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ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF THE LINEARIZED GINZBURG-LANDAU OPERATOR

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ABSTRACT: We consider the linearized operators, denoted by $\mathcal{L}_{d,1}$, of the Ginzburg-Landau operator $\Delta u + u(1- \mid u \mid^2)$ in $\mathbf{R}^2$, about the radial solutions $u_{d,1}(x) = f_d(r)e^{id\theta}$. We prove that for all $d \geq 1$ the real vector space of the bounded solutions of the equation $\mathcal{L}_{d,1}w = 0$ is spanned by the three functions that correspond to the invariance of the equation $\Delta u + u(1- \mid u \mid^2) = 0$ under the action of the rotations and the translations.

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1. INTRODUCTION

We consider the Ginzburg-Landau equation

$$-\Delta u = \frac{1}{\varepsilon^2} u(1- \mid u \mid^2) \text{ in } \mathbf{R}^2, \tag{1.1}$$

where u is a complex valued function. We know that for all $d = 1, 2, ...$, the problem (1.1) has a radial solution, that is a solution of the form

$$u_{d,\varepsilon}(r, \theta) = e^{id\theta} f_d(\frac{r}{\varepsilon}). \tag{1.2}$$

The construction and the properties of f_d can be found in Hervé and Hervé [12]. Let us recall them. The function f_d is the unique non constant solution of the problem

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$$\begin{cases} -f'' - \dfrac{f'}{r} + d^2 \dfrac{f}{r^2} = f(1-f^2) \text{ in } [0,+\infty[\\ \lim_{r\to+\infty} f(r) = 1, \quad f(0)=0, \quad f \geq 0. \end{cases} \tag{1.3}$$

Moreover, we have $f_d'(r) > 0$ for all $r \in [0,+\infty[$ and there exists some constant $a > 0$ such that

$$f_d(r) = 1 - \frac{d^2}{2r^2} + O(\frac{1}{r^4}) \text{ near } +\infty$$

and

$$f_d(r) = ar^d - \frac{a}{4(d+1)} r^{d+2} + O(r^{d+4}) \text{ near } 0.$$

We define the nonlinear mapping

$$\mathcal{N}(u) = \Delta u + \frac{u}{\varepsilon^2}(1 - \mid u \mid^2).$$

The linearized operator about a function u is defined by

$$Lw = \Delta w + \frac{w}{\varepsilon^2}(1 - \mid u \mid^2) - \frac{2}{\varepsilon^2} uu.w,$$

where w is any complex valued function and $2u.w = \overline{u}w + \overline{w}u$. We denote by $L_{d,\varepsilon}$ the linearized operator about the radial solution $u_{d,\varepsilon}$.

Let us define the conjugate operator by

$$\mathcal{L}_{d,\varepsilon} = e^{-id\theta} L_{d,\varepsilon} e^{id\theta}.$$

We will consider the eigenvalue problem

$$\mathcal{L}_{d,\varepsilon} w = \lambda(\varepsilon) w; \quad we^{id\theta} \in (H_0^1 \cap H^2)(B(0,1), \mathbf{C}) \tag{1.4}$$

and the following homogenous problem in $\mathbf{R^2}$:

$$\mathcal{L}_{d,1} w = 0. \tag{1.5}$$

Let us define

$$\Phi^0_{d,\varepsilon}(re^{i\theta}) = ie^{-id\theta} u_{d,\varepsilon}(re^{i\theta}) = if_d(\frac{r}{\varepsilon}). \tag{1.6}$$

The function $\Phi^0_{d,\varepsilon}$ is a bounded solution of $\mathcal{L}_{d,\varepsilon} w = 0$ and corresponds to the invariance of (1.1) under the one parameter group of transformation $u \to e^{i\alpha} u$ for $\alpha \in \mathbf{R}$. The functions

$$\begin{aligned} \Phi^{1,\mathcal{R}}_{d,\varepsilon}(re^{i\theta}) &= e^{-id\theta} \varepsilon \frac{\partial u_{d,\varepsilon}}{\partial x}(re^{i\theta}) \\ &= \frac{1}{2}[(f_d'(\frac{r}{\varepsilon}) - \varepsilon \frac{d}{r} f_d(\frac{r}{\varepsilon}))e^{i\theta} + (f_d'(\frac{r}{\varepsilon}) + \varepsilon \frac{d}{r} f_d(\frac{r}{\varepsilon}))e^{-i\theta}] \end{aligned} \tag{1.7}$$

and

$$\Phi^{1,\mathcal{I}}_{d,\varepsilon}(re^{i\theta}) = e^{-id\theta}\varepsilon\frac{\partial u_{d,\varepsilon}}{\partial y}(re^{i\theta})$$

$$= \frac{i}{2}[(-f'_d(\frac{r}{\varepsilon}) + \varepsilon\frac{d}{r}f_d(\frac{r}{\varepsilon}))e^{i\theta} + (f'_d(\frac{r}{\varepsilon}) + \varepsilon\frac{d}{r}f_d(\frac{r}{\varepsilon}))e^{-i\theta}] \quad (1.8)$$

are bounded solutions of $\mathcal{L}_{d,\varepsilon}w = 0$ and correspond to the invariance of (1.1) under the action of the group of translations $u \to u(. - b)$ for $b \in \mathbf{C}$.

We will need the following Fourier expansion

$$w(r,\theta) = \sum_{n\in\mathbf{Z}} a_n(r)e^{in\theta}, \quad (1.9)$$

where a_n and a_{-n} are complex valued functions. For all $n \in \mathbf{N}$ we study the eigenvalue problem

$$\mathcal{L}_{d,\varepsilon}w_n = \lambda(\varepsilon)w_n,$$

where

$$w_0 = a_0 \text{ and } w_n = a_{-n}e^{-in\theta} + a_ne^{in\theta} \text{ for } n \geq 1.$$

For $n \geq 1$, (a_n, a_{-n}) is solution of the following system in $]0,1]$

$$\begin{cases} a''_n + \frac{1}{r}a'_n - \frac{(n+d)^2}{r^2}a_n = \frac{1}{\varepsilon^2}f_d^2\overline{a}_{-n} - \frac{1}{\varepsilon^2}(1-2f_d^2)a_n - \lambda(\varepsilon)a_n \\ a''_{-n} + \frac{1}{r}a'_{-n} - \frac{(n-d)^2}{r^2}a_{-n} = \frac{1}{\varepsilon^2}f_d^2\overline{a}_n - \frac{1}{\varepsilon^2}(1-2f_d^2)a_{-n} - \lambda(\varepsilon)a_{-n} \end{cases} \quad (1.10)$$

$$a_n(1) = a_{-n}(1) = 0\,.$$

We consider

$$Q_{d,\varepsilon}(w) = \int_{B(0,1)} \mathcal{L}_{d,\varepsilon}(w).w.$$

We have

$$\frac{1}{2\pi}Q_{d,\varepsilon}(w) = \int_0^1 [r \mid a'_0 \mid^2 + \frac{d^2}{r} \mid a_0 \mid^2 - \frac{r}{\varepsilon^2}(1 - f_d^2(\frac{r}{\varepsilon})) \mid a_0 \mid^2 + \frac{r}{2\varepsilon^2}f_d^2(\frac{r}{\varepsilon}) \mid a_0$$

$$+ \overline{a}_o \mid^2]dr + \sum_{n=1}^{+\infty}\int_0^1 [r(\mid a'_{-n} \mid^2 + \mid a'_n \mid^2) + \frac{(d-n)^2}{r} \mid a_{-n} \mid^2$$

$$+ \frac{(d+n)^2}{r} \mid a_n \mid^2]dr + \sum_{n=1}^{+\infty}\int_0^1 [-\frac{r}{\varepsilon^2}(1 - f_d^2(\frac{r}{\varepsilon}))(\mid a_n \mid^2 + \mid a_{-n} \mid^2)$$

$$+ \frac{r}{\varepsilon^2}f_d^2(\frac{r}{\varepsilon}) \mid a_n + \overline{a}_{-n} \mid^2]dr.$$

Let us remark that the integrals at 0 are well defined, because

$$ae^{-i(n-d)\theta} + be^{i(n+d)\theta} \in H^1_0(B(0,1)).$$

Let us define

$$V_0 = \{a(r)\backslash a(r)e^{id\theta} \in H_0^1(B(0,1),\mathbf{C})\}$$

and for $n \in \mathbf{N}, n \neq 0$

$$V_n = \{a(r)e^{-in\theta} + b(r)e^{in\theta}\backslash a(r)e^{-i(n-d)\theta} + b(r)e^{i(n+d)\theta} \in H_0^1(B(0,1),\mathbf{C})\}.$$

We set for $n \in \mathbf{N}$

$$\lambda_n^d(\varepsilon) = \min_{\{w\in V_n;||w||^2=2\pi\}} \frac{1}{2\pi} Q_{d,\varepsilon}(w)\,,$$

where

$$|| w ||^2 = \int_{B(0,1)} | w |^2 = 2\pi \int_0^1 r(| a |^2 + | b |^2)dr.$$

Let us recall some well known facts (see Mironescu [9] and Comte and Mironescu [4]). It is classical that the minima λ_n^d are attained. For $n \geq 1$, the quantity we have to minimize decreases when we replace the complex valued functions a and b by $\max\{| a |, | b |\}$ and $-\min\{| a |, | b |\}$. Thus we have

$$\lambda_n^d(\varepsilon) = \min_{\mathcal{E}} \int_0^1 (r(| a' |^2 + | b' |^2) + \frac{(d-n)^2}{r} | a |^2 + \frac{(d+n)^2}{r} | b |^2$$

$$- \frac{r}{\varepsilon^2}(1 - f_d^2(\frac{r}{\varepsilon}))(| a |^2 + | b |^2) + \frac{r}{\varepsilon^2} f_d^2(\frac{r}{\varepsilon}) | a + b |^2)dr\,,$$

where

$$\mathcal{E} = \left\{ (a,b); ae^{-in\theta} + be^{in\theta} \in V_n; a : [0,1] \to [0,+\infty[; b : [0,1] \to]-\infty, 0] \right.$$

$$\left. \int_0^1 r(a^2 + b^2)dr = 1 \right\}.$$

By Mironescu [9], Lin [7], Lin [8] and Beaulieu [2], we know that for all $d \geq 1$ there exists $C > 0$ such that $\lambda_0^d(\varepsilon) \geq C$ for ε sufficiently small and that $\lambda_1^d(\varepsilon)$ is positive and tends to 0 as ε tends to 0. Moreover $\lambda_1^d(\varepsilon)$ is the only eigenvalue of the restriction of $\mathcal{L}_{d,\varepsilon}$ to V_1 that tends to 0. For $n \geq 2d-1$, we know that the eigenvalues of the restriction of $\mathcal{L}_{d,\varepsilon}$ to V_n are positive and do not tend to 0. These results are related to the fact that the functions $\Phi_{d,1}^0$, $\Phi_{d,1}^{1,\mathcal{R}}$ and $\Phi_{d,1}^{1,\mathcal{I}}$ span the real vector space of the bounded solutions in $\mathbf{R^2}$ of $\mathcal{L}_{d,1}w = 0$ that are of the form

$$w(r,\theta) = a_0(r) + a_1(r)e^{i\theta} + a_{-1}(r)e^{-i\theta} + \sum_{|n|\geq 2d-1} a_n(r)e^{in\theta},$$

where the $a_i(r)$ are complex valued functions. For $n = 2, ..., 2d-2$, we know that $\lambda_n^d(\varepsilon)$ are negative and do not tend to 0. In this paper we improve the result for $n = 2, ..., 2d-2$. We prove that the eigenvalues of the restriction of $\mathcal{L}_{d,\varepsilon}$ to V_n do not tend to 0, despite the fact that they may be positive or negative. This is related to the fact that for all $d \geq 1$ the real vector space of the bounded solutions of the equation $\mathcal{L}_{d,1}w = 0$ is spanned by the three functions that correspond to the invariance of the equation $\Delta u + u(1- \mid u \mid^2) = 0$ under the action of the rotations and the translations.

Theorem 1.1. *For $d \geq 1$, the functions $\Phi^0_{d,1}$, $\Phi^{1,\mathcal{R}}_{d,1}$ and $\Phi^{1,\mathcal{I}}_{d,1}$ span the real vector space of the bounded solutions in $\mathbf{R^2}$ of $\mathcal{L}_{d,1}w = 0$.*

Corollary 1.1. *For $d \geq 1$ the only eigenvalue that tends to 0 as ε tends to 0 is $\lambda^d_1(\varepsilon)$.*

2. THE PROOFS

In the following lemma we recall the statement of Lemma 2.2 in Beaulieu [2].

Lemma 2.1. *The real vector space of the solutions of (1.5) that are bounded at 0 is spanned by four independent solutions (a^j, b^j), $(ia^j, -ib^i)$, $j = 3, 4$. Moreover we have 4 independent solutions (a^j, b^j), $(ia^j, -ib^i)$, $j = 1, 2$ that blow up at 0. For $n < d$ the behaviors at 0 are given by*

$$a^4(r) = \frac{a^2}{4(2d+1)(n+d+1)} r^{n+3d+2}(1 + O(r^2)),$$

$$b^4 = r^{n+d} - \frac{1}{4(d+n+1)} r^{n+d+2}(1 + O(r^2)),$$

$$a^3(r) = r^{d-n} - \frac{1}{4(d-n+1)} r^{d-n+2}(1 + O(r^2)),$$

$$b^3 = \frac{a^2}{4(2d+1)(-n+d+1)} r^{3d-n+2}(1 + O(r^2)),$$

$$a^2(r) = \frac{a^2}{4(d-n+1)} r^{d-n+2} + O(r^{d-n+4}); \quad b^2(r) = r^{-n-d} + O(r^{-n-d+2}),$$

$$a^1(r) = r^{n-d} + O(r^{n-d+2}); \quad b^1(r) = \frac{a^2}{4(n+d+1)} r^{n+d+2} + O(r^{n+d+4}).$$

For $n > d$ the behaviors at 0 are the same, except that (a^3, b^3) is called (a^1, b^1). For $n = d$, we replace only the behavior of (a^1, b^1) by

$$a^1(r) = \log r - \frac{1}{4} r^2 \log r + \frac{1}{4} r^2 + O(r^4 \log r),$$

$$b^1(r) = \frac{a^2}{4(2d+1)} r^{2d+2} \log r - \frac{a^2(d+1)}{4(2d+1)^2} r^{2d+2} + O(r^{2d+4} \log r).$$

We have a four dimensional real vector space of solutions (a,b) that decay at $+\infty$ and it is spanned by four independent solutions whose behaviors at $+\infty$ are respectively

$$(r^{-n}(1+O(r^{-1})) \quad ; \quad -r^{-n}(1+O(r^{-1})))$$

and

$$(J_n^-(r)(1+O(r^{-1})) \quad ; \quad J_n^-(r)(1+O(r^{-1}))),$$

and the complex associated solutions. Moreover, if w blows up at $+\infty$, then it blows up like (r^n, r^n) or like (J_n^+, J_n^+), where J_n^- is a Bessel function that decays exponentially at $+\infty$ and J_n^+ is a Bessel function that blows up exponentially at $+\infty$.

In all what follows, we will have $d \geq 2$ and $n = 2, .., 2d-2$.

We will prove first the following lemma.

Lemma 2.2. *Let us define $\overline{\lambda}_n^d$ by*

$$\overline{\lambda}_n^d = \lim_{\varepsilon \to 0} \varepsilon^2 \lambda_n^d(\varepsilon).$$

There exists a nontrivial solution $w = a(r)e^{-in\theta} + b(r)e^{in\theta}$ of the system

$$\begin{cases} \mathcal{L}_{d,1} w = \overline{\lambda}_n^d w \ \ in\ \mathbf{R}^2 \\ a \geq -b \geq 0 \ \ in\ [0, +\infty[\end{cases} \tag{2.1}$$

and w is bounded at $+\infty$.

Proof. By [B], Lemma 1.1, we know that $\overline{\lambda}_n^d$ does exist and that there exists $C > 0$ such that

$$\overline{\lambda}_n^d \in [-1, -C].$$

Let $w_\varepsilon = a_\varepsilon e^{-in\theta} + b_\varepsilon e^{in\theta}$ be a non trivial solution of

$$\Big\{ \mathcal{L}_{d,1} w_\varepsilon = \varepsilon^2 \lambda_n^d(\varepsilon) w_\varepsilon \text{ in } B_{\frac{1}{\varepsilon}} e^{id\theta} w_\varepsilon \in (H^2 \cap H_0^1)(B_{\frac{1}{\varepsilon}}) a_\varepsilon \geq -b_\varepsilon \geq 0 \text{ in } [0, \frac{1}{\varepsilon}]. \tag{2.2}$$

We may suppose that the behavior at 0 of w_ε is the same as the behavior at 0 of $A_\varepsilon w^3 + B_\varepsilon w^4$, where $A_\varepsilon > 0$, $B_\varepsilon < 0$ and $\max\{A_\varepsilon, -B_\varepsilon\} = 1$.

Let us prove that w_ε tends to a limit w and let us suppose by contradiction that w blows up at $+\infty$. We will reach to a contradiction by a proof adapted from the proof of Lemma 3.1 in Beaulieu [2] (see also Lin [8]). We define

$$\tilde{w}_\varepsilon = \tilde{a}_\varepsilon e^{-in\theta} + \tilde{b}_\varepsilon e^{in\theta}$$

that has the same behavior than w_ε at 0 and that verifies the system

$$\mathcal{L}_{d,1} \tilde{w}_\varepsilon = \overline{\lambda}_n^d \tilde{w}_\varepsilon$$

and we define $\hat{w}_\varepsilon$ by

$$w_\varepsilon = \tilde{w}_\varepsilon + (\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d)\hat{w}_\varepsilon. \tag{2.3}$$

For all $R > 0$ there exists $M(R)$ and $\varepsilon_0(R)$ such that

$$\mid \hat{w}_\varepsilon(x) \mid \leq M(R) \text{ for all } x \in B(0,R) \text{ and all } 0 < \varepsilon < \varepsilon_0(R). \tag{2.4}$$

Moreover, $\tilde{w}_\varepsilon$ tends w in $\mathcal{C}^2(B(0,R))$ for all $R > 0$. We postpone the justification of (2.4) to the end of the proof. Now we multiply (2.2) by w_ε and we integrate on $B_{\frac{1}{\varepsilon}}$. We get

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} = \int_{B_{\frac{1}{\varepsilon}}} \mathcal{L}_{d,1} w_\varepsilon w_\varepsilon - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} . \tag{2.5}$$

We distinguish the cases $n = d$ and $n \neq d$ because of the convergence of the integrals at 0. For $n \neq d$ we deduce from (2.5) that

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} \geq \varepsilon^2 \lambda_0^d(\varepsilon) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})}$$

$$+ \int_0^{\frac{1}{\varepsilon}} (\frac{(d-n)^2 - d^2}{r} a_\varepsilon^2 + \frac{(d+n)^2 - d^2}{r} b_\varepsilon^2 + f_d^2(r) \mid a_\varepsilon + b_\varepsilon \mid^2) dr. \tag{2.6}$$

We set $b_\varepsilon = \tau \tilde{a}_\varepsilon$ and

$$H_n(r) = \frac{1}{r} \frac{n^2(n^2 - 4d^2) + 2r^2 n^2 f_d^2}{n(2d+n) + r^2 f_d^2}.$$

We easily obtain

$$\frac{n(-2d+n)}{r} a_\varepsilon^2 + \frac{n(2d+n)}{r} b_\varepsilon^2 + r f_d^2 \mid a_\varepsilon + b_\varepsilon \mid^2 \geq a_\varepsilon^2 H_n(r),$$

Thus

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} \geq \varepsilon^2 \lambda_0^d(\varepsilon) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})}$$

$$+ \int_0^{\frac{1}{\varepsilon}} a_\varepsilon^2(r) H_n(r) dr.$$

There exists $K > 0$ and $R > 0$ such that for all $r \geq R$,

$$H_n(r) \geq \frac{K}{r}.$$

Using (2.3), we are led to, for $0 < \varepsilon < \frac{1}{R}$,

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} \geq \varepsilon^2 \lambda_0^d(\varepsilon) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})}$$

$$+ \int_0^R (\tilde{a}_\varepsilon(r) + (\lambda_n^d(\varepsilon) - \overline{\lambda}_n^d)\hat{a}_\varepsilon)^2 H_n(r) dr.$$

Let us suppose that w blows up at $+\infty$, at least like r^n, thus there exists $R_0 > 0$ and $\varepsilon_0 > 0$ such that we have for all $R > R_0$ and $0 < \varepsilon < \varepsilon_0$,

$$\int_0^R H_n(r) a^2(r) dr > 0.$$

Thus, letting $R > R_0$ and $0 < \varepsilon < \varepsilon_0(R)$ we are led to

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} \geq \varepsilon^2 \lambda_0^d(\varepsilon) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})},$$

that gives

$$\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d > -\overline{\lambda}_n^d,$$

and this is in contradiction with $\lambda_n^d(\varepsilon) < 0$. Thus, for $n \neq d$, w is bounded at $+\infty$. For $n = d$, we define

$$K_n(r) = \frac{r}{n(2d+n) + r^2 f^2} (-r^2 f^2 + r^2 f^4 + n(2d+n)(2f^2 - 1)).$$

At $+\infty$ we have

$$K_n(r) = \frac{2d^2}{r} + O(\frac{1}{r^2}).$$

By (2.5), we are led to

$$(\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d) \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} \geq \varepsilon^2 \lambda_0^d(\varepsilon) \mid\mid b_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} - \overline{\lambda}_n^d \mid\mid w_\varepsilon \mid\mid^2_{L^2(B_{\frac{1}{\varepsilon}})} + \int_0^R a_\varepsilon^2(r) K_n(r) dr.$$

As in the previous proof, if we suppose that w blows up at $+\infty$, we may choose $R > 0$ such that $K_n(r) > 0$ for all $r > R$ and such that

$$\int_0^R a^2(r) K_n(r) dr > 0.$$

Thus there exists $\varepsilon_0(R)$ such that for all $0 < \varepsilon < \varepsilon_0(R)$ we have

$$\varepsilon^2 \lambda_n^d(\varepsilon) - \overline{\lambda}_n^d \geq -\overline{\lambda}_n^d$$

that is in contradiction with $\lambda_n^d(\varepsilon) < 0$.

In order to justify (2.4), let us remark that there exists four independent solutions for the system

$$\mathcal{L}_{d,1} w - \overline{\lambda}_n^d w = 0,$$

denoted by

$$\tilde{w}^j = \tilde{a}^j e^{-in\theta} + \tilde{b}^j e^{in\theta}, \ j = 1, ..., 4,$$

where $\tilde{a}^j$ and $\tilde{b}^j$ are real valued. The behaviors at 0 of $(\tilde{a}^j, \tilde{b}^j)$ are the same as the behaviors at 0 of (a^j, b^j), given in Lemma 2.1. Then we have, as in (3.7) in Beaulieu [2],

$$w_\varepsilon = \tilde{w}_\varepsilon - (\varepsilon^2\lambda_n^d(\varepsilon) - \overline{\lambda}_n^d)\frac{1}{W}\sum_{i=1}^{4} w^i(r)\int_0^r t(....)dt$$

and the expression in the integral has exactly the same behavior at 0 as in (3.7) of Beaulieu [2]. Thus we have (2.4). □

Now we have the following lemma.

Lemma 2.3. *There exists a real positive number $\tilde{R}$ and a non trivial solution $\overline{w} = \overline{a}e^{-in\theta} + \overline{b}e^{in\theta}$ of the system*

$$\begin{cases} \mathcal{L}_{d,1}\overline{w} = 0 \ in\ \mathbf{R}^2\backslash\{0\} \\ e^{id\theta}\overline{w} \in (H^2 \cap H_0^1)(\mathbf{R}^2\backslash B(0,\tilde{R})) \\ \overline{a} \geq -\overline{b} \geq 0 \ in\ ]0,+\infty[. \end{cases} \tag{2.7}$$

Moreover, $\overline{w}$ blows up at 0.

Proof. The function $\overline{w}$ is defined in Beaulieu [2], (0.12). Its existence is proved in Part 5 of Beaulieu [2] and it is proved that $\overline{w}$ is bounded at $+\infty$, that $\overline{w} \in (H^2 \cap H_0^1)(\mathbf{R}^2\backslash B(0,\tilde{R}))$ and that $\overline{a} \geq -\overline{b} \geq 0$ in $[\tilde{R}, +\infty[$. Let us verify that $\overline{a}$ and $\overline{b}$ are in fact well defined in $]0,+\infty[$ and that the inequality $\overline{a} \geq -\overline{b} \geq 0$ remains true in $]0,+\infty[$. Let us come back to the following definition for all $0 < R < \frac{1}{\varepsilon}$:

$$\varepsilon^2\lambda_{n,R}^d(\varepsilon) = \min_{\mathcal{E}_R}\int_R^{\frac{1}{\varepsilon}}(r(\mid a'\mid^2 + \mid b'\mid^2) + \frac{(d-n)^2}{r}\mid a\mid^2 + \frac{(d+n)^2}{r}\mid b\mid^2$$
$$- r(1-f^2)(\mid a\mid^2 + \mid b\mid^2) + rf^2\mid a+b\mid^2)dr\,, \tag{2.8}$$

where

$$\mathcal{E}_R = \Bigg\{(a,b) : [R,\frac{1}{\varepsilon}]^2 \to [0,+\infty[\times]-\infty,0],$$

$$a(r)e^{-i(n-d)\theta} + b(r)e^{i(n+d)\theta} \in H_0^1(B(0,\frac{1}{\varepsilon})\backslash B(0,R),\mathbf{C});\ \int_R^{\frac{1}{\varepsilon}} r(a^2+b^2)dr = 1\Bigg\}.$$

In fact we have

$$\varepsilon^2\lambda_{n,R}^d(\varepsilon) = \min_{\{w\in V_{n,R};||w||^2=2\pi\}}\frac{1}{2\pi}Q_{\varepsilon,R}(w),$$

where we set

$$Q_{\varepsilon,R}(w) = \int_{B(0,\frac{1}{\varepsilon})\backslash B(0,R)}\mathcal{L}_{d,1}(w).w$$

and

$$V_{n,R} = \{w = a(r)e^{-in\theta} + b(r)e^{in\theta}; we^{id\theta} \in H_0^1(B(0, \frac{1}{\varepsilon})\backslash B(0, R), \mathbf{C})\}.$$

We have proved in Beaulieu [2], Part 5, that there exists a positive real number $R(\varepsilon)$ such that $\lambda^d_{n,R(\varepsilon)}(\varepsilon) = 0$. Let a_ε and b_ε, defined in $[R(\varepsilon), \frac{1}{\varepsilon}]$, be functions that realize $\varepsilon^2\lambda^d_{n,R(\varepsilon)}(\varepsilon) = 0$. They verify the system

$$\begin{cases} a''_\varepsilon + \frac{1}{r}a'_\varepsilon - \frac{(n+d)^2}{r^2}a_\varepsilon = f_d^2 b_\varepsilon - (1-2f_d^2)a_\varepsilon \\ b''_\varepsilon + \frac{1}{r}b'_\varepsilon - \frac{(n-d)^2}{r^2}b_\varepsilon = f_d^2 a_\varepsilon - (1-2f_d^2)b_\varepsilon \end{cases} \tag{2.9}$$

and

$$a_\varepsilon(\frac{1}{\varepsilon}) = b_\varepsilon(\frac{1}{\varepsilon}) = 0.$$

It follows from (2.9) that a_ε and b_ε are in fact defined in $]0, +\infty[$. Indeed let us first denote $r = e^s$, $a(r) = x(s)$ and $b(r) = y(s)$. The system (2.9) gives

$$\begin{cases} x''(s) = (n-d)^2 x(s) + e^{2s}(f^2(e^s)y(s) - (1-2f^2(e^s))x(s)) \\ y''(s) = (n+d)^2 y(s) + e^{2s}(f^2(e^s)x(s) - (1-2f^2(e^s))y(s)) \end{cases}. \tag{2.10}$$

Let us define

$$X(s) = \begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix}.$$

The system (2.10) is equivalent to

$$X'(s) = M(s)X(s), \quad s \in [\log R(\varepsilon), \log\frac{1}{\varepsilon}],$$

where $M(s)$ is the 4×4 matrix, defined in $]-\infty, +\infty[$. It follows that a_ε and b_ε are defined in $]R(\varepsilon), +\infty[$. Now we define $r = e^{-s}$, $a(r) = x(s)$ and $b(r) = y(s)$. We are led to a system of the form

$$X'(s) = \tilde{M}(s)X(s), \quad s \in [\log\varepsilon, \log\frac{1}{R(\varepsilon)}]$$

where $\tilde{M}(s)$ is the 4×4 matrix, defined in $]-\infty, +\infty[$. It follows that a_ε and b_ε are defined in $]0, R(\varepsilon)[$.

Now, if we replace a_ε and b_ε respectively by

$$A = \max\{a_\varepsilon, -b_\varepsilon\} \text{ and } B = -\min\{a_\varepsilon, -b_\varepsilon\},$$

we find that A and B are defined in $]0, +\infty[$ and that their restrictions to $[R(\varepsilon), \frac{1}{\varepsilon}]$ are in $\mathcal{E}_R$. Moreover they realize the infimum (2.8) (see Mironescu [9]). Thus A and B

verify the system (2.9) in $[R(\varepsilon), \frac{1}{\varepsilon}[$ and consequentely in $]0, +\infty[$. Now let us denote $a_\varepsilon = A$ and $b_\varepsilon = B$. The function

$$w_{R(\varepsilon)} = a_\varepsilon e^{-in\theta} + b_\varepsilon e^{in\theta}$$

realizes

$$\begin{cases} \mathcal{L}_{d,1} w_{R(\varepsilon)} = 0 \text{ in } \mathbf{R}^2 \backslash \{0\} \\ e^{id\theta} w_{R(\varepsilon)} \in (H^2 \cap H^1_0)(\mathbf{R}^2 \backslash B_{R(\varepsilon)}) \\ a_\varepsilon \geq -b_\varepsilon \geq 0 \text{ in }]0, +\infty[. \end{cases} \tag{2.11}$$

The same proof as in Beaulieu [2], Part 5 gives a positive real number $\tilde{R}$ such that (up to a subsequence) $R(\varepsilon)$ tends to $\tilde{R}$ and $w_{R(\varepsilon)}$ tends in $\mathcal{C}^2(B_B \backslash B_A)$ for all $0 < A < B$ to $\overline{w}$ that is a non trivial solution of (2.7) and that is bounded at $+\infty$.

Now let us suppose that $\overline{w}$ is bounded at 0. Let us use the function w defined in Lemma 2.2 above. We have

$$-\overline{\lambda}_n^d \int_{\mathbf{R}^2} \overline{w} w = \int_{\mathbf{R}^2} \mathcal{L}_{d,1} w \overline{w}.$$

The behaviors of $\overline{w}$ and w are respectively the same as the behaviors of $\overline{A} w^3 + \overline{B} w^4$ and $Aw^3 + Bw^4$ with $A > 0, \overline{A} > 0$ and $B < 0, \overline{B} < 0$, where w^3 and w^4 are defined in Lemma 2.1. Thus we may integrate by parts and we get

$$-\overline{\lambda}_n^d \int_0^{+\infty} r(\overline{a}a + \overline{b}b) dr = 0,$$

that is impossible. □

We use the behaviors at 0 and at $+\infty$ given in Lemma 2.1 to deduce the following corollary from Lemma 2.3.

Corollary 2.1. *There exists a solution $\overline{w}$ of (2.7) and positive real numbers A and B such that $(\overline{a}, \overline{b})$ behaves as $(r^{-n}, -r^{-n})$ at $+\infty$ and such that the behavior at 0 is $(Ar^{-|n-d|}, -Br^{-n-d})$ for $n \neq d$ and $(A \log \frac{1}{r}, -Br^{-2d})$ for $n = d$.*

Finally, let us prove the following lemma.

Lemma 2.4. *Let $\eta_1 = \alpha_1 e^{-in\theta} + \beta_1 e^{in\theta}$ be a solution of $\mathcal{L}_{d,1} w = 0$ that is bounded at 0 and such that α_1 and β_1 are positive near 0. Then α_1 and β_1 are positive in $[0. + \infty[$ and they blow up exponentially at $+\infty$. Let $\eta_2 = \alpha_2 e^{-in\theta} + \beta_2 e^{in\theta}$ be a solution of $\mathcal{L}_{d,1} w = 0$ that is bounded at $+\infty$ and such that α_2 and β_2 are positive near $+\infty$. Then α_2 and β_2 are positive in $]0. + \infty[$ and they blow up at 0.*

Proof. The functions (α_1, β_1) and $(\overline{a}, \overline{b})$ verify the system (2.9). We multiply the first ligns respectively by $r\overline{a}$ and $r\alpha_1$ and we do a substraction. We obtain (as in Pacard and Rivière [10], Chapter 3)

$$(r(\overline{a}'\alpha_1 - \overline{a}\alpha_1'))' = rf^2(\overline{b}\alpha_1 - \overline{a}\beta_1). \tag{2.12}$$

In the same way we obtain

$$(r(\beta_1\overline{b}' - \beta_1'\overline{b}))' = rf^2(\beta_1 a - \alpha_1 b). \tag{2.13}$$

Let us suppose that there exists $r_0 > 0$ such that $\alpha_1(r) > 0$ and $\beta_1(r) > 0$ for all r in $]0, r_0[$, and $\alpha_1(r_0) = 0$. Integrating (2.12) on $[0, r_0]$ we get

$$r_0(\overline{a}'\alpha_1 - \overline{a}\alpha_1')(r_0) = \lim_{r\to 0}(r(\alpha_1'\overline{a} - \alpha_1\overline{a}')(r)) + \int_0^{r_0} tf^2(\overline{b}\alpha_1 - \overline{a}\beta_1)dt. \tag{2.14}$$

But we have

$$r_0(\overline{a}'\alpha_1 - \overline{a}\alpha_1')(r_0) = -r_0\overline{a}(r_0)\alpha_1'(r_0) > 0$$

and

$$\int_0^{r_0} tf^2(\overline{b}\alpha_1 - \overline{a}\beta_1)dt < 0.$$

Moreover the behavior of (α_1, β_1) at 0 is the behavior of $A_1 w^3 + B_1 w^4$ with $A_1 \geq 0$ and $B_! \geq 0$. Let us suppose that $A_1 > 0$ and $B_1 > 0$. We have the following behaviors at 0, for $n \neq d$

$$(r\overline{a}', r\overline{b}') \sim (-A \mid n-d \mid r^{-|n-d|}, B(n+d)r^{-n-d})$$

$$(\alpha_1, \beta_1) \sim (A_1 r^{|n-d|}, B_1 r^{n+d})$$

$$(r\alpha_1', r\beta_1') \sim (A_1 \mid n-d \mid r^{|n-d|}, B_1(n+d)r^{n+d})$$

and for $n = d$

$$(r\overline{a}', r\overline{b}') \sim (-A, 2dBr^{-2d})$$

$$(\alpha_1, \beta_1) \sim (A_1 - \frac{A_1}{4}r^2, B_1 r^{2d})$$

$$(r\alpha_1', r\beta_1') \sim (-\frac{A_1}{2}r^2, 2dB_1 r^{2d}).$$

Consequentely we have

$$\lim_{r\to 0}(r\overline{a}'\alpha_1 - r\overline{a}\alpha_1') \leq 0,$$

and this is true also if $A_1 = 0$ or $B_1 = 0$. Thus (2.14) leads to a contradiction. We have proved that $\alpha_1 > 0$ in $[0, +\infty[$. In the same way we can prove, using (2.13), that $\beta_1 > 0$ in $[0, +\infty[$. Let us prove now that α_1 blows up exponentially at $+\infty$. We proceed as in Pacard and Rivière [10], Chapter 3. We set

$$X(r) = r^{2n}\alpha_1(r)\overline{a}(r).$$

We have the following expansion at $+\infty$:

$$X'(r) = r^{2n}(\alpha_1'\overline{a} - \alpha_1\overline{a}') + O(r^{n-2}\alpha_1(r)).$$

We use (2.12) to get

$$\frac{X'(r)}{r^{2n-1}} + O(\frac{X(r)}{r^{2n+1}}) \geq \int_0^r tf^2(t)\overline{a}(t)\beta_1(t)dt.$$

This leads to

$$\frac{X'(r)}{r^{2n-1}} + O(\frac{X(r)}{r^{2n+1}}) \geq \int_0^r f^2(t)\frac{X(t)}{t^{2n-1}}\frac{\beta_1(t)}{\alpha_1(t)}dt.$$

But $\frac{\beta_1}{\alpha_1}f^2$ behaves as a constant at $+\infty$. Thus X, and consequentely α_1, blow up exponentially at $+\infty$.

Now let us prove that $\alpha_2 > 0$ in $]0, +\infty[$. Let us suppose that there exists $r_0 > 0$ such that $\alpha_2(r) > 0$ and $\beta_2(r) > 0$ for all r in $]r_0, +\infty[$, and $\alpha_2(r_0) = 0$. Let us remark that

$$\lim_{r\to+\infty} r(\overline{a}'\alpha_2 - \overline{a}\alpha_2') = 0.$$

Integrating (2.12) on $[r_0, +\infty]$ we get

$$-r_0(\overline{a}'\alpha_2 - \overline{a}\alpha_2')(r_0) = \int_{r_0}^{+\infty} t(f^2(\overline{b}\alpha_2 - \overline{a}\beta_2)dt \qquad (2.15)$$

and this gives a contradiction. In the same way we get that $\beta_2 > 0$ in $]0, +\infty[$. Then we may use the first part of the present lemma. If (α_2, β_2) were bounded near 0, it would blow up at $+\infty$. Thus (α_2, β_2) blows up at 0. □

The proofs of Theorem 1.1 and Corollary 1.1 are now completed. We deduce from Lemma 2.4 that w^3 and w^4 blow up exponentially at $+\infty$. More precisely, there exist $A_3 > 0$ and $A_4 > 0$ such that (a^3, b^3) and (a^4, b^4) behave respectively at $+\infty$ like $(A_3J_n^+, A_3J_n^+)$ and $(A_4J_n^+, A_4J_n^+)$. Now let us suppose that there exist 2 real numbers A and B such that $Aw^3 + Bw^4$ is bounded at $+\infty$. By the second part of Lemma 2.4, the behavior of $(Aa^3 + Ba^4, Ab^3 + Bb^4)$ at $+\infty$ cannot be $C(J_n^-, J_n^-)$, for a real number C. Thus we may suppose that it is $C(t^{-n}, -t^{-n})$, with the same C as in Corollary 2.1. We infer that $(Aa^3 + Ba^4 - \overline{a}, Ab^3 + Bb^4 - \overline{b})$ behaves at $+\infty$ as $D(J_n^-, J_n^-)$, for some D. But, by Lemma 2.4, this implies that $Aa^3 + Ba^4 - \overline{a}$ and $Ab^3 + Bb^4 - \overline{b}$ remain both of the sign of D in $]0, +\infty[$. We know that this is false, by Corollary 2.1 and by the behaviors of (a^3, b^3) and (a^4, b^4) near 0. Thus, for all real numbers A and B, the function $Aw^3 + Bw^4$ is not bounded at $+\infty$. The proof of Corollary 1.1 comes from Beaulieu [2], Theorem 1.

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