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PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS WITH SMALL PERTURBATIONS

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ABSTRACT: Let $A : R \times E \to E^*$ be a limit mapping of class (S_+), where E is a real reflexive Banach space. We study the existence of periodic solutions of the following evolution equation

$$\begin{cases} x'(t) = -A(t, x(t)), \quad a.e.\, t \in (0, T); \\ x(0) = x(T). \end{cases}$$

As a special case, we derive the existence of periodic solutions when A is pseudo-monotone.

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1. INTRODUCTION

Let E be a real reflexive Banach space and $A : E \to E^*$ be a nonlinear mapping. Periodic solutions of the following type equation

$$\begin{cases} x'(t) \in -Ax(t) + F(t, x(t)), \quad t \in (0, T); \\ x(0) = x(T) \end{cases} \tag{E1.1}$$

has been studied by many authors under the assumption that A is maximal monotone, accretive or pseudo-monotone operator and other conditions on F. For references, see Browder [8], Chen [10], Deimling [12], Hirano [13], Pruss [16], Shioji [17], Vrabie [18] and the references therein.

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The purpose of the present paper is to study the periodic solution of the following type equation

$$\begin{cases} u'(t) = -A(t, u(t)), & a.e.\, t \in [0, T]; \\ u(0) = u(T), \end{cases} \tag{E1.2}$$

where $A(t, u) : R \times E \to E^*$ is a limit mapping of class (S_+). The limit mapping of class (S_+) includes monotone mapping, pseudo-monotone mapping, class (S_+) mapping and their compact perturbations. Therefore $(E1.1)$ is a special case of $(E1.2)$. By using the topological degree of class (S_+) mapping, which is constructed by Browder [7], we prove the existence of periodic solutions of $(E1.2)$ with a small perturbation. As a special case, we get the main result of Shioji [17]. Browder's degree has also been used in Chen [10] to prove the existence of periodic solutions without coercive condition. We also mention here that, in Mustonen and Berkovitz [4], they claim that their result is true for periodic case, but they need the assumption that both E and H are separable, in order to apply their degree theory, Berkovits and Mustonen [5]. Similar problems of Mustonen and Berkovitz [4] but in multivalued cases have also been studied in Chang et al [9]. For convinience, we recall the definitions of class (S_+) and limit mappings of class (S_+) as follows:

Definition 1.1. A mapping $B : E \to E^*$ is said to be class (S_+) if $\{x_n\}$ converges weakly to x_0 and $\limsup_{n\to\infty}(Bx_n, x_n - x_0) \le 0$ implies that $\{x_n\}$ converges strongly to x_0. A mapping $B : E \to E^*$ is said to be a *limit mapping* of class (S_+) if $B + \epsilon J$ is class (S_+) for $\epsilon > 0$, where $J : E \to E^*$ is the duality mapping.

2. PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

In this section, we study the periodic solution of the following evolution equation

$$\begin{cases} u'(t) = -A(t, u(t)), & a.e.\, t \in [0, T]; \\ u(0) = u(T). \end{cases} \tag{E2.1}$$

In what follows, let $\|\cdot\|$ be the norm in E, $\|\cdot\|_*$ be the norm in E^*,

$$L^p(0, T; E) = \{f : \int_0^T \|f(s)\|^p ds < +\infty\},$$

$$L^q(0, T; E^*) = \{g : \int_0^T \|g(s)\|_*^q ds < +\infty\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$W^{1,p}(0, T; E) = \{u : u(0) = u(T), u \in L^p(0, T; E), u' \in L^q(0, T; E^*)\},$$

endowed with norm

$$\|u\|_T^p = \int_0^T \|u(s)\|^p ds + \int_0^T \|u'(s)\|_*^q ds.$$

W^* is the dual space of $W^{1,p}(0,T;E)$. The pairs between E and E^*, $L^p(0,T;E)$ and $L^q(0,T;E^*)$, $W^{1,p}(0,T;E)$ and W^* are denoted by $(\cdot,\cdot)$, $\langle\cdot,\cdot\rangle$, and $\langle\langle\cdot,\cdot\rangle\rangle$, respectively.

Definition 2.1. $u(t)$ is called a *periodic solution* of $(E2.1)$ if $u(\cdot)\in W^{1,p}(0,T;E)$ and satisfy (E 2.1) for almost all $t\in[0,T]$.

Theorem 2.1. *Let E be a real reflexive Banach space such that both E and E^* are locally uniformly convex. $A(\cdot,\cdot):R\times E\to E^*$ is an operator satisfying the following condtions:*

(1) $A(t,\cdot)$ is a limit mapping of class (S_+) for $t\in R$,

(2) $\|A(t,u)\|_*\le M\|u\|^{p-1}+\gamma(t)$ for all $t\in[0,T], u\in E$, where $\|\cdot\|_*$ is the norm in E^*, $M>0, p>1$, $\gamma(\cdot)\in L^q(0,T)$ and $\frac{1}{p}+\frac{1}{q}=1$,

(3) $(A(t,u),u)\ge\alpha\|u\|^p-\beta(t)\|u\|-g(t)$, a.e. $t\in[0,T]$ and $u\in E$, where $\alpha>0$ is a constant, $\beta(\cdot)\in L^q(0,T)$ and $g(\cdot)\in L^p(0,T)$,

(4) $A(t,v(t))$ is measurable for $v(\cdot)\in W^{1,p}(0,T;E)$.

Then the following evolution equation

$$\begin{cases} u'(t)=-A(t,u(t))-\epsilon J, & a.e.\ \ t\in[0,T];\\ u(0)=u(T) \end{cases} \tag{E2.2}$$

has a solution, where $\epsilon>0$ and $J:E\to E^*$ is the duality mapping.

Proof. By definition of limit mapping of class (S_+), $A(t,\cdot)+\epsilon J$ is class (S_+). We prove that the following equation

$$\begin{cases} u'(t)=-A(t,u(t))-\epsilon Ju(t), & a.e.\ \ t\in[0,T];\\ u(0)=u(T) \end{cases} \tag{E2.3}$$

has a solution in $W^{1,p}(0,T;E)$. Let $B_n:W^{1,p}(0,T;E)\to W^*$ be defined as follows:

$$\langle\langle B_nu,v\rangle\rangle=\frac{1}{n}\langle\langle\mathcal{J}u,v\rangle\rangle+\langle u',v\rangle+\langle A(\cdot,u)+\epsilon Ju,v\rangle$$

for all $u,v\in W^{1,p}(0,T;E)$, where $\mathcal{J}:W^{1,p}(0,T;E)\to W^*$ is the duality mapping. By the assumptions (1), (4) and (6), it is easy to see that B_n is well defined and bounded continuous.

Claim 1: B_n is a demi-continuous mapping of class of $(S)_+$.

Let $u_j\rightharpoonup u_0$ weakly in $W^{1,p}(0,T;E)$ such that

$$\limsup_{j\to\infty}\langle\langle B_nu_j,u_j-u_0\rangle\rangle\le 0.$$

We claim that

$$\liminf_{j\to\infty}\langle A(\cdot,u_j)+\epsilon_n Ju_j, u_j-u_0\rangle \geq 0. \tag{2.1}$$

To prove (2.1), we show that

$$\liminf_{j\to\infty}(A(t,u_j(t))+\epsilon_n Ju_j(t), u_j(t)-u_0(t)) \geq 0, \quad a.e.\ \ t\in[0,T]. \tag{2.2}$$

If (2.2) is not true, then

$$P=\{t:\liminf_{j\to\infty}(A(t,u_j(t))+\epsilon Ju_j(t), u_j(t)-u_0(t))<0\}$$

has positive measure.

By the assumptions (2) and (3), we have

$$\begin{aligned}&(A(t,u_j(t))+\epsilon_n Ju_j(t), u_j(t)-u_0(t))\\&\geq \alpha\|u_j(t)\|^p+[M\|u_j(t)\|^{p-1}+\gamma(t)]\|u_0(t)\|-\beta(t)\|u_j(t)\|\\&\quad -g(t)+\epsilon\|u_j(t)\|\|u_j(t)-u_0(t)\|, \quad a.e.\ \ t\in[0,T].\end{aligned}$$

Hence $(u_j(t))$ is bounded for almost all $t\in P$ and so $u_j(t)\rightharpoonup u_0(t)$ for almost all $t\in P$. But $A(t,\cdot)+\epsilon_n J$ is class (S_+), so $u_j(t)\to u_0(t)$ for almost all $t\in P$.

Hence

$$\lim_{j\to\infty}(A(t,u_j(t))+\epsilon_n Ju_j(t), u_j(t)-u_0(t))=0, \quad a.e.\ \ t\in P,$$

which is a contradiction to the fact that P has positive measure. Therefore (2.2) is true and so is (2.1). Now, by (2.1), we get

$$\limsup_{j\to\infty}\langle\langle \mathcal{J}u_j, u_j-u_0\rangle\rangle \leq 0.$$

$\mathcal{J}$ is a mapping of class (S_+) and so we have $u_j\to u_0$ in $W^{1,p}(0,T;E)$. Therefore, B_n is a mapping of class of $(S)_+$.

Claim 2: For each $n\geq 1$, there exists $r_n>0$ such that $\langle\langle B_n u,u\rangle\rangle>0$ for all $u\in W^{1,p}(0,T;E)$, $\|u\|_T\geq r_n$. Claim 2 follows from the following fact:

$$\begin{aligned}\langle\langle B_n u,u\rangle\rangle &= \frac{1}{n}\langle\langle \mathcal{J}u,u\rangle\rangle+\langle A(t,u)+\epsilon_n Ju,u\rangle\\&\geq \frac{1}{n}\|u\|_T^p+\alpha\int_0^T\|u(t)\|^p dt-\int_0^T\beta(t)\|u\|dt-\int_0^T g(t)dt.\end{aligned}$$

Claim 3: For each $n\geq 1$, there exists $u_n\in W^{1,p}(0,T;E)$ such that $B_n u_n=0$.

Since $\mathcal{J}$ and B_n are operators of class (S_+), by Berkovits and Mustonen [5], $\{tB_n+(1-t)\mathcal{J}\}$ is a homotopy of class (S_+). In view of Claim 2, we know

$$0\neq tB_n u+(1-t)\mathcal{J}u$$

for all $t \in [0,1]$ and $u \in W^{1,p}(0,T;E)$ with $\|u\| = r_n$. By Berkovits and Mustonen [5], we have

$$deg(B_n, S(0,r_n), 0) = deg(\mathcal{J}, S(0,r_n), 0) = 1,$$

where $S(0,r_n) = \{u, \|u\|_T < r_n\}$. Therefore, $B_n u = 0$ has a solution u_n in $S(0,r_n)$.

Now, we prove that $\{u_n\}$ is bounded in $L^p(0,T;E)$. Since $\langle\langle B_n u_n, u_n\rangle\rangle = 0$, we have

$$\begin{aligned} 0 &\geq \int_0^T (A(t,u_n(t)) + \epsilon J u_n(t), u_n(t))dt \\ &\geq \alpha \int_0^T \|u_n\|^p dt - \int_0^T \beta(t)\|u_n\|dt - \int_0^T g(t)dt. \end{aligned}$$

Thus $\int_0^T \|u_n(t)\|^p dt < +\infty$. By the assumption (2), we have $\int_0^T \|u_n'(t)\|_*^q dt < +\infty$. Consequently, $\{u_n\}$ is bounded in $W^{1,p}(0,T;E)$. Thus we may assume $u_n \rightharpoonup u_0$ in $W^{1,p}(0,T;E)$. By the condition (3), we may assume $A(\cdot, w, u_n) \rightharpoonup f_0$ in $L^q(0,T;E^*)$. Since $\langle\langle B_n u_n, u_n - u_0\rangle\rangle = 0$, by letting $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^T (A(t,u_n(t)) + \epsilon J u_n(t), u_n(t) - u_0(t))dt = 0. \tag{2.3}$$

$A(t,\cdot) + \epsilon J$ is class (S_+) for each $t \in [0,T]$ and we have

$$\liminf_{n\to\infty} (A(t,u_n(t)) + \epsilon J u_n(t), u_n(t) - u_0(t)) \geq 0,$$

$a.e.\, t \in [0,T]$. Therefore, it follows from (2.1) that $(A(\cdot,u_n(\cdot)) + \epsilon J u_n(t), u_n(\cdot) - u_0(\cdot)) \to 0$ in measure. Hence there exists a subsequence (n_k) of $\{u_n\}$ such that

$$(A(t,u_{n_k}(t)) + \epsilon J u_n(t), u_{n_k}(t) - u_0(t)) \to 0$$

for almost all $t \in [0,T]$.

It follows from (2) and (3) that $\{u_{n_k}(t)\}$ is bounded for almost all $t \in [0,T]$. Therefore $u_{n_k} \rightharpoonup u_0(t)$, $a.e.\, t \in [0,T]$. $A(t,\cdot) + \epsilon J$ is class (S_+) and $u_{n_k}(t) \to u_0(t)$ for almost all $t \in [0,T]$. It follows from the demi-continuity of $A(t,\cdot)$ that $u_0(\cdot)$ is a solution of $(E2.2)$. This completes the proof. □

Corollary 2.1. *Suppose that all the conditions of Theorem 2.1 are satisfied. Then there exists $u_n(\cdot) \in W^{1,p}(0,T;E)$ such that $\{u_n(\cdot)\}$ is bounded in $W^{1,p}(0,T;E)$ and*

$$\int_0^T \|u_n'(t) + A(t,u_n(t))\|_*^q dt \to 0 \text{ as } n \to \infty.$$

Proof. Taking a positive sequence (ϵ_n) with $\epsilon_n \to 0$, it follows from Theorem 2.1 that the following equation

$$\begin{cases} u'(t) = -A(t,u(t)) - \epsilon_n J u(t), & a.e. t \in [0,T]; \\ u(0) = u(T) \end{cases} \tag{E2.4}$$

has a solution $u_n(\cdot) \in W^{1,p}(0,T;E)$. Since $\langle u_n'(\cdot), u_n(\cdot)\rangle = 0$, it follows from (3) that $\{u_n(\cdot)\}$ is bounded in $L^p(0,T;E)$. Therefore it follows from (2) that $\{u_n(\cdot)\}$ is bounded in $W^{1,p}(0,T;E^*)$. So $u_n'(\cdot) + A(\cdot, u_n(\cdot)) \to 0$ in $L^q(0,T;E^*)$ as $n \to \infty$. This completes the proof. □

Corollary 2.2. *Let E be a real reflexive Banach space such that both E and E^* are locally uniformly convex. Suppose that $A(\cdot,\cdot) : R \times E \to E^*$ is an operator satisfying the following condtions:*

(1) $A(t,\cdot)$ is a pseudo-monotone for $t \in R$,

(2) $\|A(t,u)\|_* \le M\|u\|^{p-1} + \gamma(t)$ for all $t \in [0,T], u \in E$, where $\|\cdot\|_*$ is the norm in E^*, $M > 0, p > 1$, $\gamma(.) \in L^q(0,T)$ and $\frac{1}{p} + \frac{1}{q} = 1$,

(3) $(A(t,u),u) \ge \alpha\|u\|^p - \beta(t)\|u\| - g(t)$, a.e. $t \in [0,T]$ and $u \in E$, where $\alpha > 0$ is a constant, $\beta(\cdot) \in L^q(0,T)$ and $g(.) \in L^p(0,T)$,

(4) $A(t, v(t))$ is measurable for $v(\cdot) \in W^{1,p}(0,T;E)$.

Then the following evolution equation

$$\begin{cases} u'(t) = -A(t,u(t)), & a.e.t \in [0,T]; \\ u(0) = u(T) \end{cases} \tag{E2.5}$$

has a solution.

Proof. Since $A(t,\cdot)$ is pseudo-monotone, we know that $A(t,\cdot)$ is a limit mapping of class (S_+). By Corollary 2.1, there exists $u_n(\cdot) \in W^{1,p}(0,T;E)$ such that $\{u_n(\cdot)\}$ is bounded and $\int_0^T \|u_n'(t) + A(t,u_n(t))\|_*^q dt \to 0$ as $n \to \infty$. For simplicity, we may assume that $u_n(\cdot) \rightharpoonup u_0(\cdot)$ in $W^{1,p}(0,T;E)$. Since $\lim_{n\to\infty}\langle u_n', u_n - u_0\rangle = 0$, we have

$$\lim_{n\to\infty} \int_0^T (A(t,u_n(t)), u_n(t) - u_0(t))dt = 0.$$

By the pseudo-monotonicity of $A(t,\cdot)$ for each t, we have

$$\liminf_{n\to\infty} (A(t,u_n(t)), u_n(t) - u_0(t)) \ge 0, \quad a.e.\, t \in [0,T].$$

Hence we have $(A((\cdot),u(\cdot)), u_n(\cdot) - u_0(\cdot)) \to 0$ in measure and so $(A(t,u_{n_k}(t)), u_{n_k}(t) - u_0(t)) \to 0$ for almost all $t \in [0,T]$. By (3), $\{u_{n_k}(t)\}$ is bounded for almost all $t \in [0,T]$. So $u_{n_k}(t) \rightharpoonup u_0(t)$ for almost all $t \in [0,T]$. Again by the pseudo-monotonicity of $A(t,\cdot)$ for each t, we get

$$u_0'(t) = A(t,u_0(t)) \quad a.e.\, t \in [0,T].$$

u_0 is a periodic solution of $(E2.5)$. This completes the proof. □

Let us now consider a nonlinear operator $A : E \times H \to E^*$ which has been used by Amann [1], Kato [14], Kato [15], Crandall and Souganidis [11] in studying quasilinear or fully nonlinear equations.

Proposition 2.1. *Let $A : E \times H \to E^*$ be a nonlinear operator. Suppose that the following conditions are satisfied:*

(1) E is densely compact embedded in H,

(2) $A(\cdot, u) : E \to E^*$ is pseudo-monotone,

(3) $A(u, \cdot) : H \to E^*$ is uniformly continuous for u in bounded subset of E.

Then $Bu = A(u, u) : E \to E^*$ is pseudo-monotone.

Proof. Suppose $u_j \rightharpoonup u_0$ in E and $\limsup_{j\to\infty}(A(u_j, u_j), u_j - u_0) \le 0$.

By (1) and (3), we have $A(u_j, u_j) - A(u_j, u_0) \to 0$ in E^*. Therefore it follows that

$$\limsup_{j\to\infty}(A(u_j, u_0), u_j - u_0) \le 0.$$

By (2), we have

$$(A(u_0, u_0), u_0 - v) \le \liminf_{j\to\infty}(A(u_j, u_j), u_j - v)$$

for all $v \in E$. Hence B is pseudo-monotone. This completes the proof. □

To end this paper, we briefly sketch an example by using Proposition 2.1 and Corollary 2.2. Let $\Omega \subset R^N$ be an open bounded subset with smooth boundary. Assume that a_i, $b_i : R^2 \to [0, +\infty)$ are continuous functions for $i = 1, 2, \cdots, N$ and $f : R^2 \to R$ is continuous. Suppose that the following conditions are satisfied:

(1) $c_1 \le a_i(t, x) \le c_2$ for all $(t, x) \in R^2$, where $c_1, c_2 > 0$ are constants,

(2) $\Sigma_i[b_i(x_i) - b_i(y_i)](x_i - y_i) \ge 0$, where $x = (x_i), y = (y_i) \in R^N$,

(3) $|b_i(x)| \le \beta|x| + \gamma$ for all $x \in R$ and $i = 1, 2, \cdots, N$,

(4) $\Sigma_i b_i(x_i)x_i \ge \alpha\|x\|^2 - c_0$ for all $x = (x_i) \in R^n$,

(5) $|f(t, x)| \le M + g(t)$ for all $(t, x) \in R^2$, where $M > 0$ and $g(\cdot) \in L^2(0, T)$.

Consider the following partial differential equation

$$\begin{cases} u_t(t,x) = \Sigma_{i=1}^N D_i[a_i(t,u)b_i(D_i u(t,x))] + f(t,u(t,x)), & t \in (0,T), x \in \Omega, \\ u(0,x) = u(T,x), & x \in \Omega, \\ u(t,x) = 0, & t \in [0,T], x \in \partial\Omega. \end{cases} \tag{2.6}$$

Let $A : R \times L^2(\Omega) \times H_0^1(\Omega) \to H^*$ be defined as follows:

$$(A(t, u, v), w) = \int_\Omega \Sigma_{i=1}^N [a_i(t, u)b_i(D_i v)D_i w + f(t, u)w(x)]dx$$

for all $u, w \in H_0^1(\Omega)$. Then

(1) $(A(t,v,u_1) - A(t,v,u_2), u_1 - u_2) \geq 0$ for all $t \in R$, $u_1, u_2, \in H_0^1(\Omega)$, $v \in L^2(\Omega)$, and $A(t,v,\cdot)$ is continuous monotone for each $(t,v) \in R \times L^2(\Omega)$ and so pseudo-monotone. $A(t,\cdot,u)$ is uniformly continuous for each t and u in bounded subset of $H_0^1(\Omega)$.

(2) $H_0^1(\Omega))$ is compactly embedded in $L^2(\Omega)$.

(3) $\|A(t,u,u)\|_* \leq c_2\beta M(\int_\Omega \|Du\|^2 dx)^{\frac{1}{2}} + (\gamma + M + g(t))\sqrt{meas\Omega}$ for all $u \in H_0^1(\Omega)$.

(4) $(A(t,u,u),u) \geq c_1\alpha\|Du\|^2 - (M + g(t))\sqrt{meas\Omega}\|u\| - c_0c_1 meas\Omega$.

By Proposition 2.1 and Corollary 2.2, (2.6) has a periodic solution.

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