ON SOME CLASS OF VARIATIONAL INEQUALITIES AND THEIR REGULARITY PROPERTIES

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ABSTRACT: In this paper we develop a regularity theory for variational inequalities of nonlocal type. Using the Lagrange multipliers idea the solution of the variational inequality is shown to be also solution of an equation.

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1. INTRODUCTION

Let Ω be a bounded domain in \( \mathbb{R}^n \), \( n \geq 1 \) with boundary \( \Gamma \). Let \( \beta \) be a maximal graph on \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0) \).

\[ 0 \in \beta(0). \quad (1.1) \]

It is well known that there exists a unique convex function \( j \) such that

\[ j(0) = 0, \quad \partial j = \beta \quad (1.2) \]

(see for instance Brézis [1] for details).

For \( f \in L^p(\Omega) \), \( p \geq 2 \) and for any \( \varepsilon \geq 0 \) there exists also a unique solution \( u_\varepsilon \) to the problem

\[
\begin{cases}
-\Delta u_\varepsilon + \varepsilon \beta(u_\varepsilon) \ni f & \text{in } \Omega, \\
u_\varepsilon \in H^1_0(\Omega)
\end{cases}
\quad (1.3)
\]

(see for instance Brézis [3], recall that \( \Delta \) denotes here the usual Laplace operator).

Let us briefly explain the meaning of (1.3) – it is equivalent to the fact that there exists a couple

\[ (u_\varepsilon, g_\varepsilon) \in H^1_0(\Omega) \times L^2(\Omega) \quad (1.4) \]
such that
\[ -\Delta u_\varepsilon + \varepsilon g_\varepsilon = f \quad \text{in } \Omega \] (1.5)
in a weak sense. Moreover one has
\[ g_\varepsilon(x) \in \beta(u_\varepsilon(x)) \quad \text{a.e. in } \Omega. \] (1.6)

The meaning of (1.5) is simply
\[ \int_{\Omega} \nabla u_\varepsilon \nabla v \, dx + \varepsilon \int_{\Omega} g_\varepsilon v \, dx = \int_{\Omega} f \cdot v \, dx \quad \forall v \in H^1_0(\Omega). \] (1.7)

Then, one can show that for \( f \in L^p(\Omega) \), the solution \( u_\varepsilon \) to (1.3) belongs to \( W^{2,p}(\Omega) \) provided \( \Gamma \) is smooth enough – we refer the reader to Brézis [3] for these results.

Let us now define \( J[u] \) by
\[ J[u] = \begin{cases} +\infty & \text{if } j(u) \notin L^1(\Omega), \\ \int_{\Omega} j(u) \, dx & \text{else}. \end{cases} \] (1.8)

It is easy to see that \( J \) is a convex function lower semicontinuous on \( H^1_0(\Omega) \). So for any \( \alpha \geq 0 \) the set
\[ K_\alpha = \{ v \in H^1_0(\Omega) \mid J[v] \leq \alpha \} \] (1.9)
is a closed convex set in \( H^1_0(\Omega) \) and by the usual theory of variational inequalities – there exists a unique \( u \) solution to
\[ \begin{cases} \langle -\Delta u, v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in K_\alpha, \\ u \in K_\alpha. \end{cases} \] (1.10)
\( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \) in other words for any \( u, v \in H^1_0(\Omega) \)
\[ \langle -\Delta u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx. \] (1.11)

What we would like to show here is that the solution \( u \) to (1.10) is such that
\[ u = u_\varepsilon \] (1.12)
for some \( \varepsilon \). Then clearly the theory of regularity for (1.3) carries out for the variational inequality.

Of course this kind of result is not restricted to (1.3) and (1.10) but to a large class of problems. First remark that if we set
\[ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \]
then (1.10) reads
\[
\begin{cases}
  a(u, v - u) \geq \langle f, v - u \rangle & \forall v \in K_\alpha, \\
  u \in K_\alpha.
\end{cases}
\] (1.13)

Next, by definition of the subdifferential for \( u, v \in \mathbb{R} \)
\[
j(v) - j(u) \geq \partial j(u) \cdot (v - u).
\] (1.14)

Thus by (1.6) and (1.7) one deduces – replacing \( v \) by \( v - u_\varepsilon \) that
\[
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon (J[v] - J[u_\varepsilon]) \geq \langle f, v - u_\varepsilon \rangle \quad \forall v \in H^1_0(\Omega).
\] (1.15)

Now, it is easy to see that \( u_\varepsilon \) is the unique solution to (1.15) and for some \( \varepsilon \), we will see that \( u \) is equal to \( u_\varepsilon \).

So in what follows we would like to address problems of the type (1.13) and (1.15). More precisely the fact they have the same solution will permit us to obtain regularity results. The idea to derive regularity for variational inequalities from the regularity of the solution of an equation is not new – see for instance Brézis [2] for several examples – however the equation was always giving an approximated solution to the variational inequality. In the examples below the equation is satisfied by the solution of the variational inequality itself. Another approach regarding these problems could be based on a penalisation technique (see for instance Chipot and Ellalifi [7]).

The paper is organized as follows. In Section 2 we develop a general theory for problems of the type (1.13) and (1.15) showing in particular that (1.12) holds for some \( \varepsilon \). Next we pass to the applications and give various examples where the theory applies. One should note that all of them are of the nonlocal type – except for the last example of the paper.

## 2. SOME GENERAL RESULTS

Let \( H \) be a real Hilbert space. We denote by \((\cdot, \cdot)\) the scalar product in \( H \) and by \(|\cdot|\) the norm associated.

Let \( a(u, v) \) be a bilinear form on \( H \) that will be assumed continuous and coercive i.e. such that for some positive constants \( \nu \) and \( \lambda \)
\[
|a(u, v)| \leq \lambda |u| |v| \quad \forall u, v \in H, \quad \text{(2.1)}
\]
\[
a(u, u) \geq \nu |u|^2 \quad \forall u \in H. \quad \text{(2.2)}
\]

Let \( J[\cdot] \) be a proper convex function on \( H \) into \([0, +\infty]\) such that
\[
J[\cdot] \text{ is lower semicontinuous for the weak topology,} \quad \text{(2.3)}
\]
\[
\text{there exists } v_0 \in H \text{ such that } J[v_0] = 0. \quad \text{(2.4)}
\]
Let $\alpha \geq 0$, $\varepsilon \geq 0$ be two constants – for $f \in H'$ the strong dual of $H$ we consider the problem of finding $u_\varepsilon$ such that
\[
\begin{aligned}
\begin{cases}
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon \{J[v] - J[u_\varepsilon]\} \geq \langle f, v - u_\varepsilon \rangle \quad &\forall v \in H, \\
u_\varepsilon \in H,
\end{cases}
\end{aligned}
\] (2.5)
and for
\[
K_\alpha = \{ v \in H \mid J[v] \leq \alpha \}
\] (2.6)
u the solution to the variational inequality
\[
\begin{aligned}
\begin{cases}
a(u, v - u) \geq \langle f, v - u \rangle \quad &\forall v \in K_\alpha, \\
u \in K_\alpha.
\end{cases}
\end{aligned}
\] (2.7)
First we have
\textbf{Theorem 2.1.} There exists a unique solution $u_\varepsilon$ to (2.5) and a unique solution $u$ to (2.7).
\textbf{Proof.} The existence of a solution to (2.5) follows from standard arguments (see for instance Chipot [6], Kinderlehrer and Stampacchia [10]). It is easy to see that $K_\alpha$ is convex and non empty since $v_0 \in K_\alpha$. Moreover, due to (2.3) $K_\alpha$ is weakly closed – and so also closed for the topology of $H$ since convex. Due to (2.1) and (2.2) the existence of a unique solution to (2.7) follows from the standard theory of variational inequalities – see Kinderlehrer and Stampacchia [10].

In what follows we would like to show that for $\alpha > 0$ there exists always an $\varepsilon = \varepsilon_\alpha$ such that:
\[
u = u_{\varepsilon_\alpha}.
\] (2.8)
For that let us first prove
\textbf{Theorem 2.2.} Let $u_\varepsilon$ be the solution to (2.5) then the mapping
\[
\varepsilon \mapsto u_\varepsilon
\] [0, +\infty) \rightarrow H
is continuous when $H$ is equipped with its strong topology.
\textbf{Proof.} Taking $v = v_0$ in (2.5) one gets
\[
\begin{aligned}
a(u_\varepsilon, u_\varepsilon) + \varepsilon J[u_\varepsilon] \leq \langle f, u_\varepsilon \rangle + a(v_0, u_\varepsilon) - \langle f, v_0 \rangle.
\end{aligned}
\] (2.10)
Recall that $\langle \cdot \rangle$ is the duality bracket between $H'$, $H$. Since $\varepsilon, J \geq 0$ it follows from (2.2) that
\[
\nu |u_\varepsilon|^2 \leq |f| |u_\varepsilon| + \lambda |v_0| |u_\varepsilon| + |f| |v_0|
\leq \frac{(|f| + \lambda |v_0|)^2}{2\nu} + \frac{\nu |u_\varepsilon|^2}{2} + \frac{(|f| + \lambda |v_0|)^2}{2\nu}
\]
where $| \cdot |_*$ denotes the norm of strong dual on $H'$ (we used the Young inequality). This leads easily to

$$
|u_\varepsilon| \leq \frac{\sqrt{2}\{|f|_* + \lambda|v_0|\}}{\varepsilon}. \tag{2.11}
$$

Going back to (2.10) one gets also

$$
0 \leq J[u_\varepsilon] \leq \frac{(|f|_* + \lambda|v_0|)|u_\varepsilon|}{\varepsilon} \leq \frac{3(|f|_* + \lambda|v_0|)^2}{\nu \varepsilon}. \tag{2.12}
$$

So, let us consider $\varepsilon, \varepsilon', \varepsilon \to \varepsilon'$. From (2.5) one has

$$
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon\{J[v] - J[u_\varepsilon]\} \geq \langle f, v - u_\varepsilon \rangle \quad \forall v \in H, \tag{2.13}
a(u_{\varepsilon'}, v - u_{\varepsilon'}) + \varepsilon'\{J[v] - J[u_{\varepsilon'}]\} \geq \langle f, v - u_{\varepsilon'} \rangle \quad \forall v \in H. \tag{2.14}
$$

Taking $v = u_{\varepsilon'}$ in (2.13), $v = u_\varepsilon$ in (2.14) and adding – it comes

$$
a(u_\varepsilon - u_{\varepsilon'}, u_{\varepsilon'} - u_\varepsilon) + (\varepsilon - \varepsilon')\{J[u_{\varepsilon'}] - J[u_\varepsilon]\} \geq 0.
$$

Hence

$$
\nu|u_\varepsilon - u_{\varepsilon'}|^2 \leq a(u_\varepsilon - u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) \leq (\varepsilon - \varepsilon')\{J[u_{\varepsilon'}] - J[u_\varepsilon]\}. \tag{2.15}
$$

Clearly when $\varepsilon' > 0$, then due to (2.12), $J[u_\varepsilon], J[u_{\varepsilon'}]$ remain bounded when $\varepsilon \to \varepsilon'$. Then passing to the limit in (2.15) leads to

$$
u_\varepsilon \to u_{\varepsilon'} \quad \text{in } H
$$

which is what we wanted. In the case $\varepsilon' = 0$, (2.15) reads

$$
\nu|u_\varepsilon - u_0|^2 \leq a(u_\varepsilon - u_0, u_\varepsilon - u_0) \leq \varepsilon\{J[u_0] - J[u_\varepsilon]\} \leq \varepsilon J[u_0] \tag{2.16}
$$

and the result holds also in this case when $J[u_0] < +\infty$. In the case where $J[u_0] = +\infty$ one can proceed differently. In fact the proof below holds also for $J[u_0] < +\infty$. One notices that by (2.11) $u_\varepsilon$ is bounded independently of $\varepsilon, \varepsilon \to 0$. Thus up to a subsequence one can assume that when $\varepsilon \to 0$

$$
u_\varepsilon \to \bar{u}_0 \quad \text{in } H.
$$

From (2.5) one deduces

$$
a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, v) - \langle f, v - u_\varepsilon \rangle + \varepsilon J[v] \quad \forall v \in H.
$$

Taking the lim inf of both sides it comes

$$
a(\bar{u}_0, \bar{u}_0) \leq \liminf_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) \leq a(\bar{u}_0, v) - \langle f, v - \bar{u}_0 \rangle
$$
i.e. \( \bar{u}_0 = u_0 \). Since \( u_\varepsilon \) has only one possible limit it holds
\[
  u_\varepsilon \rightharpoonup u_0 \quad \text{when } \varepsilon \to 0.
\]
To show strong convergence it is enough to notice that from above one gets
\[
  \limsup_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) \leq a(u_0, v) - \langle f, v - u_0 \rangle \quad \forall v \in H.
\]
Taking \( v = u_0 \) it holds
\[
  \limsup_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) \leq a(u_0, u_0).
\]
Since
\[
  \liminf_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) \geq a(u_0, u_0)
\]
on one deduces that
\[
  a(u_\varepsilon, u_\varepsilon) \to a(u_0, u_0)
\]
when \( \varepsilon \to 0 \) and the strong convergence follows. This completes the proof of the theorem. \( \square \)

**Remark 2.1.** For \( \varepsilon = 0 \), as we saw (2.5) is understood in the sense
\[
  a(u_0, v - u_0) \geq \langle f, v - u_0 \rangle \quad \forall v \in H.
\]
Taking \( v = u_0 \pm v \) in this inequality – it becomes clear that \( u_0 \) is the solution to
\[
  \begin{cases}
    a(u_0, v) = \langle f, v \rangle \quad \forall v \in H, \\
    u_0 \in H.
  \end{cases}
\]  \( (2.17) \)

**Theorem 2.3.** The mapping
\[
  [0, +\infty) \to [0, +\infty] \\
  \varepsilon \mapsto J[u_\varepsilon]
\]  \( (2.18) \)
is nonincreasing, continuous. Moreover it holds
\[
  0 \leq J[u_\varepsilon] \leq \frac{3|\|f\|_* + \lambda|v_0|}{\varepsilon \nu}
\]  \( (2.19) \)

**Proof.** Suppose \( \varepsilon > \varepsilon' \). From (2.15) one deduces
\[
  (\varepsilon - \varepsilon') \{ J[u_{\varepsilon'}] - J[u_\varepsilon] \} \geq 0
\]
i.e. \( J[u_{\varepsilon'}] \geq J[u_\varepsilon] \) and \( \varepsilon \mapsto J[u_\varepsilon] \) is nonincreasing. Taking \( v = u_{\varepsilon'} \) in (2.5) it comes
\[
  \varepsilon J[u_\varepsilon] \leq \langle f, u_\varepsilon - u_{\varepsilon'} \rangle + \varepsilon J[u_{\varepsilon'}] + a(u_\varepsilon, u_{\varepsilon'} - u_\varepsilon) \\
  \leq |\|f\|_*|u_\varepsilon - u_{\varepsilon'}| + \varepsilon J[u_{\varepsilon'}] + \lambda|u_\varepsilon| |u_{\varepsilon'} - u_\varepsilon|.
\]  \( (2.20) \)
From (2.11) it is clear that $|u_\varepsilon|$ is uniformly bounded. Thus letting $\varepsilon \to \varepsilon'$ in (2.20) and using Theorem 2.2 one gets

$$\limsup_{\varepsilon \to \varepsilon'} \varepsilon J[u_\varepsilon] \leq \varepsilon' J[u_{\varepsilon'}]. \tag{2.21}$$

Using the lower semicontinuity of $J$ for the weak topology one would get

$$\liminf_{\varepsilon \to \varepsilon'} \varepsilon J[u_\varepsilon] \geq \varepsilon' J[u_{\varepsilon'}]. \tag{2.22}$$

Thus combining (2.21) and (2.22) one gets

$$\lim_{\varepsilon \to \varepsilon'} \varepsilon J[u_\varepsilon] = \varepsilon' J[u_{\varepsilon'}]$$

when $\varepsilon' > 0$ this implies also

$$\lim_{\varepsilon \to \varepsilon'} J[u_\varepsilon] = J[u_{\varepsilon'}] \tag{2.23}$$

and the continuity of $\varepsilon \to J[u_\varepsilon]$ follows. When $\varepsilon' = 0$, from the monotony of $\varepsilon \mapsto J[u_\varepsilon]$ one has

$$J[u_\varepsilon] \leq J[u_0].$$

Hence

$$\limsup_{\varepsilon \to 0} J[u_\varepsilon] \leq J[u_0].$$

This joint to

$$\liminf_{\varepsilon \to 0} J[u_\varepsilon] \geq J[u_0]$$

due to the lower semicontinuity of $J$ proves the continuity in 0 (when $J[u_0] = +\infty$ the last inequality shows that $J[u_\varepsilon] \to +\infty = J[u_0]$). (2.19) is nothing but (2.12). This completes the proof of the theorem.

Then we can now show:

**Theorem 2.4.** For $\alpha > 0$ let $u$ be the solution to (2.7). Then there exists $\varepsilon = \varepsilon_\alpha$ such that

$$u = u_{\varepsilon_\alpha}. \tag{2.24}$$

**Proof.** Let $u_0$ be the solution to (2.5) for $\varepsilon = 0$. By Remark 2.1 we have seen that $u_0$ is also solution to (2.17). Suppose that

$$J[u_0] \leq \alpha.$$

Then $u_0 \in K_\alpha$ and by (2.17) $u_0$ satisfies

$$a(u_0, v - u_0) = \langle f, v - u_0 \rangle \quad \forall v \in K_\alpha$$
so that $u_0$ is the solution of (2.7) in this case and $\varepsilon_\alpha = 0$. So, we assume now that
\begin{equation}
J[u_0] > \alpha > 0. \tag{2.25}
\end{equation}
By Theorem 2.3 the mapping $\varepsilon \to J[u_\varepsilon]$ is continuous on the interval $[0, +\infty)$ and non increasing. Moreover by (2.19) one has
\begin{equation}
\lim_{\varepsilon \to +\infty} J[u_\varepsilon] = 0. \tag{2.26}
\end{equation}
Also by Theorem 2.2 and the continuity of $\varepsilon \mapsto J[u_\varepsilon]$
\begin{equation}
\lim_{\varepsilon \to 0} J[u_\varepsilon] = J[u_0] > \alpha. \tag{2.27}
\end{equation}
So, clearly for $\varepsilon$ small enough one would have
\begin{equation}
J[u_\varepsilon] > \alpha > 0. \tag{2.27}
\end{equation}
Then combining (2.26) and (2.27) and applying the intermediate value theorem one can find $\varepsilon = \varepsilon_\alpha$ such that
\begin{equation}
J[u_\varepsilon] = \alpha. \tag{2.28}
\end{equation}
For this $\varepsilon$, (2.5) reads
\begin{equation}
a(u_\varepsilon, v - u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle + \varepsilon \{ J[u_\varepsilon] - J[v] \}
= \langle f, v - u_\varepsilon \rangle + \varepsilon \{ \alpha - J[v] \}
\geq \langle f, v - u_\varepsilon \rangle \quad \forall v \in K_\alpha. \tag{2.29}
\end{equation}
Since by (2.28), $u_\varepsilon \in K_\alpha$, $u_\varepsilon$ is the solution to (2.7). This completes the proof of the theorem. \hfill \Box

**Remark 2.2.** Assume $J$ to be differentiable – replacing in (2.5) $v$ by $u_\varepsilon + t(v - u_\varepsilon)$ it comes
\begin{equation}
ta(u_\varepsilon, v - u_\varepsilon) + \varepsilon \{ J[u_\varepsilon] + t(v - u_\varepsilon)] - J[u_\varepsilon] \} \geq t\langle f, v - u_\varepsilon \rangle.
\end{equation}
Assuming $t > 0$, dividing by $t$ and letting $t \to 0$ we obtain
\begin{equation}
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon \langle J'(u_\varepsilon), v - u_\varepsilon \rangle \geq \langle f, v - u_\varepsilon \rangle \quad \forall v \in H.
\end{equation}
Changing $v$ into $u_\varepsilon \pm v$ one gets
\begin{equation}
a(u_\varepsilon, v) + \varepsilon \langle J'(u_\varepsilon), v \rangle = \langle f, v \rangle \quad \forall v \in H. \tag{2.30}
\end{equation}
In the case where $a$ is symmetric – then the solution to (2.7) is also the minimizer of
\begin{equation}
v \mapsto \frac{1}{2} a(v, v) - \langle f, v \rangle
\end{equation}
(cf. Chipot [6], Kinderlehrer and Stampacchia [10]) and $\varepsilon$ appears in (2.30) as a Lagrange multiplier.
To end this section we turn also to the behavior of $u_\varepsilon$ when $\varepsilon \to +\infty$.

**Theorem 2.5.** Let $u_\infty$ the solution to the variational inequality

$$
\begin{cases}
a(u_\infty, v - u_\infty) \geq \langle f, v - u_\infty \rangle & \forall v \in K_0 \\
u_\infty \in K_0 = \{ v \in H \mid J(v) = 0 \}
\end{cases}
$$

(2.31)

then it holds

$$
u_\varepsilon \to u_\infty \text{ in } H$$

(2.32)

when $\varepsilon \to +\infty$.

**Proof.** From (2.11) one deduces that $u_\varepsilon$ is uniformly bounded independently of $\varepsilon$ and up to a subsequence one can assume that there exists a $u_\infty \in H$ such that

$$
u_\varepsilon \to u_\infty \text{ in } H.$$  

(2.33)

From (2.12) and the lower semicontinuity of $J$ one has

$$0 = \lim_{\varepsilon \to +\infty} J[u_\varepsilon] \geq J[u_\infty] \geq 0.$$  

So, $u_\infty \in K_0$. Moreover, from (2.5) one has for any $v \in K_0$

$$a(u_\varepsilon, v - u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle + \varepsilon J[u_\varepsilon] \geq \langle f, v - u_\varepsilon \rangle.$$  

(2.34)

This implies

$$a(u_\varepsilon, v) - \langle f, v - u_\varepsilon \rangle \geq a(u_\varepsilon, u_\varepsilon).$$

Taking the lim inf when $\varepsilon \to +\infty$ and using the lower semicontinuity of $a(u, u)$ it comes

$$a(u_\infty, v) - \langle f, v - u_\infty \rangle \geq \liminf_{\varepsilon \to +\infty} a(u_\varepsilon, u_\varepsilon) \geq a(u_\infty, u_\infty).$$

In other words $u_\infty$ satisfies

$$
\begin{cases}
a(u_\infty, v - u_\infty) \geq \langle f, v - u_\infty \rangle & \forall v \in K_0, \\
u_\infty \in K_0.
\end{cases}
$$

Thus this is the unique solution to (2.31). By uniqueness of its limit this is the whole sequence $u_\varepsilon$ that satisfies (2.33).

Next we show that the convergence is strong. Indeed from (2.34) one has

$$a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, v) - \langle f, v - u_\varepsilon \rangle, \quad \forall v \in K_0.$$  

Thus taking $v = u_\infty$

$$a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, u_\infty) - \langle f, u_\infty - u_\varepsilon \rangle.$$
Taking the lim sup when \( \varepsilon \to +\infty \) and using (2.33) we get

\[
\limsup_{\varepsilon \to +\infty} a(u_\varepsilon, u_\varepsilon) \leq a(u_\infty, u_\infty).
\]

Joint with

\[
\liminf_{\varepsilon \to +\infty} a(u_\varepsilon, u_\varepsilon) \geq a(u_\infty, u_\infty)
\]

we obtain

\[
\lim_{\varepsilon \to +\infty} a(u_\varepsilon, u_\varepsilon) = a(u_\infty, u_\infty).
\]

Then it is easy to show that

\[
\lim_{\varepsilon \to +\infty} a(u_\varepsilon - u_\infty, u_\varepsilon - u_\infty) = 0
\]

and the strong convergence follows by (2.2). This completes the proof of the theorem. \( \square \)

**Remark 2.3.** The results developed here extend of course when \( a(u, v) \) is replaced by

\[
\langle A u, v \rangle,
\]

\( A \) being a suitable nonlinear operator from \( H \) into \( H' \) – eventually from a Banach space into its dual. For the sake of simplicity we restricted ourselves to the linear case.

### 3. APPLICATIONS TO SECOND ORDER VARIATIONAL INEQUALITIES

In this section \( \Omega \) is a bounded open subset of \( \mathbb{R}^n, n \geq 1 \), with boundary \( \Gamma \). Let us denote by \( a, a_{ij} \) functions in \( \Omega \) such that

\[
\begin{align*}
a, a_{ij} &\in L^\infty(\Omega) \quad i, j = 1, \ldots, n, \quad (3.1) \\
a_{ij} \xi_i \xi_j &\geq \nu |\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad (3.2) \\
a &\geq 0 \quad \text{a.e. } x \in \Omega \quad (3.3)
\end{align*}
\]

\( \nu \) is a positive constant, in (3.2) we make the summation convention of repeated indices, \( | \cdot | \) denotes the Euclidean norm in \( \mathbb{R}^n \). For \( u, v \in H^1(\Omega) \) we will set

\[
a(u, v) = \int_{\Omega} \left\{ a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + auv \right\} \, dx. \quad (3.4)
\]
3.1. DIRICHLET BOUNDARY CONDITIONS

In this paragraph we will consider problems in $H^1_0(\Omega)$. We will denote by $A$ the operator

$$A = \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial}{\partial x_i}) + a$$

such that

$$a(u, v) = \langle -Au, v \rangle \quad \forall u, v \in H^1_0(\Omega)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. We will assume $A$, $\Omega$ smooth enough in such a way that if $u$ denotes the weak solution to the Dirichlet problem

$$\begin{cases}
    -Au = f & \text{in } \Omega, \\
    u \in H^1_0(\Omega),
\end{cases}$$

then for any $f \in L^p(\Omega)$, $p \geq 2$, (resp. $W^{-1,p}(\Omega)$) one has $u \in W^{2,p}(\Omega)$ (resp. $u \in W^{1,p}(\Omega)$) and the estimates

$$\|u\|_{2,p} \leq C|f|_p, \quad \text{(resp. } \|u\|_{1,p} \leq C|f|_{1,p}).$$

In the above inequalities $| \cdot |_p$, $| \cdot |_{-1,p}$, $\| \cdot \|_{1,p}$, $\| \cdot \|_{2,p}$ denote the norms in $L^p(\Omega)$, $W^{-1,p}(\Omega)$, $W^{1,p}(\Omega)$, $W^{2,p}(\Omega)$ respectively, $C$ constants depending on $A$, $\Omega$ only (for Sobolev spaces see Brézis [4], for conditions on $A$ and $\Omega$ for this to hold see for instance Gilbarg and Trudinger [9]).

Let first consider an example of the type evoked in the introduction. So, let $\beta$ be a maximal monotone graph of $\mathbb{R} \times \mathbb{R}$ satisfying

$$0 \in \beta(0).$$

Let $j$ be the positive convex function such that

$$j(0) = 0, \quad \partial j = \beta$$

and define $J[u]$ on $H^1_0(\Omega)$ by

$$J[u] = \begin{cases}
    \int_{\Omega} j(u) \, dx & \text{if } j(u) \in L^1(\Omega), \\
    +\infty & \text{else.}
\end{cases}$$

For $\alpha > 0$ set

$$K_\alpha = \{ v \in H^1_0(\Omega) \mid J[v] \leq \alpha \}.$$
Theorem 3.1. Let $f \in H^{-1}(\Omega)$. Then the variational inequality
\begin{equation}
\begin{cases}
\langle -Au, v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in K_{\alpha}, \\
u \in K_{\alpha},
\end{cases}
\tag{3.13}
\end{equation}
has a unique solution. Moreover, assuming that (3.7) and (3.8) hold, if $f \in L^p(\Omega)$, $p \geq 2$ then $u \in W^{2,p}(\Omega)$ and one has an estimate
\begin{equation}
\|u\|_{2,p} \leq C|f|_p.
\tag{3.14}
\end{equation}

Proof. Let us first prove that $J$ is lower semicontinuous for the weak topology of $H^1_0(\Omega)$. For that consider a sequence $u_n \in H^1_0(\Omega)$ such that
\begin{equation}
u_n \rightharpoonup u \quad \text{in} \quad H^1_0(\Omega).
\tag{3.15}
\end{equation}
Extracting if necessary a subsequence one can assume that
\begin{equation}
\lim_{n \to +\infty} J[u_n] = \liminf_{n \to +\infty} J[u_n].
\tag{3.16}
\end{equation}
Then from this subsequence still denoted $u_n$ one can extract a subsequence such that
\begin{equation}
u_n \to u \quad \text{in} \quad L^2(\Omega),
\tag{3.17}
u_n \to u \quad \text{a.e. in} \quad \Omega.
\tag{3.18}
\end{equation}
Applying Fatou’s lemma one gets – recall that $j$ is continuous –
\begin{equation}
J[u] = \int_{\Omega} j(u) \, dx = \int_{\Omega} \liminf_{n \to +\infty} j(u_n) \leq \liminf_{n \to +\infty} J[u_n].
\tag{3.19}
\end{equation}

$J$ being lower semicontinuous – one can apply Theorem 2.1 with $a(u,v)$ defined by (3.4) and $H = H^1_0(\Omega)$. $K_{\alpha}$ is not empty since by (3.10), (3.11) $J[0] = 0$ and $0 \in K_{\alpha}$. This shows existence and uniqueness for a solution to (3.13).

Next for $\varepsilon \geq 0$, $f \in L^2(\Omega)$ we consider the solution $u_\varepsilon$ to
\begin{equation}
\begin{cases}
-Au_\varepsilon + \varepsilon \beta(u_\varepsilon) \ni f, \\
u_\varepsilon \in H^1_0(\Omega).
\end{cases}
\tag{3.20}
\end{equation}
The meaning of the first equation was explained in the introduction. For existence of a solution to this problem one proceeds similarly to Brézis [3]. Multiplying the first equation of (3.20) by $v - u_\varepsilon$, $v \in H^1_0(\Omega)$ and integrating on $\Omega$ we get
\begin{equation}
\langle -Au_\varepsilon, v - u_\varepsilon \rangle + \varepsilon \int_{\Omega} \beta(u_\varepsilon)(v - u_\varepsilon) \, dx \geq \langle f, v - u_\varepsilon \rangle
\tag{3.21}
\end{equation}
where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2(\Omega)$. Since $\beta = \partial j$ one derives
\begin{equation}
\int_{\Omega} j(v) \, dx - \int_{\Omega} j(u_\varepsilon) \, dx \geq \int_{\Omega} \beta(u_\varepsilon)(v - u_\varepsilon) \, dx
\end{equation}
and (3.21) leads to
\[
\begin{aligned}
\left\{
\begin{array}{l}
\langle -Au_\varepsilon, v - u_\varepsilon \rangle + \varepsilon \{J[v] - J[u_\varepsilon]\} \geq (f, v - u_\varepsilon) \quad \forall v \in H^1_0(\Omega), \\
u_\varepsilon \in H^1_0(\Omega).
\end{array}
\right.
\end{aligned}
\]
(3.22)
Thus \(u_\varepsilon\) is also solution to (3.22). Applying then Theorem 2.4 for some \(\varepsilon = \varepsilon_\alpha\) one has \(u_\varepsilon = u\) but due to the regularity theory developed in Brézis [3] for \(f \in L^p(\Omega)\) one has \(u_\varepsilon \in W^{2,p}(\Omega)\) and an estimate of the type (3.14).

This completes the proof. \(\square\)

**Remark 3.4.** One can apply Theorem 3.1 with for instance
\[
j(u) = |u|^q \quad q > 1
\]
i.e. for convex sets
\[
K_\alpha = \left\{ v \in H^1_0(\Omega) \ \bigg| \ \int_\Omega |v|^q \, dx \leq \alpha \right\}.
\]
It is possible to consider also convex sets of the type
\[
K_\alpha = \left\{ v \in H^1_0(\Omega) \ \bigg| \ \int_\Omega j(x, v) \, dx \leq \alpha \right\}
\]
where for a.e. \(x\), \(j(x, u)\) is a convex function such that
\[
j(x, 0) = 0, \quad \partial j = \beta(x, \cdot) \quad \text{a.e. } x \in \Omega \quad (3.23)
\]
\(\beta(x, \cdot)\) being for a.e. \(x\) a maximal monotone graph. Indeed the \(W^{2,p}(\Omega)\)-regularity theory of Brézis [3] can be extended in some cases (see Chipot and Kis [8], Kis [11]). For example assuming that \(\psi \in L^\infty(\Omega)\)
\[
0 < a \leq \psi \leq b \quad \text{a.e. in } \Omega
\]
for some constants \(a, b\) then one can show that \(f \in L^p(\Omega)\) implies that the solution \(u\) to (3.13) with
\[
K_\alpha = \left\{ v \in H^1_0(\Omega) \ \bigg| \ \int_\Omega \psi(x)j(v) \, dx \leq \alpha \right\}
\]
belongs to \(W^{2,p}\) provided \(j\) is a convex function such that
\[
j(0) = 0, \quad \partial j = \beta
\]
where \(\beta\) is a maximal monotone graph. The proof goes like in Theorem 3.1 using Chipot and Kis [8].

Let \(j(x, \xi)\) be a Carathéodory function – i.e. measurable in \(x\) for every \(\xi\) and continuous in \(\xi\) for a.e. \(x \in \Omega\) – defined on \(\Omega \times \mathbb{R}^n\) and convex in \(\xi\) for a.e. \(x\). Moreover, suppose that \(j\) is differentiable in \(\xi\) with
\[
\left| \frac{\partial j}{\partial \xi_i}(x, \xi) \right| \leq C \quad \forall i = 1, \ldots, n, \ \forall \xi \in \mathbb{R}^n, \ \text{a.e. } x \in \Omega \quad (3.24)
\]
where $C$ denotes some positive constant. Let us assume also that
\[
j(x, 0) = 0, \quad j(x, \xi) \geq 0 \quad \forall \xi, \text{ a.e. } x.
\] (3.25)
Due to (3.24) and (3.25) one has
\[
|j(x, \nabla u)| \leq C|\nabla u|
\] for some constant $C$ – $|\nabla u|$ denotes the Euclidean norm of $\nabla u$ – and thus
\[
\int_{\Omega} j(x, \nabla u) \, dx < +\infty \quad \forall u \in H^{1}_{0}(\Omega).
\] (3.26)
Then one can show

**Theorem 3.2.** Let $f \in H^{-1}(\Omega)$. Then the variational inequality
\[
\begin{aligned}
&\langle -Au, v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in K_{\alpha}, \\
&u \in K_{\alpha} = \left\{ v \in H^{1}_{0}(\Omega) \mid \int_{\Omega} j(x, \nabla u) \, dx \leq \alpha \right\},
\end{aligned}
\] (3.27)
has for any $\alpha > 0$ a unique solution. Moreover, assuming that (3.7) and (3.8) hold, if $f \in W^{-1,p}$, $p \geq 2$ then
\[
u \in W^{1,p}(\Omega).
\] (3.28)
In particular if $f \in W^{-1,\infty}(\Omega)$ then
\[
u \in W^{1,p}(\Omega) \quad \forall p \geq 2.
\] (3.29)

**Proof.** Since $j(x, \cdot)$ is uniformly Lipschitz continuous
\[
J[u] = \int_{\Omega} j(x, \nabla u) \, dx
\]
is continuous for the strong topology in $H^{1}_{0}(\Omega)$. So, due to its convexity $J$ is also weakly lower semicontinuous. Then existence of a solution follows directly from Theorem 2.1. (Note that $0 \in K_{\alpha}$ since $J[0] = 0$). Next, due to Theorem 2.4 $u$ is solution for some $\varepsilon > 0$ to
\[
\langle -Au_{\varepsilon}, v - u_{\varepsilon} \rangle + \varepsilon \{J[v] - J[u_{\varepsilon}]\} \geq \langle f, v - u_{\varepsilon} \rangle \quad \forall v \in H^{1}_{0}(\Omega).
\]
Using the Remark 2.2 we see that $u_{\varepsilon}$ satisfies also
\[
\langle -Au_{\varepsilon}, v \rangle + \varepsilon \int_{\Omega} j(\xi)(x, \nabla u_{\varepsilon}) \frac{\partial v}{\partial x_{i}} \, dx = \langle f, v \rangle \quad \forall v \in H^{1}_{0}(\Omega),
\]
i.e. $u_{\varepsilon}$ is the solution to
\[
\begin{aligned}
-\frac{\partial}{\partial x_{i}}(j(\xi)(x, \nabla u_{\varepsilon})) + f, \\
u_{\varepsilon} \in H^{1}_{0}(\Omega).
\end{aligned}
\] (3.30)
The result follows then from (3.7) and (3.8) since the right hand side of the equation (3.30) belongs to $W^{-1,p}(\Omega)$.

Consider now $b_{ij}, i, j = 1, \ldots, n$ measurable functions such that

$$b_{ij} \in L^\infty(\Omega),$$

$$b_{ij} \xi_i \xi_j \geq 0 \quad \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n. \quad (3.31)$$

Then set

$$J[u] = \int_\Omega b_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \quad \forall u \in H^1_0(\Omega),$$

$$K_\alpha = \{ v \in H^1_0(\Omega) \mid J[v] \leq \alpha \}. \quad (3.33)$$

One has

**Theorem 3.3.** Let $f \in H^{-1}(\Omega)$. Then the variational inequality

$$\begin{cases}
\langle -Au, v - u \rangle \geq \langle f, v - u \rangle & \forall v \in K_\alpha, \\
u \in K_\alpha,
\end{cases} \quad (3.35)$$

has a unique solution. Moreover assuming $b_{ij}$ sufficiently smooth and that (3.7) and (3.8) hold if $f \in L^p(\Omega), p \geq 2$ then $u \in W^{2,p}(\Omega)$ and one has an estimate

$$\|u\|_{2,p} \leq C|f|_p. \quad (3.36)$$

**Proof.** Writing

$$J[u] - J[v] = \int_\Omega b_{ij} \frac{\partial}{\partial x_i} (u - v) \frac{\partial u}{\partial x_j} \, dx + \int_\Omega b_{ij} \frac{\partial v}{\partial x_i} \frac{\partial (u - v)}{\partial x_j} \, dx$$

it is clear that $J$ is continuous on $H^1_0(\Omega)$. $J$ being convex, $J$ is also lower semicontinuous for the weak topology of $H^1_0(\Omega)$. The existence of a solution to (3.35) follows then from Theorem 2.1 taking into account that $0 \in K_\alpha$ – i.e. $J[0] = 0$. Using then Theorem 2.4 and Remark 2.2 one sees that $u = u_\varepsilon$ where $u_\varepsilon$ is the solution to

$$\begin{cases}
\langle -Au_\varepsilon, v \rangle + \varepsilon \int_\Omega b_{ij} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx = \langle f, v \rangle & \forall v \in H^1_0(\Omega), \\
u_\varepsilon \in H^1_0(\Omega).
\end{cases}$$

So, $u_\varepsilon$ is the weak solution of an elliptic Dirichlet problem and (3.36) follows from the $W^{2,p}(\Omega)$-regularity theory for such a problem. \hfill \Box

### 3.2. Neumann Boundary Conditions

In this paragraph in addition to (3.1) and (3.2) we assume

$$a(x) \geq \nu > 0 \quad \text{a.e. } x \in \Omega. \quad (3.37)$$
Let $\beta$ be a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$ with

$$0 \in \beta(0) \quad (3.38)$$

and let $j$ be the positive convex function such that

$$j(0) = 0, \quad \partial j = \beta. \quad (3.39)$$

For $v \in H^1(\Omega)$ set

$$J[u] = \begin{cases} \int_\Gamma j(u) \, d\Gamma & \text{if } j(u) \in L^1(\Gamma), \\ +\infty & \text{else,} \end{cases} \quad (3.40)$$

where $d\Gamma$ denotes the superficial measure on $\Gamma$. Moreover for $\alpha > 0$ set

$$K_\alpha = \{ v \in H^1(\Omega) \mid J[v] \leq \alpha \}. \quad (3.41)$$

**Theorem 3.4.** For $f \in L^2(\Omega)$ there exists a unique solution to the variational inequality

$$\begin{cases} a(u, v - u) \geq (f, v - u) & \forall v \in K_\alpha, \\ u \in K_\alpha. \end{cases} \quad (3.42)$$

Moreover, assuming $a, a_{ij}, \Omega$ smooth for $f \in L^p(\Omega), \frac{n}{2} < p < n$ one has

$$u \in W^{1,p^*}(\Omega), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \quad (3.43)$$

**Proof.** Let $u_n$ be a sequence in $H^1(\Omega)$ such that

$$u_n \rightharpoonup u. \quad (3.44)$$

Then $u_n$ is bounded in $H^1(\Omega)$ and due to the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Gamma)$ one can find a subsequence still denoted $u_n$ such that

$$\begin{aligned} u_n &\to u \quad \text{in } L^2(\Gamma), \\ u_n &\to u \quad \text{a.e. on } \Gamma, \\ J[u_n] &\to \liminf_{n \to +\infty} J[u_n]. \end{aligned} \quad (3.45)$$

Then by Fatou’s lemma

$$\liminf_{n \to +\infty} \int_\Gamma j(u_n) \, d\Gamma \geq \int_\Gamma \liminf_{n \to +\infty} j(u_n) \, d\Gamma = \int_\Gamma j(u) \, d\Gamma$$

i.e. one has – due to the last convergence in (3.45)

$$\liminf_{n \to +\infty} J[u_n] \geq J[u].$$
Thus $J$ is lower semicontinuous and the existence of a solution follows from Theorem 2.1 applied with $H = H^1(\Omega)$ taking into account that $J[0] = 0$, i.e. $0 \in K_\alpha$. ((3.1), (3.2), (3.37) imply (2.1) and (2.2)). Following Brézis [2] for $f \in L^p$, $\frac{n}{2} < p < n$ the weak solution to
\[
\begin{align*}
- Au_\varepsilon &= f \in \Omega \quad \text{in } \Omega, \\
\frac{\partial u_\varepsilon}{\partial n} &\in -\varepsilon \beta(u_\varepsilon) \quad \text{on } \Gamma,
\end{align*}
\]
(3.46) belongs to $W^{1,p'}(\Omega)$. ($\frac{\partial u_\varepsilon}{\partial n}$ denotes the usual conormal derivative). Now, the solution to (3.46) is also the solution to
\[
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon \int_\Gamma \beta(u_\varepsilon) \cdot v - u_\varepsilon \geq \int_\Omega f \cdot (v - u_\varepsilon) \quad \forall v \in H^1(\Omega)
\]
and due to (3.39) satisfies also
\[
a(u_\varepsilon, v - u_\varepsilon) + \varepsilon \{J[v] - J[u_\varepsilon]\} \geq \int_\Omega f \cdot (v - u_\varepsilon) \, dx \quad \forall v \in H^1(\Omega).
\]
But it results from Theorem 2.4 that $u$ – the solution to (3.42) – is equal to such $u_\varepsilon$ for some $\varepsilon$. This completes the proof of the theorem. \hfill \square

**Remark 3.5.** If $\beta$ is Lipschitz continuous – and with the same proof as above one has in fact $u \in W^{2,p}(\Omega)$ – see Brézis [2].

### 4. APPLICATIONS TO FOURTH ORDER VARIATIONAL INEQUALITIES

We assume here that $a$, $a_{ij}$, $\Omega$ are smooth and we denote by $A$ the elliptic operator defined in (3.5).

We denote by $j$ a nonnegative convex function such that
\[
\begin{align*}
j(0) &= 0, \\
j \text{ is differentiable and } |j'(u)| &\leq C \quad \forall u \in \mathbb{R}.
\end{align*}
\]
(4.1) (4.2)

We consider here $H = H^2(\Omega) \cap H^1_0(\Omega)$ – see Kinderlehrer and Stampacchia [10], Rodrigues [12] for the definition of $H^2(\Omega)$. It is well known that
\[
\|u\|_{2,2} = \left\{ \int_\Omega (Au)^2 \, dx \right\}^{1/2}
\]
(4.3) is a norm on $H$ equivalent to the $H^2(\Omega)$-norm.

For $u \in H$ we set
\[
J[u] = \int_\Omega j(Au) \, dx.
\]
(4.4)
(Since by (4.1), (4.2) \( |j(t)| \leq C|t| \) one has clearly \( j(Au) \in L^2(\Omega) \subset L^1(\Omega) \) and the definition makes sense). Then set for \( \alpha > 0 \)

\[
K_\alpha = \left\{ v \in H^2(\Omega) \cap H^1_0(\Omega) \mid \int_\Omega j(Au) \, dx \leq \alpha \right\}.
\] (4.5)

One can then show:

**Theorem 4.1.** For \( f \in L^2(\Omega) \) let consider the variational inequality

\[
\begin{cases}
(Au, A(v - u)) \geq (f, v - u) & \forall v \in K_\alpha, \\
u \in K_\alpha,
\end{cases}
\] (4.6)

where \((\cdot, \cdot)\) denotes the scalar product in \( L^2(\Omega) \). Then (4.6) has a unique solution. Moreover, if \( f \in L^p(\Omega) \) then \( u \in W^{3,p^*}(\Omega) \), \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \) and one has an estimate

\[
\|u\|_{3,p^*} \leq C|f|_p
\] (4.7)

if in addition \((I + \varepsilon j_0')^{-1}\) has for every \( \varepsilon > 0 \) its second derivative bounded then \( u \in W^{4,p}(\Omega) \) and it holds

\[
\|u\|_{4,p} \leq C|f|_p
\] (4.8)

for some positive constant \( C \).

**Proof.** Since \( j \) is Lipschitz continuous one has clearly

\[
|J[v] - J[u]| \leq C \int_\Omega |Av - Au| \, dx \leq C|\Omega|^{\frac{1}{2}}|Av - Au|_2.
\]

It follows that \( J[\cdot] \) is continuous on \( H = H^2(\Omega) \cap H^1_0(\Omega) \). Since \( J \) is convex, \( J \) is also weakly lower semicontinuous on \( H \) and Theorem 2.1 applies which gives existence to a solution to (4.6). (Note that \( J[0] = 0 \).

Next applying Theorem 2.4 and the Remark 2.2 we obtain that \( u \) is the solution for some \( \varepsilon > 0 \) of the problem

\[
\begin{cases}
(Au_\varepsilon, Av) + \varepsilon(j'(Au_\varepsilon), Av) = (f, v) & \forall v \in H^2(\Omega) \cap H^1_0(\Omega), \\
u_\varepsilon \in H^2(\Omega) \cap H^1_0(\Omega).
\end{cases}
\] (4.9)

The first equation to (4.9) reads also

\[
((I + \varepsilon j')(Au_\varepsilon), Av) = (f, v) & \forall v \in H^2(\Omega) \cap H^1_0(\Omega). \] (4.10)

Let us denote \( A^* \) the adjoint operator of \( A \) – i.e.

\[
A^* = \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right) + a
\] (4.11)
(see (3.5)). Let $u^*$ be the weak solution to the Dirichlet problem
\[
\begin{aligned}
-A^*u^* &= f & \text{in } \Omega, \\
u^* &\in H^1_0(\Omega).
\end{aligned}
\]

By the definition of $A^*$ one has in particular
\[
(u^*, Av) = (f,v) \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega).
\]

Moreover, $j$ being convex and $j'$ monotone it is well known that for $\varepsilon > 0$
\[
(I + \varepsilon j')^{-1}
\]
is a contraction on $\mathbb{R}$ ($I$ is the identity, see Brézis [1]). Thus $(I + \varepsilon j')^{-1}(u^*) \in H^1_0(\Omega)$ and there exists a unique $\tilde{u}_\varepsilon$ solution to
\[
\begin{aligned}
A\tilde{u}_\varepsilon &= (I + \varepsilon j')^{-1}u^* & \text{in } \Omega, \\
\tilde{u}_\varepsilon &\in H^2(\Omega) \cap H^1_0(\Omega).
\end{aligned}
\]
Of course $(I + \varepsilon j')A\tilde{u}_\varepsilon = u^*$ and by (4.13) it holds
\[
((I + \varepsilon j')A\tilde{u}_\varepsilon, Av) = (f,v) \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega).
\]
Thus – see (4.10) – by uniqueness of the solution to (4.9), $\tilde{u}_\varepsilon = u_\varepsilon = u$. Then if $f \in L^p(\Omega)$ one has $u^* \in W^{2,p}(\Omega) \subset W^{1,p'}(\Omega)$ with
\[
\|u^*\|_{1,p'} \leq C|f|_p.
\]
Since $(I + \varepsilon j')^{-1}$ is a contraction it follows from (4.15) that
\[
\|Au\|_{1,p'} \leq C|f|_p
\]
and (4.7) follows. In the case where $(I + \varepsilon j')^{-1}$ has a second derivative bounded one has for $f \in L^p(\Omega)$, $(I + \varepsilon j')^{-1}(u^*) \in W^{2,p}(\Omega)$ and (4.8) follows easily from (4.15).

\textbf{Remark 4.6.} It is easy to see that $W^{4,p}(\Omega)$-regularity holds for $j(u) = u^2$ – i.e. for
\[
J[u] = \int_{\Omega} (Au)^2 \, dx.
\]

Let us consider now two functions satisfying
\[
\begin{aligned}
\varphi_1, \varphi_2 &\in L^2(\Omega) , \quad (4.16) \\
\varphi_1 &\leq \varphi_2 \quad \text{a.e. in } \Omega. \quad (4.17)
\end{aligned}
\]

Then one can show
Theorem 4.2. Let $u^*$ the solution to (4.12) and let $u$ be the solution to the variational inequality:

$$
\begin{aligned}
(Au, A(v - u)) &\geq (f, v - u) \quad \forall v \in K_0, \\
u &\in K_0 = \{ v \in H^2(\Omega) \cap H_0^1(\Omega) \mid \varphi_1 \leq Au \leq \varphi_2 \textit{ a.e. in } \Omega \}.
\end{aligned}
$$

(4.18)

If $f \in L^p(\Omega)$, $p \geq 2$ (resp. $f \in W^{-1,p}(\Omega)$) then (4.18) has a unique solution. Moreover, if

$$
\partial j(x, u^*) \in W^{2,p}(\Omega), \quad (\text{resp. } W^{1,p}(\Omega))
$$

(4.19)

then $u$ belongs to $W^{4,p}(\Omega)$, (resp. $W^{3,p}(\Omega)$).

Proof. If $(\cdot)^+$, $(\cdot)^-$ denote the positive and negative parts of functions set

$$
\begin{aligned}
j(x, t) &= \frac{1}{2}[(t - \varphi_2)^+]^2 + \frac{1}{2}[(t - \varphi)^-]^2, \\
\partial j(x, t) &= (t - \varphi_2)^+ - (t - \varphi_1)^-, \\
J[u] &= \int_{\Omega} j(x, Au) \, dx.
\end{aligned}
$$

It is easy to see that $J[\cdot]$ is continuous on $H^2(\Omega) \cap H_0^1(\Omega)$ and thus also weakly lower semicontinuous. Now, clearly

$$
K_0 = \{ v \in H^2(\Omega) \cap H_0^1(\Omega) \mid J[v] = 0 \}.
$$

Thus it follows from Theorem 2.1 that (4.18) admits a unique solution. From Theorem 2.1 it results also that for every $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ to

$$
(Au_\varepsilon, A(v - u_\varepsilon)) + \varepsilon \{ J[v] - J[u_\varepsilon] \} \geq (f, v - u) \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).
$$

Moreover, replacing $v$ by $u_\varepsilon + t(v - u_\varepsilon)$ it comes – see Remark 2.2 –

$$
(t(Au_\varepsilon, A(v - u_\varepsilon)) + \varepsilon \{ J[u_\varepsilon] - J[u_\varepsilon] \} - J[u_\varepsilon]) \geq t(f, v - u_\varepsilon)
$$

(4.20)

this for any $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Now, one has clearly, due to the convexity of $j(x, \cdot)$

$$
J[u_\varepsilon + t(v - u_\varepsilon)] - J[u_\varepsilon] \leq t \int_{\Omega} \partial j(x, Au_\varepsilon + t(Av - Au_\varepsilon)) \cdot (Av - Au_\varepsilon) \, dx.
$$

(4.21)

Combining (4.20) and (4.21) dividing by $t$ and letting $t \to 0$ it comes

$$
(Au_\varepsilon, A(v - u_\varepsilon)) + \varepsilon (\partial j(x, Au_\varepsilon), Av - Au_\varepsilon) \geq (f, v - u_\varepsilon).
$$

This for any $v \in H^2(\Omega) \cap H_0^1(\Omega)$. It follows easily that

$$
(Au_\varepsilon + \varepsilon \partial j(x, Au_\varepsilon), Av) = (f, v) \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).
$$
Thus $Au_\varepsilon$ is the solution to

$$Au_\varepsilon + \varepsilon \partial j(x, Au_\varepsilon) = u^*.$$ 

An easy computation shows that

$$(I + \varepsilon \partial j(x, \cdot))^{-1} = I - \frac{\varepsilon}{1 + \varepsilon} \partial j(x, \cdot)$$

and thus

$$Au_\varepsilon = u^* - \frac{\varepsilon}{1 + \varepsilon} \partial j(x, u^*).$$

Thus we obtain

$$\|Au_\varepsilon\|_{2,p} \leq C \quad \text{(resp. } \|Au_\varepsilon\|_{1,p} \leq C), \quad (4.22)$$

where $C$ is independent of $\varepsilon$. Letting $\varepsilon \to +\infty$ it follows from Theorem 2.5 that $u_\varepsilon \to u$ in $H^2(\Omega) \cap H^1_0(\Omega)$. Passing to the limit in (4.22) leads -- by the lower semicontinuity of the norm -- to

$$\|Au\|_{2,p} \leq C \quad \text{(resp. } \|Au\|_{1,p} \leq C)$$

and the result is proved. \hfill \Box

**Remark 4.7.** If $f \in L^p(\Omega)$, $p \geq 2$ then $u^* \in W^{2,p}(\Omega) \subset W^{1,p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. If one assumes that $\varphi_1, \varphi_2 \in W^{1,p^*}(\Omega)$ then clearly

$$\partial j(x, u^*) \in W^{1,p^*}(\Omega)$$

and then $u \in W^{3,p^*}(\Omega)$. (Compare with Brézis and Stampacchia [5]).

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**ADDED IN PROOFS**

After this paper was completed we heard about the paper:


In this work -- although the goal was not a regularity theory -- one can find ideas close to ours.
REFERENCES


