUTION OF CAUCHY-TYPE PROBLEM FOR HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT: The Cauchy-type problem for a nonlinear differential equation involving Hilfer (generalized Riemann-Liouville) fractional derivative is considered. The equivalence between this Cauchy-type problem and a nonlinear Volterra integral equation in the space of weighted continuous functions is established. Using this result, existence, uniqueness and continuous dependence of solution for Cauchy-type problem are obtained by using successive approximations and the generalized Gronwall inequality.

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1. INTRODUCTION

Nowadays, the fractional differential equations have received much attention as they have emerged in various applications of science and engineering from the study of exact description of nonlinear phenomena. It has been found that the models using mathematical tools from fractional calculus describe various complex phenomena such as control, viscoelasticity, electrochemistry, porous media, and many other branches of science. For detailed account on the applications, see [4, 5, 6, 9, 14, 15] and references therein. However the investigation of basic theory of fractional differential equations involves the existence and uniqueness of solutions on the finite interval $[a, b]$. The existence and uniqueness of solution of fractional differential equations under different conditions are studied in [1]-[3],[8]-[15].

Recently, in [3], Furati et.al. obtained the existence and uniqueness of the initial value problem (IVP)

$$D_{a^+}^{\alpha,\beta} y(x) = f(x, y), \qquad 0 < \alpha < 1, \, 0 \leq \beta \leq 1 \tag{1.1}$$

$$I_{a^+}^{1-\gamma} y(a) = y_a, \qquad \gamma = \alpha + \beta(1 - \alpha) \tag{1.2}$$

by using Banach fixed point theorem and with the help of equivalent Volterra integral equation of the second kind

$$y(x) = \frac{y_a}{\Gamma(\gamma)}(x - a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t, y(t))dt. \tag{1.3}$$

In this paper, we study the existence, uniqueness and continuous dependence of more general problem by Picard's successive approximations. Clearly, the problem is well posed and investigation covers the qualitative properties of Cauchy-type problem associated to Hilfer fractional differential equations.

The rest of the paper is organized as follows: in Section 2, some preliminary results and notations are provided. The existence and uniqueness results are proved in Section 3. The dependence of solutions on order and initial conditions is studied in the last section.

2. PRELIMINARIES

In this section, we present some definitions and weighted spaces which are needed in subsequent sections.

Let $-\infty < a < b < \infty$. Let $C[a, b], AC[a, b]$ and $C^n[a, b]$ be the spaces of continuous, absolutely continuous and continuously differentiable functions on $[a, b]$, respectively. We denote $L^p(a, b), p \geq 1$, the space of Lebesgue integrable functions on (a, b). Further more we recall some definitions of following weighted spaces [9]:

$$\begin{aligned} C_\gamma[a,b] &= \{f:(a,b]\to\mathbb{R}:(x-a)^\gamma f(x)\in C[a,b]\},\\ C_\gamma^n[a,b] &= \{f\in C^{n-1}[a,b]:f^{(n)}(x)\in C_\gamma[a,b]\},\ n\in\mathbb{N},\\ C_{n-\gamma}^{\alpha,\beta}[a,b] &= \{f\in C_{n-\gamma}[a,b]:D_{a^+}^{\alpha,\beta}f(x)\in C_{n-\gamma}[a,b]\}. \end{aligned} \tag{2.1}$$

Note that, $D_{a^+}^{\alpha,\beta}f = I_{a^+}^{\beta(1-\alpha)}D_{a^+}^{\gamma}f$ and $C_{n-\gamma}^{\gamma}[a,b]\subset C_{n-\gamma}^{\alpha,\beta}[a,b]$, $\gamma=\alpha+n\beta-\alpha\beta$, $n-1<\alpha<n, 0\le\beta\le1$, for details see [3]. Consider the space $C_\gamma^0[a,b]$ with the norm

$$\|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1}\|f^{(k)}\|_C + \|f^{(n)}\|_{C_\gamma}. \tag{2.2}$$

Definition 2.1. [9] Let $f\in L^1(a,b)$. Then Riemann-Liouville fractional integral of order α of function f is defined as

$$I_{a^+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a,\ \alpha>0, \tag{2.3}$$

where Γ is the Euler's Gamma function.

Definition 2.2. [9] The left-sided Riemann-Liouville fractional derivative of order $\alpha, n-1<\alpha<n$, of function $f\in L^1(a,b)$ is expressed as

$$D_{a^+}^{\alpha}f(x) = D^n I_{a^+}^{n-\alpha}f(x), \qquad D^n = \frac{d^n}{dx^n}. \tag{2.4}$$

Definition 2.3. [5] The left-sided Hilfer fractional derivative of order α, and type β, $(n-1<\alpha<n, 0\le\beta\le1)$, of a function $f\in L^1(a,b)$ is defined as

$$D_{a^+}^{\alpha,\beta}f(x) = I_{a^+}^{\beta(n-\alpha)}D^n I_{a^+}^{(1-\beta)(n-\alpha)}f(x), \quad n-1<(1-\beta)(n-\alpha)<n. \tag{2.5}$$

Definition 2.4. Let $G\subset\mathbb{R}$ and $f:(a,b]\times G\to\mathbb{R}$ satisfies Lipschitz condition

$$|f(x,y_1)-f(x,y_2)|\le A|y_1-y_2|, \tag{2.6}$$

for all $x\in(a,b]$ and for any $y_1,y_2\in G$, where $A>0$ does not depend on $x\in(a,b]$ is Lipschitz constant.

For the power functions we have the following lemma.

Lemma 2.1. *[9] For $\alpha\ge0,\beta>0,x>a$, we have*

(i) $I^{\alpha}_{a+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a).^{\beta+\alpha-1}$

(ii) $D^{\alpha}_{a+}(x-a)^{\alpha-k} = 0, \quad \alpha \in (n-1, n), k = 1, 2, .., n.$

Lemma 2.2. *[3] Let $\alpha > 0$ and $0 \leq \gamma < 1$, then I^{α}_{a+} is bounded from $C_\gamma[a,b]$ into $C_\gamma[a,b]$. In addition, if $\gamma \leq \alpha$, then I^{α}_{a+} is bounded from $C_\gamma[a,b]$ into $C[a,b]$.*

Lemma 2.3. *[3] Let $0 < \alpha < 1, 0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$ and $I^{a^+}_{1-\alpha}f \in C^1_\gamma[a,b]$,*

$$\textit{then} \qquad I^{\alpha}_{a+}D^{\alpha,\beta}_{a^+}f(x) = f(x) - \frac{I^{1-\alpha}_{a^+}f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} \qquad \textit{for all } x \in (a,b].$$

3. EXISTENCE AND UNIQUENESS RESULTS

Here we prove the result which generalizes the result of ([3], Theorem 23).

Corollary 3.1. *Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(.,y(.)) \in C_{n-\gamma}[a,b]$ for any $y \in C_{n-\gamma}[a,b]$. If $y \in C^{\gamma}_{n-\gamma}[a,b]$, then y satisfies the Cauchy-type problem*

$$D^{\alpha,\beta}_{a^+}y(x) = f(x,y), \qquad n-1 < \alpha < n,\ 0 \leq \beta \leq 1 \tag{3.1}$$

$$I^{n-\gamma}_{a^+}y(a) = b_k, \qquad k = 1, 2, .., n = -[-\alpha],\ \gamma = \alpha + \beta(n-\alpha) \tag{3.2}$$

if and only if, y satisfies the Volterra integral equation of the second kind

$$y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma-k+1)}(x-a)^{\gamma-k} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t,y(t))dt. \tag{3.3}$$

Proof: To prove the necessary condition, let $y \in C^{\gamma}_{n-\gamma}[a,b]$ be a solution of Cauchy-type problem (3.1)-(3.2). Since $f(x,y(x)) \in C_{n-\gamma}[a,b]$ for any $y \in C_{n-\gamma}[a,b]$, by Definition 2.2 and Lemma 2.2, we obtain

$$I^{n-\gamma}_{a^+}y \in C[a,b] \text{ and } D^{\gamma}_{a^+}y = D^n I^{n-\gamma}_{a^+}y \in C_{n-\gamma}[a,b].$$

Then clearly, we have $I_{a^+}^{n-\gamma}y \in C_{n-\gamma}^n[a,b]$. Applying $I_{a^+}^{\alpha}$ on both sides of equation (3.1), we obtain

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} y(x) = I_{a^+}^{\gamma} D_{a^+}^{\gamma} y(x) = I_{a^+}^{\alpha} f(x, y(x))$$
$$y(x) - \sum_{k=1}^{n} \frac{I_{a^+}^{k-\gamma} y(a)}{\Gamma(\gamma - k + 1)} (x-a)^{\gamma-k} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y(t)) dt$$

Using initial conditions (3.2), we get the required integral equation (3.3).

Now to prove sufficiency, let $y \in C_{n-\gamma}^{\gamma}[a,b]$ satisfy integral equation (3.3). Applying $D_{a^+}^{\alpha,\beta}$ to both sides of integral equation (3.3), we get

$$D_{a^+}^{\alpha,\beta} y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma - k + 1)} D_{a^+}^{\alpha,\beta} (x-a)^{\gamma-k} + D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f(x, y(x)),$$

From Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} D_{a^+}^{\alpha,\beta} y(x) =& \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma - k + 1)} I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\gamma} (x-a)^{\gamma-k} \\ &+ I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\beta(n-\alpha)} f(x, y(x)) \\ =& I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\beta(n-\alpha)} f(x, y(x)) \\ D_{a^+}^{\alpha,\beta} y(x) =& f(x, y(x)) \\ &- \sum_{k=1}^{n} \frac{I_{a^+}^{k-\beta(n-\alpha)} f(x, y(x))|_{x=a}}{\Gamma(\beta(n-\alpha))} (x-a)^{\beta(n-\alpha)-k} \end{aligned} \tag{3.4}$$

Since $n - \gamma < n - \beta(n-\alpha)$, by Lemma 13 in [3], we have

$$\sum_{k=1}^{n} I_{a^+}^{k-\beta(n-\alpha)} f(x, y(x))|_{x=a} = 0.$$

Thus equation (3.4) yields the relation

$$D_{a^+}^{\alpha,\beta} y(x) = f(x, y).$$

To show that the initial conditions (3.2) holds, apply $I_{a^+}^{n-\gamma}$ to both sides of equation (3.3),

$$I_{a^+}^{n-\gamma} y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma - k + 1)} I_{a^+}^{n-\gamma} (x-a)^{\gamma-k} + I_{a^+}^{n-\gamma} I_{a^+}^{\alpha} f(x, y(x))$$

$$= b_k + \frac{1}{\Gamma(n-\beta(n-\alpha))} \int_a^x (x-t)^{n-\beta(n-\alpha)-1} f(t, y(t))dt. \tag{3.5}$$

Taking limit as $x \to a^+$ in equation (3.5), we obtain

$$I_{a^+}^{n-\gamma} y(a) = b_k$$

This proves sufficiency and the proof is complete.

Now we prove the existence and uniqueness result for Cauchy-type problem (3.1)-(3.2) in the space $C_{n-\gamma}^{\alpha,\beta}[a, b]$.

Theorem 3.1. *Let $f(x, (y)) \in C_{n-\gamma}[a, b]$ satisfies the conditions in Corollary 3.1. Moreover $f(x, y(x))$ satisfies Lipschitz condition* (2.6)*, then there exists a unique solution $y(x)$ for Cauchy-type problem* (3.1)-(3.2) *in the space $C_{n-\gamma}^{\alpha,\beta}[a, b]$.*

Proof: The equation (3.3) makes sense in any interval $[a, x_1] \subset [a, b]$. Choose x_1 such that $y \in C_{n-\gamma}[a, x_1]$,

$$A \frac{(x_1 - a)^\alpha \Gamma(\gamma)}{\Gamma(\alpha + \gamma)} < 1 \tag{3.6}$$

and set

$$y_0(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma - k + 1)} (x-a)^{\gamma - k}, \qquad \gamma = \alpha + n\beta - \alpha\beta \tag{3.7}$$

$$y_m(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y_{m-1}(t))dt, \qquad m \in \mathbb{N}. \tag{3.8}$$

By Lemma 2.2, $y_m(x) \in C_{n-\gamma}[a, b]$. Estimate $\|y_m(x) - y_{m-1}(x)\|_{C_{n-\gamma}}$, $m \in \mathbb{N}$. By equations (3.7) and (3.8), we have

$$\begin{aligned}\|y_1(x) - y_0(x)\|_{C_{n-\gamma}[a,x_1]} &= \|\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y_0(t))dt\|_{C_{n-\gamma}[a,x_1]} \\ &\le \|I_{a^+}^{\alpha}[\max_{x \in (a,x_1]} |f(x, y_0)|]\|_{C_{n-\gamma}[a,x_1]}\end{aligned}$$

$$\|y_1(x) - y_0(x)\|_{C_{n-\gamma}[a,x_1]} \le M \frac{(x_1 - a)^\alpha \Gamma(\gamma)}{\Gamma(\gamma + \alpha)}. \tag{3.9}$$

We deduce that

$$\|y_2(x) - y_1(x)\|_{C_{n-\gamma}[a,x_1]} \le M \frac{(x_1 - a)^\alpha \Gamma(\gamma)}{\Gamma(\gamma + \alpha)} \left(A \frac{(x_1 - a)^\alpha \Gamma(\gamma)}{\Gamma(\gamma + \alpha)} \right). \tag{3.10}$$

Continuing in this way m-times, one obtain

$$\|y_m(x) - y_{m-1}(x)\|_{C_{n-\gamma}[a,x_1]} \leq \frac{M(x_1-a)^\alpha\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{A(x_1-a)^\alpha\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\right)^{m-1} \tag{3.11}$$

By equation (3.8), clearly the sequence $\{y_m(x)\}$ tends to $y(x) \in C_{n-\gamma}[a, x_1]$:

$$\|y_m(x) - y(x)\|_{C_{n-\gamma}[a,x_1]} \to 0, \qquad \text{as} \quad m \to \infty. \tag{3.12}$$

By Lemma 2.2, it follows that

$$\|I^\alpha_{a^+} f(x, y_m(x)) - I^\alpha_{a^+} f(x, y(x))\|_{C_{n-\gamma}[a,x_1]} \leq A\frac{(x_1-a)^\alpha\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\|y_m(x) - y(x)\|_{C_{n-\gamma}[a,x_1]}$$

and hence

$$\|I^\alpha_{a^+} f(x, y_m(x)) - I^\alpha_{a^+} f(x, y(x))\|_{C_{n-\gamma}[a,x_1]} \to 0, \qquad \text{as} \quad m \to \infty. \tag{3.13}$$

From equations (3.12) and (3.13), $y(x)$ is the solution of integral equation (3.3) in the space $C_{n-\gamma}[a, x_1]$.

Now to show that the solution $y(x)$ is unique, consider there exists two solutions $y(x)$ and $z(x)$ of the equation (3.3) on $[a, x_1]$. Substituting them into equation (3.3) and using Lemma 2.2 with Lipschits condition (2.6), we get

$$\begin{aligned}\|y(x) - z(x)\|_{C_{n-\gamma}[a,x_1]} &= \|I^\alpha_{a^+} f(x, y(x)) - I^\alpha_{a^+} f(x, z(x))\|_{C_{n-\gamma}[a,x_1]} \\ &\leq A\frac{(x_1-a)^\alpha\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\|y_m(x) - y(x)\|_{C_{n-\gamma}[a,x_1]}\end{aligned} \tag{3.14}$$

This yields

$$A\frac{(x_1-a)^\alpha\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \geq 1,$$

which contradicts to condition (3.6). Thus there exists $y(x) = y_1(x) \in C_{n-\gamma}[a, x_1]$ as a unique solution on $[a, x_1]$.

Next, consider the interval $[x_1, x_2]$, where $x_2 = x_1 + h_1, h_1 > 0$ such that $x_2 < b$. Rewrite equation (3.3) in the form

$$y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\gamma-k+1)}(x-a)^{\gamma-k} + \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x}(x-t)^{\alpha-1}f(t, y(t))dt$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^{x_1}(x-t)^{\alpha-1}f(t,y(t))dt, \qquad x\in[x_1,x_2]. \tag{3.15}$$

Since the function $y(x)$ is uniquely defined on $[a, x_1]$, the last integral can be considered as the known function and rewrite the equation (3.15) in the form

$$y(x)=y_0^*(x)+\frac{1}{\Gamma(\alpha)}\int_a^{x_1}(x-t)^{\alpha-1}f(t,y(t))dt, \qquad x\in[x_1,x_2]. \tag{3.16}$$

where

$$y_0^*(x)=\sum_{k=1}^{n}\frac{b_k}{\Gamma(\gamma-k+1)}(x-a)^{\gamma-k}+\frac{1}{\Gamma(\alpha)}\int_{x_1}^{x}(x-t)^{\alpha-1}f(t,y(t))dt \tag{3.17}$$

is the known function. Using the same arguments as above, we deduce that there is a unique solution $y(x) = y_2(x) \in C_{n-\gamma}[x_1, x_2]$ on $[x_1, x_2]$. Taking the interval $[x_2, x_3]$, where $x_3 = x_2 + h_2, h_2 > 0$ such that $x_3 < b$, and repeating the above process, we obtain a unique solution $y(x) \in C_{n-\gamma}[a, b]$ of equation (3.3) such that $y(x) = y_j(x) \in C_{n-\gamma}[x_{j-1}, x_j]$, $j = 1, 2, .., l$, and $a = x_1 < x_2 < .. < x_l = b$.

Thus there exists a unique solution $y(x) \in C_{n-\gamma}[a, b]$ of the equation (3.3). In accordance with equations (3.10), (3.1) and Lipschitz condition (2.6), we obtain

$$\begin{aligned}\|D_{a^+}^{\alpha,\beta}y_m(x)-D_{a^+}^{\alpha,\beta}y(x)\|_{C_{n-\gamma}[a,b]} &= \|f(x,y_m(x))-f(x,y(x))\|_{C_{n-\gamma}[a,b]}\\ &\leq A\|y_m(x)-y(x)\|_{C_{n-\gamma}[a,b]}\end{aligned} \tag{3.18}$$

Clearly by (3.12), $D_{a^+}^{\alpha,\beta}y(x) \in C_{n-\gamma}[a, b]$. This completes the proof.

Remark 3.1. For $\beta = 0$, Theorem 3.1 reduces to the Theorem 3.5, in [10] for Cauchy-type problem for Riemann-Liouville fractional differential equation.

Remark 3.2. For $\beta = 1$, Theorem 3.1 reduces to the Theorem 2, in [8] for Cauchy-type problem for Caputo fractional differential equation.

4. CONTINUOUS DEPENDANCE

In this section we study the continuous dependence of solution of Cauchy-type problem for Hilfer fractional differential equation. Using generalized Gronwall inequality as a handy tool, we show that the small change in order and small change in initial conditions cause the small change in solution of Cauchy-type problem for Hilfer fractional differential equation.

Consider the IVP (1.1)-(1.2), for $0 < \alpha < 1, 0 \leq \beta \leq 1$, $a \leq x < b$, $(b \leq +\infty)$ and $f : [a, b) \times \mathbb{R} \to \mathbb{R}$. Here we present the dependence of the solution on the order, for this, let us consider the solutions of two IVPs with the neighbouring orders. To prove the results we need the following theorem.

Theorem 4.1. *[18] Suppose $\beta > 0$, $a(t)$ is nonnegative function locally integrable on $0 \leq t < T$ for some $(T \leq +\infty)$ and $g(t)$ is nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds$$

on the interval. Then

$$u(t) \leq a(t) + \int_0^t \left[\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T.$$

Theorem 4.2. *Let $\alpha > 0, \delta > 0$ such that $0 < \alpha - \delta < \alpha \leq 1$. Let f be a continuous function satisfying Lipschitz condition (2.6) in $\mathbb{R}$. For $a \leq x \leq h < b$, assume that y and $\hat{y}$ are the solutions of IVPs (1.1)-(1.2) and*

$$D_{a^+}^{\alpha-\delta,\beta} \hat{y}(x) = f(x, \hat{y}), \qquad 0 < \alpha < 1, \, 0 \leq \beta \leq 1, \tag{4.1}$$

$$I_{a^+}^{1-\gamma-\delta(\beta-1)} \hat{y}(x)\big|_{x=a} = \hat{y}_a, \qquad \gamma = \alpha + \beta(1-\alpha), \tag{4.2}$$

respectively. Then for $a < x \leq h$,

$$|\hat{y}(x) - y(x)| \leq B(x) + \int_a^x \left[\sum_{n=1}^{\infty} \left(\frac{A}{\Gamma(\alpha)} \Gamma(\alpha-\delta) \right)^n \frac{(x-t)^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))} B(t) \right] dt$$

hold, where

$$\begin{aligned} B(x) = & \left| \frac{\hat{y}_a (x-a)^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma+\delta(\beta-1))} - \frac{y_a (x-a)^{\gamma-1}}{\Gamma(\gamma)} \right| \\ & + \|f\| \left| \frac{(x-a)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} - \frac{(x-a)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} \right| \\ & + \|f\| \left| \frac{(x-a)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right| \end{aligned} \tag{4.3}$$

and

$$\|f\| = \max_{a\leq x\leq h} |f(x, y(x))|.$$

Proof: The equivalent integral solutions of IVP's (1.1)-(1.2) and (4.1)-(4.2) are

$$y(x) = \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t, y(t))dt$$

and

$$\hat{y}(x) = \frac{\hat{y}_a}{\Gamma(\gamma+\delta(\beta-1))}(x-a)^{\gamma+\delta(\beta-1)-1} + \frac{1}{\Gamma(\alpha-\delta)}\int_a^x (x-t)^{\alpha-\delta-1} f(t, \hat{y}(t))dt,$$

respectively. It follows that

$$\begin{aligned}
|\hat{y}(x) - y(x)| \leq& \Bigg|\frac{\hat{y}_a}{\Gamma(\gamma+\delta(\beta-1))}(x-a)^{\gamma+\delta(\beta-1)-1} - \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} \\
&+ \int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f(t, \hat{y}(t))dt - \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt\Bigg| \\
\leq& \left|\frac{\hat{y}_a}{\Gamma(\gamma+\delta(\beta-1))}(x-a)^{\gamma+\delta(\beta-1)-1} - \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}\right| \\
&+ \left|\int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f(t, \hat{y}(t))dt - \int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha)} f(t, \hat{y}(t))dt\right| \\
&+ \left|\int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha)} f(t, \hat{y}(t))dt - \int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha)} f(t, y(t))dt\right| \\
&+ \left|\int_a^x \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt\right| \\
\leq& \left|\frac{\hat{y}_a}{\Gamma(\gamma+\delta(\beta-1))}(x-a)^{\gamma+\delta(\beta-1)-1} - \frac{y_a}{\Gamma(\gamma)}(x-a)^{\gamma-1}\right| \\
&+ \left|\int_a^x \left[\frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} - \frac{(x-t)^{\alpha-\delta-1}}{\Gamma(\alpha)}\right] |f(t, \hat{y}(t))|dt\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-\delta-1} |f(t, \hat{y}(t)) - f(t, y(t))|dt\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_a^x \left[(x-t)^{\alpha-\delta-1} - (x-t)^{\alpha-1}\right] |f(t, y(t))|dt\right| \\
\leq& B(x) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-\delta-1} A|\hat{y}(t) - y(t)|dt,
\end{aligned}$$

where $B(x)$ is defined by (4.3). Applying Theorem 4.1, we obtain

$$|\hat{y}(x) - y(x)| \leq B(x) + \frac{A}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-\delta-1} |\hat{y}(t) - y(t)|dt$$

$$\leq B(x) + \int_a^x \left[\sum_{n=1}^{\infty} \left(\frac{A}{\Gamma(\alpha)} \Gamma(\alpha - \delta) \right)^n \frac{(x-t)^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))} B(t) \right] dt,$$

this proves the theorem.

Next, we consider the fractional differential equation (3.1) with the small changes in the initial conditions (3.2),

$$I_{a^+}^{n-\gamma} y(x)|_{x=a} = b_k + \epsilon_k, \quad \gamma = \alpha + \beta(n - \alpha), \tag{4.4}$$

where $\epsilon_k, k = 1, 2, .., n$, are arbitrary constants. We prove the results as follows.

Theorem 4.3. *Let the assumptions of Theorem 3.1 hold. If $y(x)$ is a solution of IVP (3.1)-(3.2) and $\hat{y}(x)$ is a solution of IVP (3.1)-(4.4). Then for $x \in (a, b]$, the following relation holds:*

$$|y(x) - \hat{y}(x)| \leq \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} E_{\alpha,\gamma-k+1}(A(x-a)^{\alpha}), \tag{4.5}$$

where $E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu+\nu)}$ is the Mittag-Leffler function (see [7, 16, 17]).

Proof: In accordance with Theorem 3.1, we have $y(x) = \lim_{m\to\infty} y_m(x)$, with $y_0(x)$ and $y_m(x)$ are as defined in (3.7) and (3.8), respectively. Clearly, we can write $\hat{y}(x) = \lim_{m\to\infty} \hat{y}_m(x)$, and

$$\hat{y}_0(x) = \sum_{k=1}^{n} \frac{(b_k + \epsilon_k)(x-a)^{\gamma-k}}{\Gamma(\gamma - k + 1)}, \tag{4.6}$$

$$\hat{y}_m(x) = \hat{y}_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, \hat{y}_{m-1}(t)) dt. \tag{4.7}$$

From (3.7) and (4.6), directly we have

$$\left| y_0(x) - \hat{y}_0(x) \right| = \left| \sum_{k=1}^{n} \frac{b_k (x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} - \sum_{k=1}^{n} \frac{(b_k + \epsilon_k)(x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} \right|$$

$$\left| y_0(x) - \hat{y}_0(x) \right| \leq \sum_{k=1}^{n} |\epsilon_k| \frac{(x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)}. \tag{4.8}$$

By the subsequent relations (3.8) and (4.7), the Lipschitz condition (2.6) and the inequality (4.8), we obtain

$$\left| y_1(x) - \hat{y}_1(x) \right| = \left| \sum_{k=1}^{n} \frac{\epsilon_k (x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} [f(t, y_0(t)) - f(t, \hat{y}_0(t))] dt \right|$$

$$\leq \sum_{k=1}^{n} |\epsilon_k| \frac{(x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} + \frac{A}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |y_0(t) - \hat{y}_0(t)| dt$$

$$\leq \sum_{k=1}^{n} |\epsilon_k| \frac{(x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} + A \sum_{k=1}^{n} |\epsilon_k| \frac{(x-a)^{\alpha+\gamma-k}}{\Gamma(\alpha+\gamma-k+1)}$$

$$\big|y_1(x) - \hat{y}_1(x)\big| \leq \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} \sum_{j=0}^{1} \frac{A^j (x-a)^{\alpha j}}{\Gamma(\alpha j + \gamma - k + 1)}. \tag{4.9}$$

Similarly, by using (4.9), it directly follows that

$$\big|y_2(x) - \hat{y}_2(x)\big| \leq \sum_{k=1}^{n} |\epsilon_k| \frac{(x-a)^{\gamma-k}}{\Gamma(\gamma-k+1)} + \frac{A}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |y_1(t) - \hat{y}_1(t)| dt$$

$$\leq \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} \sum_{j=0}^{2} \frac{A^j (x-a)^{\alpha j}}{\Gamma(\alpha j + \gamma - k + 1)}.$$

By the induction, we obtain

$$\big|y_m(x) - \hat{y}_m(x)\big| \leq \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} \sum_{j=0}^{m} \frac{A^j (x-a)^{\alpha j}}{\Gamma(\alpha j + \gamma - k + 1)}. \tag{4.10}$$

Taking limit as $m \to \infty$ in (4.10), we have

$$\big|y(x) - \hat{y}(x)\big| \leq \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} \sum_{j=0}^{m} \frac{A^j (x-a)^{\alpha j}}{\Gamma(\alpha j + \gamma - k + 1)}$$

$$= \sum_{k=1}^{n} |\epsilon_k| (x-a)^{\gamma-k} E_{\alpha, \gamma-k+1}(A(x-a)^{\alpha}),$$

which completes the proof of Theorem 4.3.

Remark 4.1. It follows from Theorem 4.2 and 4.3 that, small change in order and initial conditions (3.2) cause only small change in the solution $[l, b]$ for l between a to b which does not contain initial point a. On the other hand,the solution may change significantly in the interval $[a, l]$.

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