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QUATERNION FUNCTIONS AND FOUR-DIMENSIONAL RIEMANNIAN METRICS

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ABSTRACT: In this paper we discuss homeomorphic transformations of Riemannian metrics in four-dimensional Riemannian manifolds, and show that these transformations are related to the solutions of Beltrami-type systems of differentiable, quaternionic functions. It is introduced the concept of quaternionic factorization of metrics, and demonstrated that monogenic functions are a particular case in a larger class of quaternionic differentiable functions. This class is formed by the solutions of an homogeneous operator equation, constructed for any factorizable, Riemannian metric.

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1. INTRODUCTION

The theory of pseudoanalytic functions (or generalized analytic functions) and quasiconformal mappings is a well developed subject in the framework of Complex Analysis, with many applications to Partial Differential Equations, Topology and Function Theory. Since the beginning of the last century, several authors have worked out relevant contributions to this subject, and it is beyond our capabilities (and the goals of this paper) to discuss them without losses of several important topics. Not too far from present days, the works of Vekua [1] and Bers [2], Bers [3] are a good source of information concerning quasiconformality and generalized analytic functions in complex domains. Moreover, the theory of quasiconformal mappings in n-spaces, introduced by Loewner [4] and developed mainly by Väsälä [5] and Gehring [6] has received more recently a great deal of attention from other mathematicians. Due

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mainly to the contributions of the Finnish School, the contemporary quasiconformal theory in $\mathbf{R}^n$ (Heinonen et al [7], and Heinonen and Koskela [8]) is based on concepts from measure and integration theory, as well as differential form calculus. A good review on the analytical foundations of quasiconformality in $\mathbf{R}^n$ may be found for instance in Bojarski and Iwaniec [9]. In the context of quaternionic analysis, very recently Cerejeiras et al [10] have demonstrated the existence of homeomorphic local solutions for quaternionic functions, which are related to the solutions of a postulated, quaternionic Beltrami-type equation. In their proof, singular integral operators are applied to solve the Beltrami equation by fixed point techniques. In this paper one intends to regard a somewhat different point of view, namely the connections between homeomorphic solutions, defined by

$$\Lambda\left(\sum_{i=1}^{4} dw_i dw_i\right) = \sum_{\alpha,\beta} g_{\alpha\,\beta} du^{\alpha} du^{\beta}, \quad \Lambda > 0, \tag{1}$$

for Riemannian metrics $g_{\alpha\,\beta}$ in $\mathbf{R_4}$, and the solvability of quaternionic, differential equations which form Beltrami-type systems. It is shown that the concept of monogenic function, presented in Fueter's theory, see Fueter [11], Sudbery [12], of quaternion analysis, is a particular case of a larger class of quaternionic differentiable functions. By introducing the concept of quaternionic factorization of Riemannian metrics, it is demonstrated that this class is formed by the solutions of the homogeneous equation $\Omega\, f = 0$, where Ω is a properly defined operator for factorizable Riemannian metrics. This work is organized as follows. Section 2 is a short review about the Beltrami equation in two dimensions, presenting some basic definitions and techniques which are useful for the discussion about generation of homeomorphic mappings. In Section 3 are introduced the concepts of induced metrics and quaternionic factorization of Riemannian metrics, and it is shown how these concepts lead directly to the construction of homeomorphic mappings in four dimensional manifolds. Two examples are given, a factorization for a canonical, general Riemannian metric and a decomposition for a Weyl-Papapetrou type metric, which is not in the canonical form. Section 4 addresses a generalization for the notion of quaternion function regularity, as well as the corresponding Beltrami-like systems for the metrics presented in Section 3. Finally, Section 5 discusses applications of the preceeding results in the framework of general relativity, by using the classical Schwarszchild metric.

2. HOMEOMORPHIC MAPPINGS AND BELTRAMI EQUATIONS IN COMPLEX DOMAINS

We will recall some basic ideas used for a fast deduction of Beltrami equations in the complex domain, since some of them are used in the next section in a four dimensional context. Given a first differential form

$$F \equiv a(x,y)\,dx^2 + 2b(x,y)\,dxdy + c(x,y)\,dy^2, \qquad \Delta \equiv ac - b^2 > 0, \tag{2}$$

then a complex factorization exists, such that

$$aF = \left[a\,dx + \left(b + i\sqrt{\Delta}\,dy\right)\right]\left[a\,dx + \left(b - i\sqrt{\Delta}\,dy\right)\right]. \tag{3}$$

If the complex functions $\alpha = \alpha(z)$ and $w = u + i\,v$ are found and satisfy the equation

$$dw = \alpha\left[a\,dx + \left(b + i\sqrt{\Delta}\,dy\right)\right], \tag{4}$$

then one has

$$\frac{\partial w}{\partial x} = \alpha\,a \qquad \frac{\partial w}{\partial y} = \alpha\left(b + i\sqrt{\Delta}\right), \tag{5}$$

and since the complex multiplication commutes,

$$\left(b + i\sqrt{\Delta}\right)\,\alpha(z)\,a - a\,\alpha(z)\left(b + i\sqrt{\Delta}\right) = 0, \tag{6}$$

what leads to

$$a\,\frac{\partial w}{\partial y} - \left(b + i\sqrt{\Delta}\right)\frac{\partial w}{\partial x} = 0. \tag{7}$$

Therefore, a solution w of (7) assures the transformation

$$F \equiv \frac{\mu\,\bar{\mu}}{a} dw\,\overline{dw} \equiv \Lambda\left(du^2 + dv^2\right), \qquad \Lambda = \frac{\mu\,\bar{\mu}}{a}, \qquad \mu = \frac{1}{\alpha}, \tag{8}$$

and the existence of *homeomorphisms* for the given quadratic form F.

With help of the operators

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{9}$$

and after some algebraic manipulations, equation (7) may be given as

$$\begin{aligned}\left(a + \sqrt{\Delta} - i\,b\right)\frac{\partial w}{\partial \bar{z}} &- \left(a - \sqrt{\Delta} + i\,b\right)\frac{\partial w}{\partial z} \\ &= i\left(a\,\frac{\partial w}{\partial y} - \left(b + i\sqrt{\Delta}\right)\frac{\partial w}{\partial x}\right) = 0,\end{aligned} \tag{10}$$

which is always given in the form

$$\frac{\partial w}{\partial \bar{z}} = q(z)\frac{\partial w}{\partial z}, \qquad q(z) = \frac{\left(a - \sqrt{\Delta} + i\,b\right)}{\left(a + \sqrt{\Delta} - i\,b\right)} \tag{11}$$

as the celebrated *Beltrami equation*. It is equivalent to the following system (Vekua [1]) for real functions:

$$\sqrt{\Delta}\frac{\partial u}{\partial x} - a\,\frac{\partial v}{\partial y} + b\,\frac{\partial v}{\partial x} = 0,$$

$$\sqrt{\Delta}\frac{\partial u}{\partial y} - b\,\frac{\partial v}{\partial y} + c\,\frac{\partial v}{\partial x} = 0. \tag{12}$$

The homeomorphisms of (2) are hence the univalent solutions of (12). Solutions of (12) are also termed generalized analytic or pseudoanalytic functions, since they satisfy a non-homogeneous extension of the classical Cauchy-Riemann system:

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\Delta}}a\,\frac{\partial v}{\partial y} - \frac{1}{\sqrt{\Delta}}b\,\frac{\partial v}{\partial x},$$

$$\frac{\partial u}{\partial y} = -c\frac{1}{\sqrt{\Delta}}\,\frac{\partial v}{\partial x} + \frac{1}{\sqrt{\Delta}}b\,\frac{\partial v}{\partial y}. \tag{13}$$

In the case of Euclidean metrics, $a = c = 1$ and $b = 0$, leading to the result

$$\frac{\partial w}{\partial \bar{z}} = 0. \tag{14}$$

The equation (14) is equivalent to the Cauchy-Riemann relations of Complex Analysis, in other words, analytical functions are homeomorphic (univalent) and conformal transformations for the complex domain. In the literature, the Beltrami equation is commonly written as $\frac{\partial w}{\partial \bar{z}} = \mu(z)\frac{\partial w}{\partial z}$, and the solutions for $\mu(z) \neq 0$ (Riemannian metrics in general) are called *μ-conformal functions*. By the way, a homeomorphism w is also called *quasiconformal* if it is μ-conformal and $\mu(z)$ is measurable, with $\|\mu\|_\infty < 1$. It is not the scope of these section to give an extended discussion of this definitions and their implications. However, in the next section it is shown that conformal mappings and Beltrami-like relations in four-dimensional spaces may be related, in some way, to quaternion functions and quaternion differential operators.

3. RIEMANNIAN METRICS AND QUATERNION FUNCTIONS

Let us consider first a subspace V_4 of a Riemmanian five dimensional manifold V_5, and their parametrical equations

$$x^j = x^j\left(u^\alpha\right), \quad j = 1,\ \dots \dots, 5, \quad \alpha = 1,\ \dots \dots, 4. \tag{15}$$

It is a well known fact that, at each point P of V_4, the components of

$$B^j_\alpha = \frac{\partial x^j}{\partial u^\alpha} = \frac{\partial}{\partial u^\alpha} = \partial_\alpha \tag{16}$$

are the components of four linearly independent vectors in the tangent space $T_5(P)$, spaning the tangent space $T_4(P)$ of V_4 at P (see for instance Lovelock and Rund [13]. A displacement in V_5 is given therefore by

$$dx^j = B^j_\alpha du^\alpha \tag{17}$$

and is called tangential to V_4, the corresponding arc length associated with this displacement may be written as

$$ds^2 = g_{hj} dx^h dx^j = g_{hj} B^h_\alpha B^j_\beta du^\alpha du^\beta. \tag{18}$$

By putting

$$G_{\alpha\beta}(u^\epsilon) = g_{hj}\left(x^l(u^\epsilon)\right) B^h_\alpha B^j_\beta, \tag{19}$$

one has in V_4 that

$$ds^2 = G_{\alpha\beta} du^\alpha du^\beta. \tag{20}$$

Since the components of the metric tensor g_{hj} of V_5 are symmetric, equation (19) assures that $G_{\alpha\beta}$ is also a symmetric tensor. One says that (19) defines the components of the metric tensor on V_4, and the resulting metric of V_4 is *induced* on V_4 by the metric of V_5. The subspace V_4 is also a Riemannian manifold, and $G_{\alpha\beta}$ are regarded as the coefficients of the *first fundamental form.* More explicity, this form (20) is given hence by

$$\begin{gathered} ds^2 = G_{11} du^1 du^1 + G_{22} du^2 du^2 + G_{33} du^3 du^3 + G_{44} du^4 du^4 \\ +2G_{12} du^1 du^2 + 2G_{13} du^1 du^3 + 2G_{14} du^1 du^4 + 2G_{23} du^2 du^3 \\ +2G_{24} du^2 du^4 + 2G_{34} du^3 du^4. \end{gathered} \tag{21}$$

Let us suppose that a quaternion factorization is possible for the form in (21), in such a way that

$$ds^2 = \left(A du^1 + B du^2 + C du^3 + D du^4\right) . \left(\bar{A} du^1 + \bar{B} du^2 + \bar{C} du^3 + \bar{D} du^4\right), \tag{22}$$

where the dot denotes quaternion multiplication, A stands for

$$A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k}, \quad \bar{A} = A_1 - A_2\mathbf{i} - A_3\mathbf{j} - A_4\mathbf{k}, \tag{23}$$

and similar definitions hold for B, C and D. In the equations above, **i**, **j** and **k** are the quaternion units and the components A_1, A_2, etc. are functions of the variables

u^1,u^2,u^3 and u^4. By using the corresponding multiplication table for the quaternionic group,

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i},$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i^2} = -\mathbf{1},$$

$$\mathbf{j^2} = -\mathbf{1}, \mathbf{k^2} = -\mathbf{1},$$

$$\mathbf{1i} = \mathbf{i1} = \mathbf{i}, \mathbf{1j} = \mathbf{j1} = \mathbf{j}, \mathbf{1k} = \mathbf{k1} = \mathbf{k}, \tag{24}$$

expanding (22) and comparing it with (21), results

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 = G_{11},$$

$$B_1^2 + B_2^2 + B_3^2 + B_4^2 = G_{22},$$

$$C_1^2 + C_2^2 + C_3^2 + C_4^2 = G_{33},$$

$$D_1^2 + D_2^2 + D_3^2 + D_4^2 = G_{44},$$

$$A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4 = G_{12},$$

$$A_1C_1 + A_2C_2 + A_3C_3 + A_4C_4 = G_{13},$$

$$A_1D_1 + A_2D_2 + A_3D_3 + A_4D_4 = G_{14},$$

$$B_1C_1 + B_2C_2 + B_3C_3 + B_4C_4 = G_{23},$$

$$B_1D_1 + B_2D_2 + B_3D_3 + B_4D_4 = G_{24},$$

$$C_1D_1 + C_2D_2 + C_3D_3 + C_4D_4 = G_{34}. \tag{25}$$

The system (25) is underdetermined, but may be solved for most practical cases by imposing further conditions on the coefficients. Taking for instance a metric dS^2 reduced to the canonical form,

$$dS^2 = g_{11}du^1du^1 + g_{22}du^2du^2 + g_{33}du^3du^3 + g_{44}du^4du^4, \tag{26}$$

a possible (and not unique) factorization is

$$A_1 = \sqrt{g_{11}}, \quad B_2 = \sqrt{g_{22}}, \quad C_3 = \sqrt{g_{33}}, \quad D_4 = \sqrt{g_{44}}, \tag{27}$$

with all remaining coefficients in (25) equal to zero. One considers now a quaternion function w, defined as

$$w = w_1 + w_2\mathbf{i} + w_3\mathbf{j} + w_4\mathbf{k}, \tag{28}$$

where as usually the components $w_i : \mathbf{R}^4 \to \mathbf{R}$ are real functions of the variables u^1, u^2, u^3 and u^4. Given two quaternion numbers x and y, the conjugation satisfies the property $\overline{xy} = \overline{y}\,\overline{x}$. Therefore, by taking a right quaternionic integrating factor function $\mu = \mu_1 + \mu_2\mathbf{i} + \mu_3\mathbf{j} + \mu_4\mathbf{k}$, one gets

$$dw = \left(\sqrt{g_{11}}du^1 + \sqrt{g_{22}}du^2\mathbf{i} + \sqrt{g_{33}}du^3\mathbf{j} + \sqrt{g_{44}}du^4\mathbf{k}\right)\mu, \tag{29}$$

and hence

$$dw\,\overline{dw} = \overline{dw}\,dw = \overline{\mu}dS^2\mu = \overline{\mu}\mu\,dS^2. \tag{30}$$

Since it is a relation between two first quadratic forms, it is clear that a constant quaternion μ results in a conformal application w. The condition for existence of homeomorphic relations may be written as

$$\Lambda\,dw\,\overline{dw} = \Lambda\left(dw_1dw_1 + dw_2dw_2 + dw_3dw_3 + dw_4dw_4\right) = dS^2,$$

$$\Lambda = \left(\mu\,\overline{\mu}\right)^{-1}. \tag{31}$$

According to (29),

$$\frac{\partial w}{\partial u^1} = \sqrt{g_{11}}\,\mu,\quad \frac{\partial w}{\partial u^2} = \sqrt{g_{22}}\;\mathbf{i}\,\mu,\quad \frac{\partial w}{\partial u^3} = \sqrt{g_{33}}\;\mathbf{j}\,\mu,\quad \frac{\partial w}{\partial u^4} = \sqrt{g_{44}}\;\mathbf{k}\,\mu, \tag{32}$$

what leads to the system

$$\frac{\partial w_1}{\partial u^1} + \frac{\partial w_2}{\partial u^1}\mathbf{i} + \frac{\partial w_3}{\partial u^1}\mathbf{j} + \frac{\partial w_4}{\partial u^1}\mathbf{k} = \sqrt{g_{11}}\left(\mu_1 + \mu_2\mathbf{i} + \mu_3\mathbf{j} + \mu_4\mathbf{k}\right),$$

$$\frac{\partial w_1}{\partial u^2} + \frac{\partial w_2}{\partial u^2}\mathbf{i} + \frac{\partial w_3}{\partial u^2}\mathbf{j} + \frac{\partial w_4}{\partial u^2}\mathbf{k} = \sqrt{g_{22}}\left(\mu_1\mathbf{i} - \mu_2 + \mu_3\mathbf{k} - \mu_4\mathbf{j}\right),$$

$$\frac{\partial w_1}{\partial u^3} + \frac{\partial w_2}{\partial u^3}\mathbf{i} + \frac{\partial w_3}{\partial u^3}\mathbf{j} + \frac{\partial w_4}{\partial u^3}\mathbf{k} = \sqrt{g_{33}}\left(\mu_1\mathbf{j} - \mu_2\mathbf{k} - \mu_3 + \mu_4\mathbf{i}\right),$$

$$\frac{\partial w_1}{\partial u^4} + \frac{\partial w_2}{\partial u^4}\mathbf{i} + \frac{\partial w_3}{\partial u^4}\mathbf{j} + \frac{\partial w_4}{\partial u^4}\mathbf{k} = \sqrt{g_{44}}\left(\mu_1\mathbf{k} + \mu_2\mathbf{j} - \mu_3\mathbf{i} - \mu_4\right). \tag{33}$$

From (33) it results finally that the required transformation should obey the following conditions:

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_1}{\partial u^1} = \frac{1}{\sqrt{g_{22}}}\frac{\partial w_2}{\partial u^2} = \frac{1}{\sqrt{g_{33}}}\frac{\partial w_3}{\partial u^3} = \frac{1}{\sqrt{g_{44}}}\frac{\partial w_4}{\partial u^4},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_2}{\partial u^1} = -\frac{1}{\sqrt{g_{22}}}\frac{\partial w_1}{\partial u^2} = \frac{1}{\sqrt{g_{44}}}\frac{\partial w_3}{\partial u^4} = -\frac{1}{\sqrt{g_{33}}}\frac{\partial w_4}{\partial u^3},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_3}{\partial u^1} = -\frac{1}{\sqrt{g_{33}}}\frac{\partial w_1}{\partial u^3} = -\frac{1}{\sqrt{g_{44}}}\frac{\partial w_2}{\partial u^4} = \frac{1}{\sqrt{g_{22}}}\frac{\partial w_4}{\partial u^2},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_4}{\partial u^1} = -\frac{1}{\sqrt{g_{44}}}\frac{\partial w_1}{\partial u^4} = \frac{1}{\sqrt{g_{33}}}\frac{\partial w_2}{\partial u^3} = -\frac{1}{\sqrt{g_{22}}}\frac{\partial w_3}{\partial u^2}. \tag{34}$$

As expected, a left quaternion integrating factor will generate entirely analogous relations:

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_1}{\partial u^1} = \frac{1}{\sqrt{g_{22}}}\frac{\partial w_2}{\partial u^2} = \frac{1}{\sqrt{g_{33}}}\frac{\partial w_3}{\partial u^3} = \frac{1}{\sqrt{g_{44}}}\frac{\partial w_4}{\partial u^4},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_2}{\partial u^1} = -\frac{1}{\sqrt{g_{22}}}\frac{\partial w_1}{\partial u^2} = -\frac{1}{\sqrt{g_{44}}}\frac{\partial w_3}{\partial u^4} = \frac{1}{\sqrt{g_{33}}}\frac{\partial w_4}{\partial u^3},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_3}{\partial u^1} = -\frac{1}{\sqrt{g_{33}}}\frac{\partial w_1}{\partial u^3} = \frac{1}{\sqrt{g_{44}}}\frac{\partial w_2}{\partial u^4} = -\frac{1}{\sqrt{g_{22}}}\frac{\partial w_4}{\partial u^2},$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial w_4}{\partial u^1} = -\frac{1}{\sqrt{g_{44}}}\frac{\partial w_1}{\partial u^4} = -\frac{1}{\sqrt{g_{33}}}\frac{\partial w_2}{\partial u^3} = \frac{1}{\sqrt{g_{22}}}\frac{\partial w_3}{\partial u^2}. \tag{35}$$

These procedures can be also extended for many Riemannian metrics which are not in the canonical form. Let us take for instance a Riemannian metric of the Weyl-Papapetrou type (Ansorg [14]),

$$ds^2 = e^{-2U}\left(e^{-2k}\left(d\rho^2 + d\zeta^2\right) + \rho^2 d\phi^2\right) - e^{2U}\left(dt + a d\phi\right)^2, \tag{36}$$

which may be given as

$$ds^2 = g_{11}du^1du^1 + g_{22}du^2du^2 + g_{33}du^3du^3 + g_{44}du^4du^4 + 2g_{34}du^3du^4 \tag{37}$$

with the correspondence

$$\rho \to u^1,\ \ \zeta \to u^2,\ \ \phi \to u^3,\ \ t \to u^4. \tag{38}$$

A possible (not unique) quaternion factorization is given by the values

$$A_1 = \sqrt{g_{11}},\ \ B_2 = \sqrt{g_{22}},\ \ C_3 = \sqrt{g_{33} - \frac{(g_{34})^2}{g_{44}}},\ \ C_4 = \frac{g_{34}}{\sqrt{g_{44}}},\ \ D_4 = \sqrt{g_{44}}, \tag{39}$$

and as before,

$$\frac{1}{A_1}\frac{\partial w_1}{\partial u^1} = \frac{1}{B_2}\frac{\partial w_2}{\partial u^2} = \frac{1}{K_2}\left(C_3\frac{\partial w_3}{\partial u^3} + C_4\frac{\partial w_4}{\partial u^3}\right) = \frac{1}{D_4}\frac{\partial w_4}{\partial u^4},$$

$$\frac{1}{A_1}\frac{\partial w_2}{\partial u^1} = -\frac{1}{B_2}\frac{\partial w_1}{\partial u^2} = \frac{1}{D_4}\frac{\partial w_3}{\partial u^4} = \frac{1}{K_2}\left(C_4\frac{\partial w_3}{\partial u^3} - C_3\frac{\partial w_4}{\partial u^3}\right),$$

$$\frac{1}{A_1}\frac{\partial w_3}{\partial u^1} = -\frac{1}{K_2}\left(C_3\frac{\partial w_1}{\partial u^3} + C_4\frac{\partial w_2}{\partial u^3}\right) = -\frac{1}{D_4}\frac{\partial w_2}{\partial u^4} = \frac{1}{B_2}\frac{\partial w_4}{\partial u^2},$$

$$\frac{1}{A_1}\frac{\partial w_4}{\partial u^1} = -\frac{1}{D_4}\frac{\partial w_1}{\partial u^4} = \frac{1}{K_2}\left(C_3\frac{\partial w_2}{\partial u^3} - C_4\frac{\partial w_1}{\partial u^3}\right) = -\frac{1}{B_2}\frac{\partial w_3}{\partial u^2}, \tag{40}$$

where $K_2 = C_3^2 + C_4^2$.

For an Euclidean metric, $g_{11} = g_{22} = g_{33} = g_{44} = 1$, and system (34) is reduced to

$$\frac{\partial w_1}{\partial u^1} = \frac{\partial w_2}{\partial u^2} = \frac{\partial w_3}{\partial u^3} = \frac{\partial w_4}{\partial u^4},$$

$$\frac{\partial w_2}{\partial u^1} = -\frac{\partial w_1}{\partial u^2} = \frac{\partial w_3}{\partial u^4} = -\frac{\partial w_4}{\partial u^3},$$

$$\frac{\partial w_3}{\partial u^1} = -\frac{\partial w_1}{\partial u^3} = -\frac{\partial w_2}{\partial u^4} = \frac{\partial w_4}{\partial u^2},$$

$$\frac{\partial w_4}{\partial u^1} = -\frac{\partial w_1}{\partial u^4} = \frac{\partial w_2}{\partial u^3} = -\frac{\partial w_3}{\partial u^2}. \tag{41}$$

Relations like (41) induce conformal mappings in Euclidean four dimensional spaces, as outlined (Machado and Borges [15]) and demonstrated by both of us in Machado and Borges [16], and therefore correspond in some sense to the Cauchy-Riemann equations (14) of the previous section. Two cases for conformal mappings in Euclidean spaces are possible: one has all partial derivatives of w and constant components of μ not identically zero, or the absence of μ constant components imposes that some partial derivatives of w are also zero. Otherwise, one supposes that a transformation w with all Jacobian components not identically zero, but related by hypothesis to a scalar function integrating factor, may generate an homeomorphic application which could be only quasiconformal or locally quasiconformal. In the next section, a more manageable form of (33) is found for the given Riemannian metrics and pure scalar integrating factors. Consequently a Beltrami-like system is deduced, which is the four-dimensional counterpart of (12).

4. RIEMANNIAN METRICS AND BELTRAMI-LIKE SYSTEMS

Given the quaternion factorization described in the previous section for a canonical Riemannian metric, one supposes that a mapping w exists for $\mu = \alpha$, where α is a pure scalar function of the coordinates. Hence one gets

$$dw = \left(\sqrt{g_{11}}du^1 + \sqrt{g_{22}}du^2\mathbf{i} + \sqrt{g_{33}}du^3\mathbf{j} + \sqrt{g_{44}}du^4\mathbf{k}\right)\alpha, \tag{42}$$

and for this case results from (33) that

$$\frac{\partial w}{\partial u^1} = \alpha A_1, \quad \frac{\partial w}{\partial u^2} = \alpha B_2\,\mathbf{i}, \quad \frac{\partial w}{\partial u^3} = \alpha C_3\mathbf{j}, \quad \frac{\partial w}{\partial u^4} = \alpha D_4\mathbf{k}. \tag{43}$$

Therefore,

$$B_2\mathbf{i}\frac{\partial w}{\partial u^1} - A_1\frac{\partial w}{\partial u^2} + C_3\mathbf{j}\frac{\partial w}{\partial u^4} + D_4\mathbf{k}\frac{\partial w}{\partial u^3}$$

$$= \alpha A_1 B_2 - \alpha A_1 B_2 + \alpha C_3 D_4\mathbf{i} - \alpha C_3 D_4\mathbf{i} = 0. \tag{44}$$

After expanding, since $w = w_1 + w_2\mathbf{i} + w_3\mathbf{j} + w_4\mathbf{k}$, one defines an operator Ω such that

$$\Omega\, w = -\left(B_2\frac{\partial w_2}{\partial u^1} + A_1\frac{\partial w_1}{\partial u^2} + C_3\frac{\partial w_3}{\partial u^4} + D_4\frac{\partial w_4}{\partial u^3}\right)$$

$$+\mathbf{i}\left(B_2\frac{\partial w_1}{\partial u^1} - A_1\frac{\partial w_2}{\partial u^2} - D_4\frac{\partial w_3}{\partial u^3} + C_3\frac{\partial w_4}{\partial u^4}\right)$$

$$+\mathbf{j}\left(-B_2\frac{\partial w_4}{\partial u^1} + C_3\frac{\partial w_1}{\partial u^4} + D_4\frac{\partial w_2}{\partial u^3} - A_1\frac{\partial w_3}{\partial u^2}\right)$$

$$+\mathbf{k}\left(B_2\frac{\partial w_3}{\partial u^1} + D_4\frac{\partial w_1}{\partial u^3} - C_3\frac{\partial w_2}{\partial u^4} - A_1\frac{\partial w_4}{\partial u^2}\right) = 0. \tag{45}$$

This homogeneous, quaternionic equation $\Omega\, w = 0$ may be written as

$$B_2\frac{\partial w_2}{\partial u^1} + A_1\frac{\partial w_1}{\partial u^2} + C_3\frac{\partial w_3}{\partial u^4} + D_4\frac{\partial w_4}{\partial u^3} = 0,$$

$$B_2\frac{\partial w_1}{\partial u^1} - A_1\frac{\partial w_2}{\partial u^2} - D_4\frac{\partial w_3}{\partial u^3} + C_3\frac{\partial w_4}{\partial u^4} = 0,$$

$$B_2\frac{\partial w_3}{\partial u^1} + D_4\frac{\partial w_1}{\partial u^3} - C_3\frac{\partial w_2}{\partial u^4} - A_1\frac{\partial w_4}{\partial u^2} = 0,$$

$$-B_2\frac{\partial w_4}{\partial u^1} + C_3\frac{\partial w_1}{\partial u^4} + D_4\frac{\partial w_2}{\partial u^3} - A_1\frac{\partial w_3}{\partial u^2} = 0. \tag{46}$$

This is clearly a four-dimensional generalization of the Beltrami system (12). For an Euclidean metric with a factorization $A_1 = B_2 = D_4 = 1$, $C_3 = -1$, system (46) corresponds to the definition of a left-regular function in the Fueter theory, $\Gamma w = 0$, where

$$\Gamma = \left(\frac{\partial}{\partial u^1} + \mathbf{i}\frac{\partial}{\partial u^2} + \mathbf{j}\frac{\partial}{\partial u^3} + \mathbf{k}\frac{\partial}{\partial u^4}\right), \tag{47}$$

or

$$\frac{\partial w_2}{\partial u^1} + \frac{\partial w_1}{\partial u^2} - \frac{\partial w_3}{\partial u^4} + \frac{\partial w_4}{\partial u^3} = 0,$$

$$\frac{\partial w_1}{\partial u^1} - \frac{\partial w_2}{\partial u^2} - \frac{\partial w_3}{\partial u^3} - \frac{\partial w_4}{\partial u^4} = 0,$$

$$\frac{\partial w_3}{\partial u^1} + \frac{\partial w_1}{\partial u^3} + \frac{\partial w_2}{\partial u^4} - \frac{\partial w_4}{\partial u^2} = 0,$$

$$\frac{\partial w_4}{\partial u^1} + \frac{\partial w_1}{\partial u^4} - \frac{\partial w_2}{\partial u^3} + \frac{\partial w_3}{\partial u^2} = 0. \tag{48}$$

Each known left regular function in this case would therefore realize the homeomorphism (31), for a certain scalar function α such that $\Lambda = (\alpha^2)^{-1}$. A similar result holds for right-regular functions, with

$$\frac{\partial w}{\partial u^1}B_2\mathbf{i} - \frac{\partial w}{\partial u^2}A_1 + \frac{\partial w}{\partial u^4}C_3\mathbf{j} + \frac{\partial w}{\partial u^3}D_4\mathbf{k} = 0 \tag{49}$$

and the choice $A_1 = B_2 = C_3 = 1$, $D_4 = -1$. One comes hence to the conclusion that left regularity in Fueter sense is, in fact, a particular case for the equation $\Omega w = 0$ where Ω is given by (45), when we restrict ourselves to Euclidean metrics. Obviously another operator, similar to Ω, applies for right-regularity. Non canonical metrics like the Weyl-Papapetrou one previously mentioned result in a non-homogeneous equation for Ω: according to (39), and proceeding as before,

$$B_2\mathbf{i}\frac{\partial w}{\partial u^1} - A_1\frac{\partial w}{\partial u^2} + (C_3\mathbf{j} - C_4\mathbf{k})\frac{\partial w}{\partial u^4} + D_4\mathbf{k}\frac{\partial w}{\partial u^3} = 0, \tag{50}$$

or

$$\begin{aligned}
&- B_2\frac{\partial w_2}{\partial u^1} - A_1\frac{\partial w_1}{\partial u^2} - C_3\frac{\partial w_3}{\partial u^4} - D_4\frac{\partial w_4}{\partial u^3} + C_4\frac{\partial w_4}{\partial u^4} = 0,\\
&B_2\frac{\partial w_1}{\partial u^1} - A_1\frac{\partial w_2}{\partial u^2} - D_4\frac{\partial w_3}{\partial u^3} + C_3\frac{\partial w_4}{\partial u^4} + C_4\frac{\partial w_3}{\partial u^4} = 0,\\
&B_2\frac{\partial w_3}{\partial u^1} + D_4\frac{\partial w_1}{\partial u^3} - C_3\frac{\partial w_2}{\partial u^4} - A_1\frac{\partial w_4}{\partial u^2} - C_4\frac{\partial w_1}{\partial u^4} = 0,\\
&-B_2\frac{\partial w_4}{\partial u^1} + C_3\frac{\partial w_1}{\partial u^4} + D_4\frac{\partial w_2}{\partial u^3} - A_1\frac{\partial w_3}{\partial u^2} - C_4\frac{\partial w_2}{\partial u^4} = 0,
\end{aligned} \tag{51}$$

a system which may be given in the form

$$\Omega w + C_4\left(\frac{\partial w_4}{\partial u^4} + \frac{\partial w_3}{\partial u^4} - \frac{\partial w_2}{\partial u^4} - \frac{\partial w_1}{\partial u^4}\right) = 0. \tag{52}$$

5. APPLICATIONS

In the geometrical framework of theoretical physics, the formalism introduced in (19), also called tetrad or vierbein formalism in the four-dimensional case (Lord [17]), or fünfbein for dimensions greather than four, may lead to alternative general relativity theories. Two earlier works by Möller [18] on energy-momentum complex, and by Pellegrini and Plebanski [19] on tetrad components regarded as physical fields subject to variations on the variational principle have tried to extend Einstein's theory well beyond its original main purposes: the form of the fields in Plebanski/Pellegrini's approach makes it easier to introduce matter with intrinsic spin (a somewhat intrinsic angular momentum of matter) in the gravity theories geometrical scenario. F*ü*nfbein formalism has also been used by one of us (Borges [20]) for constructing a geometrical supersymmetric theory of Yang-Mills type. Therefore, it is expected that quaternion factorization (22) briefly discussed from now onwards, would provide even new insights to the development of Einstein's theory and its generalizations. In Einstein's theory written in the alternative language of tetrad (vierbein), rather than in the usual formalism of tensor calculus, the tetrad is interpreted to deal with the gravitational "interaction of spinors" (Bade and Jehle [21]). Spinors, defined by Cartan [22], are the most general linear representation of the Lorentz group (op. cit., Lord [17]). Tensors and even vectors are just special cases of spinors. Nevertheless, despite all the achievements obtained through the use of spinors, they are only a two-dimensional representation of the Lorentz group, which is associated to a four dimensional space-time. This fact would require the introduction of the quaternionic four dimensional factorization approach into the mainstream of general relativity. A coordinate system x^α $(\alpha = 1, 2, 3, 4)$, is set up at each V_4 space-time points. The four vectors $\vec{e}_i$ define the local frame of reference, or as they are named the local inertial frame of reference on V_4. A displacement $d\vec{P}$ is given by

$$d\vec{P} = B^i \vec{e}_i, \tag{53}$$

with B^i being the vierbein $(i = 1, 2, 3, 4)$, and

$$B^i = B^i_\mu \, dx^\mu, \tag{54}$$

where B^i_μ are the vierbein components (16).

If in (53) the frame now is to be regarded as that of tangent vectors, then

$$d\vec{P} = dx^\mu \partial_\mu. \tag{55}$$

Therefore:

$$\partial_\mu = B^i_\mu \, \vec{e}_i, \tag{56}$$

and

$$\vec{e}_i = B^\mu_i \partial_\mu; \quad \|B^i_\mu\| = \|B^\mu_i\|^{-1}. \tag{57}$$

The vierbein components may be reinterpreted as the transformation that allows the passage from a local inertial system to a general coordinate system. Up to now no metric has been introduced. Let $\eta_{ij} = (\vec{e}_i . \vec{e}_j)$ be a metric defined on the tangent space T_s on V_4, called the flat metric of general relativity, and defined as

$$\eta_{ij} = 1, \quad i = j = 1,$$

$$\eta_{ij} = -1, \quad i = j = 2, 3, 4,$$

$$\eta_{ij} = 0. \quad i \neq j. \tag{58}$$

Under transformations of SO(1,3) (Lorentz group) on T_s at a point s $\in V_4$, we have that

$$ds^2 = d\vec{P}^2 = B^i_\mu B^j_\nu dx^\mu \, dx^\nu \, \eta_{ij}. \tag{59}$$

In (59),

$$G_{\mu\nu} = B^i_\mu B^j_\nu \eta_{ij}, \tag{60}$$

or

$$\eta_{ij} = B^\mu_i B^\nu_j G_{\mu\nu}. \tag{61}$$

Suppose we change from one tetrad to another $\bar{B}^\mu_a$, then the new tetrad components can be expressed as linear combinations of the old:

$$\bar{B}^\mu_a = \Lambda^b_a B^\mu_b. \tag{62}$$

As $\bar{B}^\mu_a$ must satisfy (61) we obtain that

$$\eta_{ad} = \Lambda^a_b \eta_{bc} \Lambda^c_d, \tag{63}$$

that means Λ is the Lorentz matrix belonging to the Lorentz group SO(1,3). Then in the context of curved space-times we may reinterpret the Lorentz group as the group of rotations of the tetrad.

5.1. SCHWARSZCHILD METRIC

At this point we are ready to perform the quaternionic factorization of the so called Schwarszchild metric that is given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \,. \tag{64}$$

In (64), G is the gravitational constant; M is the mass of the body, the source of the gravitational field; c is the velocity of light and r, θ, ϕ are the spherical coordinates. Accordingly to (20), and in considering (58), (59) and (60), ds^2 may be written as

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu, \tag{65}$$

that means by comparison with (64), $G_{\mu\nu} = 0$ if $\mu \neq \nu$, and

$$B^1_1 B^1_1 - B^2_1 B^2_1 - B^3_1 B^3_1 - B^4_1 B^4_1 = G_{11}; \tag{66}$$

$$B_2^1 B_2^1 - B_2^2 B_2^2 - B_2^3 B_2^3 - B_2^4 B_2^4 = G_{22}; \tag{67}$$

$$B_3^1 B_3^1 - B_3^2 B_3^2 - B_3^3 B_3^3 - B_3^4 B_3^4 = G_{33}; \tag{68}$$

$$B_4^1 B_4^1 - B_4^2 B_4^2 - B_4^3 B_4^3 - B_4^4 B_4^4 = G_{44}. \tag{69}$$

As the tetrad components B_μ^i are arbitrary, except that they must satisfy (61), we may consider

$$B_1^i B_1^j \neq 0, \qquad \text{only if} \quad i = j = 1; \tag{70}$$

$$B_2^i B_2^j \neq 0, \qquad \text{only if} \quad i = j = 2; \tag{71}$$

$$B_3^i B_3^j \neq 0, \qquad \text{only if} \quad i = j = 3; \tag{72}$$

and

$$B_4^i B_4^j \neq 0, \qquad \text{only if} \quad i = j = 4. \tag{73}$$

Therefore,

$$B_1^1 B_1^1 = \left(1 - \frac{2GM}{c^2 r}\right); \;\; G_{11} = \left(1 - \frac{2GM}{c^2 r}\right) \eta_{11}; \tag{74}$$

$$B_2^2 B_2^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}; \;\; G_{22} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \eta_{22}; \tag{75}$$

$$B_3^3 B_3^3 = r^2; \;\; G_{33} = r^2 \, \eta_{33}; \tag{76}$$

$$B_4^4 B_4^4 = r^2 \, \sin^2 \phi; \;\; G_{44} = r^2 \, \sin^2 \phi \, \eta_{44}. \tag{77}$$

In performing the quaternionic factorization (27), it follows that a possible solution is

$$K = \sqrt{\left(\frac{2GM}{c^2 r} - 1\right)} \mathbf{1} + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k},$$

$$\overline{K} = \sqrt{\left(\frac{2GM}{c^2 r} - 1\right)} \mathbf{1} - 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k}, \tag{78}$$

$$L = 0\mathbf{1} + \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)^{-1}}\,\mathbf{i} + 0\mathbf{j} + 0\mathbf{k},$$

$$\overline{L} = 0\mathbf{1} - \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)^{-1}}\,\mathbf{i} - 0\mathbf{j} - 0\mathbf{k}, \tag{79}$$

$$Q = 0\mathbf{1} + 0\mathbf{i} + r\mathbf{j} + 0\mathbf{k},$$

$$\overline{Q} = 0\mathbf{1} - 0\mathbf{i} - r\mathbf{j} - 0\mathbf{k}, \tag{80}$$

$$U = 0\mathbf{1} + 0\mathbf{i} + 0\mathbf{j} + r\,\sin\phi\,\mathbf{k},$$

$$\overline{U} = 0\mathbf{1} - 0\mathbf{i} - 0\mathbf{j} - r\,\sin\phi\,\mathbf{k}, \tag{81}$$

leading to

$$ds^2 = -\left(K\,dt + L\,dr + Q\,d\theta + U\,d\phi\right).\left(\overline{K}\,dt + \overline{L}\,dr + \overline{Q}\,d\theta + \overline{U}\,d\phi\right). \tag{82}$$

Therefore, the corresponding system (46) may be now constructed for the Schwarszchild metric, and solved to generate homeomorphic transformations.

6. CONCLUDING REMARKS

One of the main features of this work is the introduction of the metrical quaternionic factorization concept, connected with the tetrad formalism. It is also shown that the idea of regularity, present in the Fueter theory, may be extended from the classical Euclidean metric to any other quaternionic factorizable metric. Tetrad formalism, which consists of four mutually perpendicular unit vectors (the tetrad or vierbein), was proposed by Möller (op. cit. Möller [18]) in the earlier sixties in order to modify Einstein's theory of gravitation by the enrichment of Riemannian geometry. Besides the curvilinear coordinate transformations, this theory is to admit as a further invariant operation the simultaneous Lorentz rotation of tetrads, at all world points, through an arbitrary but constant Lorentz matrix. The introduction of the tetrad in the general relativity geometrical scenario allows, for instance, the definition of a suitable "energy momentum complex", and a redefinition of Einstein's idea of absolute parallelism: two vectors were said to be parallel in the absolute sense if they have identical components with respect to the local tetrads. In this paper, we claim that a possible quaternionic extension of Einstein's general relativity may be worked out through the tetrad formalism and quaternionic factorizable metrics. The existence of homeomorphic mappings for factorizable Riemannian metrics may be of importance for a new (perhaps numeric) treatment of Einstein's equations. Since these homeomorphisms are solutions of an extended new class of Beltrami-like systems, their study presents a mathematical interest by itself.

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