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ON BOUNDEDNESS AND STABILITY IN TERMS OF TWO MEASURES FOR DISCRETE SYSTEMS OF VOLTERRA TYPE

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ABSTRACT: In this paper we consider the discrete system of Volterra type. By using Razumikhin techniques, we establish several boundedness and stability criteria for this system in terms of two measures.

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1. INTRODUCTION

Discrete systems of Volterra type have being studied for the past few years. However, there have not appeared any boundedness results for this discrete system so far by using Lyapunov functionals or functions, and the stability theory of this system is still in an initial stage of development. We know only that the comparison results for stability theory of this system have been recently established in Burton [1]. (For the relevant results of integro-differential systems of Volterra type we refer to Lakshimikantham and Liu [3], Lakshmikantham and Rama Mohana Rao [4], Fu and Liu [2].) In this paper we consider the boundedness and stability of the solutions of a general class of discrete systems of Volterra type. Apply Razumikhin techniques to this system, we establish some Razumikhin-type boundedness and stability theorems in terms of two measures. To our knowledge, so called Razumikhin technique is an important technique when discuss the stability of delay differential systems. We exploiting this technique here.

2. PRELIMINARIES

Consider the discrete system of Volterra type

$$\begin{cases} \triangle x(n) = f(n, x(n), & \sum_{s=n_0}^{n-1} G(n, s, x(s))) \quad , \\ x(n_0) = x_{n_0}, & n_0 \in Z^+, \end{cases} \tag{2.1}$$

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where $f : Z^+ \times R^K \times R^K \to R^K,\ G : Z^+ \times Z^+ \times R^K \to R^K, R^K$ is space of K vectors. For $n_0 \in Z^+$ and $x_0 \in R^K$,we denote by $x(n) = x(n, n_0, x_0)$ the solution of (2.1) satisfying the initial condition $x(n_0, n_0, x_0) = x_0$.We shall assume existence of solutions of (2.1) for all $n \geq n_0$.

Let us list the following classes of functions and definitions for convenience.

$\Re = \{\varphi \in C[R_+, R_+] : \varphi(r)$ is strictly increasing in r and $\varphi(0) = 0\}$;

$\Gamma = \left\{h \in C[R_+ \times R^K, R_+] : \inf_{(t,x)} h(t, x) = 0\right\}$;

$S(h, \rho) = \left\{(n, x) \in Z^+ \times R^K : h(n, x) < \rho\right\}$;

$S^C(h_0, \rho) = \left\{(n, x) \in Z^+ \times R^K : h_0(n, x) \geq \rho\right\}$.

Definition 2.1. Let $h_0, h \in \Gamma, x(n) = x(n, n_0, x_0)$ is any solution of (2.1).Then the system (2.1) is said to be

(*i*) (h_0, h)-uniform bounded if for each $M_1 > 0$ there is a $M_2 > 0$ such that for $n_0 \in Z^+$

$$h_0(n_0, x_0) < M_1 \text{ implies } h(n, x(n)) < M_2 \text{ for } n > n_0;$$

(*ii*) (h_0, h)-uniform ultimate bounded for bound M if for each $M_3 > 0$ there is an integer $N > 0$ such that for $n_0 \in Z^+$

$$h_0(n_0, x_0) < M_3 \text{ implies } h(n, x(n)) < M \text{ for } n \geq n_0 + N.$$

Definition 2.2. Let $h_0, h \in \Gamma, x(n) = x(n, n_0, x_0)$ is any solution of (2.1). Then the system (2.1) is said to be

(*i*) (h_0, h)-uniformly stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for $n_0 \in Z^+$

$$h_0(n_0, x_0) < \delta \text{ implies } h(n, x(n)) < \varepsilon \text{ for } n \geq n_0;$$

(*ii*) (h_0, h)-uniformly attractive if there is constant $\delta_0 > 0$ and for each $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) > 0$ such that for $n_0 \in Z^+$

$$h_0(n_0, x_0) < \delta_0 \text{ implies } h(n, x(n)) < \varepsilon \text{ for } n \geq n_0 + N(\varepsilon);$$

(*iii*) (h_0, h)-uniformly asymptotically stable if (*i*) and (*ii*) hold together.

Definition 2.3. Let $h_0, h \in \Gamma$. Then we say that h_0 is uniformly finer than h if there exists a number $\delta > 0$ and a function $\varphi \in \Re$ such that

$$h_0(n, x) < \delta \text{ implies } h(n, x) \leq \varphi(h_0(n, x)).$$

Definition 2.4. Let $h_0, h \in \Gamma, V : Z^+ \times R^K \to [0, \infty)$. Then $V(n, x)$ is said to be

(*i*) h-positive definite if there exists a constant $\delta > 0$ and a function $\varphi_1 \in \Re$ such that

$$h(t, x) < \delta \text{ implies } \varphi_1(h(n, x)) \leq V(n, x);$$

(*ii*) h_0-decrescent if there exists a constant $\delta > 0$ and a function $\varphi_2 \in \Re$ such that

$$h_0(n, x) < \delta \text{ implies } V(n, x) \leq \varphi_2(h_0(n, x)).$$

Definition 2.5. Let $V : Z^+ \times R^K \to R_+$.We define the variation of V along the solutions $x(n)$ of (2.1)

$$\triangle_{(2.1)}V(n, x(n)) = V(n + 1, x(n + 1)) - V(n, x(n)).$$

3. MAIN RESULTS

We state and prove our main results in this section. Let us establish two Razumikhin-type boundedness theorems for system (2.1) in terms of two measures.

Theorem 3.1. *Assume that* $h_0, h \in \Gamma$ *and that there exists* $V : Z^+ \times R^K \to [0, \infty)$ *satisfying the following conditions:*

(i) $\varphi_1(h(n,x)) \leq V(n,x) \leq \varphi_2(h_0(n,x))$, *where* $\varphi_1, \varphi_2 \in \Re$ *and* $\varphi_1(r) \to \infty$ *as* $r \to \infty$;

(ii) $\triangle_{(2.1)} V(n, x(n)) \leq -\varphi_3(h_0(n, x(n))$,
whenever $(n+1, x(n+1)) \in S^C(h_0, \rho)$ *and* $p(V(n+1, x(n+1))) > V(s, x(s))$ *for* $n_1 \leq s \leq n, n_1 \geq n_0$,*where* $p : (0, \infty) \to (0, \infty)$ *with* $p(s) > s$ *for* $s > 0, \varphi_3 \in \Re$ *and* *x(n) is any solution of (2.1).*

Then the system (2.1) is (h_0, h)*-uniform bounded and* (h_0, h)*-uniform ultimate bounded.*

Proof. To show the (h_0, h)-uniform bounded, let $M_1 \geq \rho$ be given. For any $n_0 \in Z^+, (n_0, x_0) \in S(h_0.M_1)$,denote $x(n) = x(n, n_0, x_0)$ and $V(n) = V(n, x(n))$.By assumption (i) we have

$$\varphi_1(h(n_0, x_0)) \leq V(n_0) \leq \varphi_2(h_0(n_0, x_0)) \leq \varphi_2(M_1).$$

Now we have to show that

$$V(n) \leq \varphi_2(M_1) \text{ for all } n \geq n_0.$$

If this is not true, then there is an $n_1 \geq n_0$ such that

$$V(n) \leq \varphi_2(M_1) \text{ for } n_0 \leq n \leq n_1 \tag{3.1}$$

but

$$V(n_1 + 1) > \varphi_2(M_1). \tag{3.2}$$

Thus from (3.1) and (3.2) it follows that

$$p(V(n_1 + 1)) > V(n_1 + 1) > V(s) \text{ for } n_0 \leq s \leq n_1.$$

Note that

$$\varphi_2(h_0(n_1 + 1, x(n_1 + 1))) \geq V(n_1 + 1) > \varphi_2(M_1)$$

or

$$h_0(n_1 + 1, x(n_1 + 1)) \geq M_1 \geq \rho,$$

i.e., $(n_1 + 1, x(n_1 + 1)) \in S^C(n_0, \rho)$. Hence, by assumption (ii) we get

$$\triangle_{(2.1)} V(n_1) \leq -\varphi_3(h_0(n_1, x(n_1)) \leq 0 \textit{ or } V(n_1 + 1) \leq V(n_1) \leq \varphi_2(M_1).$$

This contradicts (3.2). Hence we show that

$$\varphi_1(h(n, x(n))) \leq V(n) \leq \varphi_2(M_1) \text{ for all } n \geq n_0$$

or

$$h(n, x(n)) \leq \varphi_1^{-1}(\varphi_2(M_1)) \equiv M_2 \text{ for all } n \geq n_0.$$

This proves (h_0, h)-uniform bounded.
Next, we prove (h_0, h)-uniform ultimate bounded. Let $M = \varphi_1^{-1}(\varphi_2(\rho))$.
For any given $M_3 \geq \rho$, by (h_0, h)-uniform bounded we can find $M_4 > M$ such that $h_0(n_0, x_0) < M_3$ for $n_0 \in Z^+$ imply that

$$h((n, x(n)) \leq M_4 \text{ and } V(n) \leq \varphi_1(M_4) \text{ for all } n \geq n_0. \tag{3.3}$$

Setting

$$\beta = \inf_{\varphi_1(M) \leq u \leq \varphi_1(M_4)} \{p(u) - u\} > 0, \tag{3.4}$$

we obtain

$$p(u) > u + \beta \text{ for } \varphi_1(M) \leq u \leq \varphi_1(M_4).$$

Furthermore, there exists a positive integer K satisfying

$$\varphi_1(M_4) < \varphi_1(M) + K\beta. \tag{3.5}$$

Let us construct $K + 1$ numbers:

$$n_i = n_0 + i\left(\left[\frac{\varphi_1(M_4)}{\varphi_3(\rho)}\right] + 1\right),\ i = 0, 1, 2, \ldots, K,$$

where $[\cdot]$ denotes the greatest integer function. Now we claim that

$$h(n, x(n)) \leq M \text{ for all } n \geq n_0 + N,$$

where $N = K\left(\left[\frac{\varphi_1(M_4)}{\varphi_3(\rho)}\right] + 1\right)$. It is, therefore, sufficient to show that

$$V(n) \leq \varphi_1(M) + (K - i)\beta \text{ for } n \geq n_i,\ i = 0, 1, 2, \ldots, K. \tag{3.6}$$

For $i = 0$, from (3.3) and (3.5) we see that (3.6) holds. Suppose that for some $i, 0 \leq i < K$, we have

$$V(n) \leq \varphi_1(M) + (K - i)\beta,\ n \geq n_i. \tag{3.7}$$

We have to show

$$V(n) \leq \varphi_1(M) + (K - i - 1)\beta,\ n \geq n_{i+1}. \tag{3.8}$$

We assert that if there is some $\tilde{n} \geq n_i$ with

$$V(\tilde{n}) \leq \varphi_1(M) + (K - i - 1)\beta, \tag{3.9}$$

then there must hold

$$V(n) \leq \varphi_1(M) + (K - i - 1)\beta \text{ for all } n \geq \tilde{n}.$$

If it is not true, then there is some $n^* \geq \tilde{n}$ such that

$$V(n) \leq \varphi_1(M) + (K - i - 1)\beta \text{ for } \tilde{n} \leq n \leq n^*$$

but

$$V(n^* + 1) > \varphi_1(M) + (K - i - 1)\beta. \tag{3.10}$$

Then it implies that

$$p(V(n^*+1)) \geq V(n^*+1) + \beta > \varphi_1(M) + (K-i)\beta \geq V(s) \text{ for } \tilde{n} \leq s \leq n^*.$$

By (i) and (3.10), we have

$$\varphi_2(h_0(n^*+1, x(n^*+1))) \geq V(n^*+1) > \varphi_1(M) + (K-i-1)\beta \geq \varphi(M).$$

Then

$$h_0(n^*+1, x(n^*+1)) > \varphi_2^{-1}(\varphi_1(M)) = \rho,$$

i.e.$(n^*+1, x(n^*+1)) \in S^C(h_0, \rho)$.Thus, by (ii) we obtain

$$\triangle_{(2.1)} V(n^*) \leq -\varphi_3(h_0(n^*, x(n^*))) \leq 0,$$

or

$$V(n^*+1) \leq V(n^*) \leq \varphi_1(M) + (K-i-1)\beta.$$

This contradicts (3.10).
Now we want to show that there does exist some $\tilde{n} \in [n_i, n_{i+1}]$ such that

$$V(\tilde{n}) \leq \varphi_1(M) + (K-i-1)\beta.$$

In fact, suppose that it is not true, for all $n \geq n_i$ we have

$$V(n) > \varphi_1(M) + (K-i-1)\beta. \tag{3.11}$$

By (3.3) and (3.11) we can see that

$$\varphi_1(M) < V(n) \leq \varphi_1(M_4) \text{ for all } n \geq n_i.$$

Then using (3.4), (3.11) and (3.7) we obtain

$$p(V(n+1)) \geq V(n+1) + \beta > \varphi_1(M) + (K-i)\beta \geq V(s) \text{ for } n_i \leq s \leq n, n \geq n_i.$$

With the same arguments as above we have

$$h_0(n, x(n)) > \varphi_2^{-1}(\varphi_1(M)) = \rho \text{ for all } n \geq n_i.$$

Hence, by assumption (ii) we get

$$\triangle_{(2.1)} V(n) \leq -\varphi_3(h_0(n, x(n))) < -\varphi_3(\rho) \text{ for all } n \geq n_i$$

or

$$V(n+1) \leq V(n) - \varphi_3(\rho) \text{ for all } n \geq n_i$$

which implies that

$$V(n_i+m) \leq V(n_i) - \sum_{j=n_i}^{n_i+m-1} \varphi_3(h_0(j, x(j))) \leq \varphi_1(M_4) - m\varphi_3(\rho) < 0,$$

if $m = \left[\frac{\varphi_1(M_4)}{\varphi_3(\rho)}\right] + 1$. This contradiction implies that there must be some $\tilde{n} \in [n_i, n_{i+1}]$ with (3.9) holding. Thus (3.8) holds. By induction, we obtain

$$\varphi_1(h(n, x(n))) \leq V(n) \leq \varphi_1(M) \text{ for } n \geq n_K,$$

or

$$h(n, x(n)) \leq M \text{ for all } n \geq n_0 + N,$$

where $N = K\left(\left[\frac{\varphi_1(M_4)}{\varphi_3(\rho)}\right] + 1\right)$ obviously depends on M_3 only. Theorem 3.1 is proved.

The following Razumikhin-type boundedness theorem is an even more convenient version for applications.

Theorem 3.2. *Assume that V, p be the same as in Theorem 3.1 and that $h_0, h \in \Gamma$.If*

(i) $\varphi_1(h(n,x)) \leq V(n,x) \leq \varphi_2(h_0(n,x))$, where $\varphi_1, \varphi_2 \in \Re$ and $\varphi_1(r) \to \infty$ as $r \to \infty$;

(ii) $\triangle_{(2.1)} V(n, x(n)) \leq R - \varphi_3(h_0(n, x(n)))$ for some constant $R > 0$, whenever $(n+1, x(n+1)) \in S^c(h_0, \rho)$ and $p(V(n+1, x(n+1))) > V(s, x(s))$ for $n_1 \leq s \leq n, n_1 \geq n_0$, where $\varphi_3 \in \Re$ and $x(n)$ is any solution of (2.1).

Then the system (2.1) *is (h_0, h)-uniform bounded and (h_0, h)-uniform ultimate bounded.*

Proof. First, we show the (h_0, h)-uniform bounded.
Let $M_1 > max\left\{\varphi_3^{-1}(R), \rho\right\}$ be given. For any $n_0 \in Z^+, (n_0, x_0) \in S(h_0, M_1)$,denote $x(n) = x(n, n_0, x_0))$ and $V(n) = V(n, x(n))$. By (i) we have

$$\varphi_1(h(n_0, x_0)) \leq V(n_0) \leq \varphi_2(h_0(n_0, x_0)) \leq \varphi_2(M_1).$$

Now we have to show that

$$V(n) \leq \varphi_2(M_1) + R \text{ for all } n \geq n_0. \tag{3.12}$$

If this is not true, then there is an $n_1 \geq n_0$ such that

$$V(n) \leq \varphi_2(M_1) + R \text{ for all } n_0 \leq n \leq n_1 \tag{3.13}$$

but

$$V(n_1 + 1) > \varphi_2(M_1) + R. \tag{3.14}$$

We discuss two possibilities:
(I) $\quad (n_1) \leq \varphi_2(M_1)$ and $V(n_1+1) > \varphi_2(M_1) + R$.
Using (3.13) and (3.14), we have

$$p(V(n_1 + 1)) > V(n_1 + 1) > \varphi_2(M_1) + R \geq V(s) \text{ for } n_0 \leq s \leq n_1.$$

By (i) and (3.14) we see that

$$\varphi_2(h_0(n_1 + 1, x(n_1 + 1))) \geq V(n_1 + 1) > \varphi_2(M_1) + R > \varphi_2(M_1)$$

which implies that $(n_1 + 1, x(n_1 + 1)) \in S^c(h_0, \rho)$. Thus, by (ii) we obtain

$$\triangle_{(2.1)} V(n_1) \leq R - \varphi_3(h_0(n_1, x(n_1)))$$

or

$$V(n_1 + 1) \leq V(n_1) + R - \varphi_3(h_0(n_1, x(n_1))) \leq V(n_1) + R \leq \varphi_2(M_1) + R.$$

This contradicts (3.14).
(II) $\quad V(n_1) > \varphi_2(M_1)$ and $V(n_1+1) > \varphi_2(M_1) + R$.

As in the case (I), we can see that $(n_1+1, x(n_1+1)) \in S^c(n_0, \rho)$ and $p(V(n_1+1)) > V(s)$ for $n_0 \le s \le n_1$.By (ii) we have

$$V(n_1+1) \le V(n_1) + R - \varphi_3(h_0(n_1, x(n_1))).$$

Note that

$$\varphi_2(h_0(n_1, x(n_1))) \ge V(n_1) > \varphi_2(M_1) \text{ implies } h_0(n_1, x(n_1)) > M_1.$$

Thus, from the choice of M_1, we obtain

$$\varphi_3(h_0(n_1, x(n_1))) > \varphi_3(M_1) > R.$$

Then we get

$$V(n_1+1) \le V(n_1) + R - \varphi_3(h_0(n_1, x(n_1))) < V(n_1) \le \varphi_2(M_1) + R.$$

Again, leads to a contradiction. Hence (3.12) is true, and we have

$$h(n, x(n)) \le \varphi_1^{-1}(\varphi_2(M_1)) \equiv M_2 \text{ for all } n \ge n_0.$$

This proves (h_0, h)-uniform bounded.
Next, we show the (h_0, h)-uniform ultimate bounded. Let $\rho > 0$ be sufficiently large so that

$$R - \varphi_3(\rho) < 0.$$

Setting $M = \varphi_1^{-1}(\varphi_2(\rho))$, i.e. $\rho = \varphi_2^{-1}(\varphi_1(M))$. For any given $M_3 \ge \rho$, by (h_0, h)-uniform bounded, we can find $M_4 > M$ such that $h_0(n_0, x_0) < M_3$ for $n_0 \in Z^+$ imply that (3.3) holds. Then we can denote β by (3.4) and choose a positive integer K such that (3.5) holds as in the proof of Theorem 3.1. Let us construct $K+1$ numbers:

$$n_i = n_0 + i\left(\left[\frac{\varphi_1(M_4)}{\varphi_3(\rho) - R}\right] + 1\right), i = 0, 1, 2, \dots, K,$$

where $[\cdot]$ denotes the greatest integer function. We claim that

$$h(n, x(n)) \le M \text{ for all } n \ge n_0 + N,$$

where $N = K\left(\left[\frac{\varphi_1(M_4)}{\varphi_3(\rho)-R}\right] + 1\right)$. It is, therefore, sufficient to show that

$$V(n) \le \varphi_1(M) + (K - i)\beta \text{ for all } n \ge n_i, i = 0, 1, 2, \dots, K.$$

Since all the arguments in Theorem 3.1 could get through, the following proof is omitted here. The proof of Theorem 3.2 is complete.

We shall establish a Razumikhin-type stability theorem for the system (2.1).

Theorem 3.3. *Assume that*

(i) $h_0, h \in \Gamma$ and h_0 *is uniformly finer than* h;

(ii) $V : S(h, \rho) \to R^+$ *is* h*-positive definite and* h_0*-decrescent;*

(iii) $\triangle_{(2.1)} V(n, x(n)) \le -\varphi(h_0(n, x(n)))$,

whenever $p(V(n+1), x(n+1))) > V(s, x(s))$ for $n_1 \le s \le n$, $n_1 \ge n_0$, where p :

$(0,\infty) \to (0,\infty)$ with $p(s) > s$ for $s > 0, \varphi \in \Re$ and $x(n)$ *is any solution of (2.1). Then the system (2.1) is* (h_0, h)*-uniformly stable and* (h_0, h)*-uniformly asymptotically stable.*

Proof. First, we show the (h_0, h)-uniformly stable. Since $V(n,x)$ is h-positive definite and h_0-decrescent, there exists a $\sigma > 0$ and $\varphi_1 \in \Re$ such that

$$\varphi_1(h(n,x)) \le V(n,x), \text{if } h(n,x) < \sigma \tag{3.15}$$

and $\varphi_2 \in \Re, \delta_0 > 0$ such that

$$V(n,x) \le \varphi_2(h_0(n,x)), \text{if } h_0(n,x) < \delta_0. \tag{3.16}$$

Also, h_0 is uniformly finer than h implies that there exists a $\delta_1 > 0$ and $\varphi_3 \in \Re$ such that

$$h(n,x) \le \varphi_3(h_0(n,x)), \text{ whenever } h_0(n,x) < \delta_1, \tag{3.17}$$

where δ_1 is such that $\varphi_3(\delta_1) < \sigma$.

Let $0 < \varepsilon < \rho^* = min\{\sigma, \rho\}$. By the assumption on φ_2, there is a $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$\varphi_2(\delta_2) < \varphi_1(\varepsilon). \tag{3.18}$$

Let $\delta = \delta(\varepsilon) = min\{\delta_0, \delta_1, \delta_2\}$. For any $n_0 \in Z^+, x_0 \in R^K$, consider the solution $x(n) = x(n, n_0, x_0)$ of system (2.1) with $h_0(n_0, x_0) < \delta$. It is clear from (3.15) – (3.18) that when $h_0(n_0, x_0) < \delta$ we have

$$\varphi_1(h(n_0,x_0)) \le V(n_0,x_0) \le \varphi_2(h_0(n_0,x_0)) < \varphi_1(\varepsilon)$$

which implies that $h(n_0, x_0) < \varepsilon$. Now we have to show that

$$h(n, x(n)) < \varepsilon, \quad n \ge n_0,$$

sufficient to show that

$$V(n) \overset{def.}{=} V(n, x(n)) < \varphi_1(\varepsilon) \text{ for all } n \ge n_0. \tag{3.19}$$

If this is not true, then there exists $n_1 \ge n_0$ such that

$$V(n) < \varphi_1(\varepsilon) \text{ for } n_0 \le n \le n_1$$

but

$$V(n_1 + 1) \ge \varphi_1(\varepsilon). \tag{3.20}$$

Then it implies that

$$p(V(n_1+1)) > V(n_1+1) \ge \varphi_1(\varepsilon) > V(s) \text{ for } n_0 \le s \le n_1.$$

By assumption (iii), we obtain

$$\triangle_{(2.1)} V(n_1) \le -\varphi(h_0(n_1, x(n_1))) \le 0.$$

Hence

$$V(n_1+1) \le V(n_1) < \varphi_1(\varepsilon)$$

which contradicts (3.20). Therefore (3.19) holds.
Next, we shall show that the (h_0, h_0)-uniformly asymptotically stable. For this purpose, we have still to show that the system (2.1) is (h_0, h)-uniformly attractive. Since V is h-positive definite and h_0-decrescent, (3.15) and (3.16) hold. We find $\delta = \delta(\rho^*)$ by (h_0, h)-uniformly stable. Let $x(n) = x(n, n_0, x_0)$ be a solution of system (2.1) for which $h_0(n_0, x_0) < \delta^*$,where $\delta^* = min\{\delta(\rho^*), \delta_0\}$. For given $r > 0$ such that $r < \rho^*$ and $\varphi_1(r) < \varphi_2(\rho^*)$. Then from the first part of the proof it follows that

$$h(n, x(n)) < r \text{ and } V(n) < \varphi_2(\rho^*) \text{ for all } n \geq n_0. \tag{3.21}$$

Setting

$$\xi = \inf_{\varphi_1(r) \leq u \leq \varphi_2(\rho^*).} \{p(u) - u\} > 0$$

and K be a positive integer satisfying the inequality

$$\varphi_1(r) + K\xi > \varphi_2(\rho^*). \tag{3.22}$$

If, for some $n > n_0$, we have $V(n) \geq \varphi_1(r)$, then

$$\varphi_2(h_0(n, x(n))) \geq V(n) \geq \varphi_1(r) \text{ or } h_0(n, x(n)) \geq \varphi_2^{-1}(\varphi_1(r)) \overset{def.}{=} \lambda(r) = \lambda.$$

Thus

$$\varphi(h_0(n, x(n))) \geq \varphi(\lambda), \tag{3.23}$$

where $\varphi(\lambda)$ depends on r only. Let

$$n_i = n_0 + i \left[\frac{\varphi_2(\rho^*)}{\varphi(\lambda)} \right], i = 0, 1, 2, \ldots, K,$$

where $[\cdot]$ denotes the greatest integer function. Now we shall show that

$$h(n, x(n)) < r \text{ for all } n \geq n_0 + N,$$

where $N = K \left[\frac{\varphi_2(\rho^*)}{\varphi(\lambda)} \right]$ .It is, therefore, sufficient to show that

$$V(n) < \varphi_1(r) + (K - i)\xi, \ n \geq n_i, i = 0, 1, 2, \ldots, K. \tag{3.24}$$

From (3.21) and (3.22), we obtain

$$V(n) < \varphi_2(\rho^*) < \varphi_1(r) + K\xi \text{ for } n \geq n_0,$$

which implies that (3.24) holds for $n = 0$. Suppose that for some $i, 0 \leq i < K$, we have

$$V(n) < \varphi_1(r) + (K - i)\xi, \ n \geq n_i.$$

We want to show that

$$V(n) < \varphi_1(r) + (K - i - 1)\xi, \ n \geq n_{i+1}. \tag{3.25}$$

First, we claim that there exists some $n^* \in [n_i, n_{i+1}]$ such that

$$V(n^*) < \varphi_1(r) + (K - i - 1)\xi. \tag{3.26}$$

Suppose not, then

$$V(n) \geq \varphi_1(r) + (K - i - 1)\xi \text{ for all } n \geq n_i.$$

We have

$$\varphi_1(r) \leq V(n) \leq \varphi_2(\rho^*) \text{ for all } n \geq n_i, \tag{3.27}$$

and, thus,

$$p(V(n+1)) > V(n+1) + \xi \geq \varphi_1(r) + (K - i)\xi > V(s) \text{ for } n_i \leq s \leq n, n \geq n_i.$$

Note (3.27), we can see that (3.23) holds for all $n \geq n_i$.Then using assumption (iii), we get

$$V(n_i + m + 1) \leq V(n_i) - \sum_{j=n_i}^{n_i+m} \varphi(h_0(j, x(j))) \leq \varphi_2(\rho^*) - (m+1)\varphi(\lambda) < 0$$

if $m = \left[\frac{\varphi_2(\rho^*)}{\varphi(\lambda)}\right]$ .This contradiction show that there must be some $n^* \in [n_i, n_{i+1}]$ such that (3.26) holds.
Next, we have to show that (3.26) implies that

$$V(n) < \varphi_1(r) + (K - i - 1)\xi, n \geq n^*, n^* \in [n_i, n_{i+1}].$$

If this is not true, then there is some $\tilde{n} \geq n^*$ such that

$$V(n) < \varphi_1(r) + (K - i - 1)\xi \text{ for } n^* \leq n \leq \tilde{n},$$

but

$$V(\tilde{n} + 1) \geq \varphi_1(r) + (K - i - 1)\xi. \tag{3.28}$$

Thus, we have

$$p(V(\tilde{n} + 1)) > V(\tilde{n} + 1) \geq \varphi_1(r) + (K - i - 1)\xi > V(s) \text{ for } n^* \leq s \leq \tilde{n}.$$

Then using (iii) we obtain

$$V(\tilde{n} + 1) \leq V(\tilde{n}) < \varphi_1(r) + (K - i - 1)\xi.$$

This contradicts (3.28). Hence (3.25) holds, and thus, (3.24) is true for any $i = 0, 1, 2, \ldots, K$ by induction. Therefore, we have

$$V(n) < \varphi_1(r) \text{ for all } n \geq n_K$$

or

$$h(n, x(n)) < r \text{ for all } n \geq n_0 + N,$$

where $N = K\left[\frac{\varphi_2(\rho^*)}{\varphi(\lambda)}\right]$ obviously depends on r only. This completes the proof of Theorem 3.3.
Remark. Our results crucially depends on choosing an appropriate minimal class of functions along which the variation of Lyapunov functions relative to the discrete system (2.1) allows a convenient estimate.

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