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SOLUTIONS FOR A MODEL OF LOW-SPEED FLOW FOR FLUIDS WITH CAPILLARY EFFECTS

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ABSTRACT: We study the initial-value problem for a system of equations that models the low-speed flow of an inviscid, incompressible fluid with capillary stress effects. The system includes hyperbolic equations for the density and velocity, a parabolic equation for the temperature, and an algebraic equation (the equation of state). We prove the local existence of a unique, classical solution to an initial-value problem with suitable initial data. We also present a new, a priori estimate for the density, and then use this estimate, along with a bootstrapping argument, to show that if the regularity of the initial data for the temperature and velocity (but not the density) is increased, then the regularity of the solution for the density, temperature, and velocity may be increased.

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1. INTRODUCTION

In this paper, we study a system of multi-dimensional equations which govern the dynamics of a low-speed, single-phase fluid, and include forces due to capillary stresses that arise from a contribution made to the energy by strong density gradients, Denny and Pego [2]. When viscosity is neglected and when the fluid is inhomogeneous and incompressible, the equations reduce to the following system written in terms of the

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density ρ, temperature θ, and velocity $\mathbf{v}$:

$$\frac{D\rho}{Dt} = 0, \tag{1.1}$$

$$\frac{D\theta}{Dt} = (\rho c_v)^{-1}\nabla\cdot(\kappa\nabla\theta), \tag{1.2}$$

$$\rho\frac{D\mathbf{v}}{\mathbf{Dt}} + \nabla p = c\rho\nabla\Delta\rho - \rho\mathbf{g}, \tag{1.3}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{1.4}$$

Here c is a coefficient of capillarity which is constant, c_v is the specific heat capacity at constant volume, κ is the coefficient of thermal conductivity, $\mathbf{g}$ is the gravitational acceleration, and the material derivative $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$. The term $c\rho\nabla\Delta\rho$ arises from capillary stresses due to energetic contributions of density gradients, as in the theory of Kortweg-type materials developed by Dunn and Serrin [3]. Anderson et al [1] have given a review of related theories and applications to diffuse-interface modeling. The fluid's thermodynamic state is determined by the density ρ and temperature θ. The pressure p is determined from the density and temperature by an equation of state, $p = \hat{p}(\rho, \theta)$.

We now will make the simplifying assumption that the term $\mathbf{v}\cdot\nabla\mathbf{v}$ is negligible in equation (1.3). Then equation (1.3) becomes:

$$\rho\frac{\partial\mathbf{v}}{\partial\mathbf{t}} + \nabla p = c\rho\nabla\Delta\rho - \rho\mathbf{g}\,. \tag{1.5}$$

The system (1.1), (1.2), (1.4), (1.5) supplemented with the equation of state are the equations we shall use to study the initial-value problem. The key difference between these equations (which were systematically derived in a more general form in Denny and Pego [2]) and other similar equations studied previously lies in the presence of the capillary stress term $c\rho\nabla\Delta\rho$ in the momentum equation. The purpose of this paper is to study the existence of solutions to these equations.

In this paper we present a proof of local-in-time existence of a unique, classical solution to the system of equations (1.1), (1.2), (1.4), (1.5) having initial data with sufficient Sobolev regularity, under periodic boundary conditions. That is, we choose for our domain Ω the N-dimensional torus $\mathbb{T}^{\mathbb{N}}$, where $N = 2$ or $N = 3$. In this way we avoid complications with boundary terms, but still work within a bounded domain. The main result of this paper is stated in Theorem 3.1.

The proof of the existence theorem for this nonlinear system is based on the method of successive approximations, in which an iteration scheme, based on solving a linearized version of the equations, is designed and convergence of the sequence of approximating solutions to a unique solution satisfying the nonlinear equations is sought. The framework of our proof follows one used, for example, by A. Majda to prove the existence of a solution to a symmetric, hyperbolic system of conservation laws, Majda [8]. P. Embid in [4], Embid [5] also uses the same framework to prove the existence of a solution to equations for zero Mach number combustion. Under this framework, the convergence proof is presented in two steps. In the first step, we prove uniform boundedness of the approximating sequence of solutions in a high Sobolev norm. The second step is to prove contraction of the sequence in a low Sobolev norm. Standard interpolation and compactness arguments will be used to finish the proof.

We also derive a new, a priori estimate for the density which will be used, along with a bootstrapping argument, to show that if the regularity of the initial data for the velocity and temperature are increased, then the regularity of the solution for the density, temperature, and velocity may be increased.

In an appendix, we provide an existence proof for the linearized equations that appear in the iteration scheme.

2. PRELIMINARIES

The main tools utilized in the existence proof are a priori estimates. We will work with the Sobolev space $H^s(\Omega)$ (where $s \geq 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|f\|_s^2 = \sum_{|\alpha|\leq s} \int_\Omega |D^\alpha f|^2\, dx$ and inner product $(f,g)_s = \sum_{|\alpha|\leq s} \int_\Omega (D^\alpha f)\cdot(D^\alpha g) dx$. Here, we adopt the standard multi-index notation. For convenience, we will denote derivatives by $f_\alpha = D^\alpha f$. We will let Df denote the gradient of f. Also, we will denote the L^2 inner product by $(f,g) = \int_\Omega f \cdot g\ dx$. We will use standard function spaces. $L^\infty([0,T], H^s)$ is the space of bounded measurable functions from $[0,T]$ into $H^s(\Omega)$, with the norm $\|f\|_{s,T}^2 = \text{ess sup}_{0\leq t\leq T} \|f(t)\|_s^2$. $C([0,T], H^s)$ is the space of continuous functions from $[0,T]$ into $H^s(\Omega)$. $L^2([0,T], H^s)$ is the space of measurable functions from $[0,T]$ into $H^s(\Omega)$ with square-integrable norm.

We also introduce a norm that will be used for the a priori estimates for the density, temperature and velocity fields, given by:

$$|||f|||_{s,T}^2 = \|f\|_{s,T}^2 + \int_0^T \|Df\|_s^2\, dt.$$

The following technical lemmas will be needed for the proof of the existence of a unique, classical solution to the initial-value problem for the system (1.1), (1.2), (1.4), (1.5).

Lemma 2.1. (Standard Calculus Inequalities)

(a) If $f \in H^{s_1}(\Omega)$, $g \in H^{s_2}(\Omega)$ and $s_3 = \min(s_1, s_2, s_1+s_2-s_0) \geq 0$, where $s_0 = \left[\dfrac{N}{2}\right] + 1$, then $fg \in H^{s_3}(\Omega)$, and $\|fg\|_{s_3} \leq C\|f\|_{s_1}\|g\|_{s_2}$. We note that $s_0 = 2$ for $N = 2$ or $N = 3$.

(b) If $f \in H^s(\Omega)$, $g \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$, $Df \in L^\infty(\Omega)$, and $|\alpha| \le s$, then

$$\|D^\alpha(fg) - fD^\alpha g\|_0 \le C(|Df|_{L^\infty}\|g\|_{s-1} + |g|_{L^\infty}\|Df\|_{s-1}).$$

In (a) the constant C depends on s_1, s_2, and Ω, while in (b) the constant C depends on s and Ω. These inequalities are well known. Proofs may be found, for example, in Klainerman and Majda [7], Moser [9].

Lemma 2.2. (Low-Norm Commutator Estimate)
If $Df \in H^{r_1}(\Omega)$, $g \in H^{r-1}(\Omega)$, where $r_1 = \max\{r-1, s_0\}$, $s_0 = \left[\frac{N}{2}\right] + 1$, then for any $r \ge 1$, f, g satisfy the estimate $\|D^\alpha(fg) - fD^\alpha g\|_0 \le C\|Df\|_{r_1}\|g\|_{r-1}$, where $r = |\alpha|$, and the constant C depends on r, Ω.

Proof. The proof is based on the Sobolev calculus inequalities from Lemma 2.1. We consider separately the cases $r-1 < s_0$ and $r-1 \ge s_0$, where $r \ge 1$. If $r-1 < s_0$, we expand the term $D^\alpha(fg)$ using the Leibniz rule and then apply inequality (a) from Lemma 2.1 to obtain the desired estimate. If $r-1 \ge s_0$, we apply the inequality (b) from Lemma 2.1 and the Sobolev inequality $|h|_{L^\infty} \le C\|h\|_{s_0}$ for $s_0 = \left[\frac{N}{2}\right] + 1$, to obtain the estimate for this case. Combining these two results then completes the proof. □

Lemma 2.3. *If u is a sufficiently smooth solution of $Du/Dt = f$, $u(\mathbf{x},\mathbf{0}) = \mathbf{u_0}(\mathbf{x})$, where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, with $\Omega = \mathbb{T}^{\mathbb{N}}$, then for any $r \ge 1$, u satisfies the estimate $\|u\|_r^2 \le Ce^{\alpha(t)}\left(\|u_0\|_r^2 + \int_0^t \|f\|_r^2\, d\tau\right)$, where $\alpha(t) = C\int_0^t(1 + \|D\mathbf{v}\|_{r_1})d\tau$, $r_1 = \max\{r-1, s_0\}$, $s_0 = \left[\frac{N}{2}\right] + 1$, and C depends on r.*

Proof. The proof is standard, for example, see Embid [5] for a proof. □

Lemma 2.4. *If u is a sufficiently smooth solution of*

$$Du/Dt = a\Delta u + f, \qquad u(\mathbf{x},\mathbf{0}) = \mathbf{u_0}(\mathbf{x}),$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, and where $c_1 < a(\mathbf{x},\mathbf{t})$, with $0 < c_1 < 1$, and $\Omega = \mathbb{T}^{\mathbb{N}}$, then for $r \ge 1$, u satisfies the estimate

$$\|u\|_r^2 + \int_0^t \|Du\|_r^2\, d\tau \le Ce^{\alpha(t)}\left(\|u_0\|_r^2 + \int_0^t \|f\|_{r-1}^2\, d\tau\right),$$

where $\alpha(t) = C\int_0^t(1 + \|Da\|_{r_1}^2 + \|\mathbf{v}\|_{r_1}^2 + |\nabla\cdot\mathbf{v}|_{L^\infty})d\tau$, and $r_1 = \max\{r-1, s_0\}$ with $s_0 = \left[\frac{N}{2}\right] + 1$, and C depends on r, c_1.

Proof. The proof is standard, see for example, Embid [4] for a proof. □

Lemma 2.5. *If* $\mathbf{u}$ *is a sufficiently smooth solution of* $\partial \mathbf{u}/\partial t = a\nabla p + c\nabla\Delta\rho + \mathbf{f}$, $\nabla \cdot \mathbf{u} = \mathbf{0}$, $\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{u_0}(\mathbf{x})$, $\nabla \cdot \mathbf{u_0} = \mathbf{0}$, *where* c *is a constant, and* $\Omega = \mathbb{T}^{\mathbb{N}}$, *then for any* $r \geq 1$, $\mathbf{u}$ *satisfies the estimate*

$$\|\mathbf{u}\|_r^2 \leq Ce^{\alpha(t)} \|\mathbf{u_0}\|_r^2 + Ce^{\alpha(t)} \int_0^t e^{-\alpha(\tau)}(\|p\|_r^2 + \|\mathbf{f}\|_r^2)d\tau,$$

with $\alpha(t) = \int_0^t C(1 + \|Da\|_{r_1}^2)d\tau$, *where* $r_1 = \max\{r-1, s_0\}$, $s_0 = \left[\frac{N}{2}\right] + 1$, *and* C *depends on* r.

Proof. First, we obtain an L^2 estimate for $\mathbf{u}$. We compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_0^2 &= (\mathbf{u_t}, \mathbf{u}) = (\mathbf{a}\nabla\mathbf{p}, \mathbf{u}) + \mathbf{c}(\nabla\Delta\rho, \mathbf{u}) + (\mathbf{f}, \mathbf{u}) \\ &= -(ap, \nabla\cdot\mathbf{u}) - (\mathbf{p}\nabla\mathbf{a}, \mathbf{u}) - \mathbf{c}(\Delta\rho, \nabla\cdot\mathbf{u}) + (\mathbf{f}, \mathbf{u}) \\ &\leq |Da|_{L^\infty} \|\mathbf{u}\|_0 \|p\|_0 + |(\mathbf{f}, \mathbf{u})| , \end{aligned} \tag{2.1}$$

where we used the fact that $\nabla\cdot\mathbf{u} = \mathbf{0}$. After applying the operator D^α to the equation for $\mathbf{u}$, we obtain

$$\frac{\partial \mathbf{u_a}}{\partial \mathbf{t}} = a\nabla p_\alpha + c\nabla\Delta\rho_\alpha + \mathbf{F}_\alpha\,, \tag{2.2}$$

where $\mathbf{F}_\alpha = \mathbf{f}_\alpha + [(\mathbf{a}\nabla\mathbf{p})_\alpha - \mathbf{a}\nabla\mathbf{p}_\alpha]$. For (2.2), estimate (2.1) becomes

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\alpha\|_0^2 \leq |Da|_{L^\infty} \|\mathbf{u}_\alpha\|_0 \|p_\alpha\|_0 + |(\mathbf{F_a}, \mathbf{u_a})|\,. \tag{2.3}$$

Next, we estimate $|(\mathbf{F}_\alpha, \mathbf{u}_\alpha)|$. Using the commutator estimate from Lemma 2.2, we obtain

$$\begin{aligned} |(\mathbf{F}_\alpha, \mathbf{u}_\alpha)| &\leq |(\mathbf{f}_\alpha, \mathbf{u}_\alpha)| + |((a\nabla p)_\alpha - a\nabla p_\alpha, \mathbf{u}_\alpha)| \\ &\leq C\|\mathbf{f}\|_k \|\mathbf{u}\|_k + C\|Da\|_{k_1} \|Dp\|_{k-1} \|\mathbf{u}\|_k\,, \end{aligned} \tag{2.4}$$

where $k = |\alpha|$, and $k_1 = \max\{k-1, s_0\}$. Substituting estimate (2.4) into (2.3), and adding (2.3) over $|\alpha| = k \leq r$, including the L^2 estimate (2.1), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_r^2 &\leq C\|Da\|_{r_1}(\|p\|_r + \|Dp\|_{r-1})\|\mathbf{u}\|_r + C\|\mathbf{f}\|_r\|\mathbf{u}\|_r \\ &\leq C(1 + \|Da\|_{r_1}^2)\|\mathbf{u}\|_r^2 + C(\|p\|_r^2 + \|\mathbf{f}\|_r^2)\,, \end{aligned}$$

where $r_1 = \max\{r-1, s_0\}$. Here, we used the Sobolev inequality $|h|_{L^\infty} \leq C\|h\|_{s_0}$, where $s_0 = \left[\frac{N}{2}\right] + 1$. After applying Gronwall's inequality, we obtain the desired estimate. □

Lemma 2.6. *If g is sufficiently smooth and if $\Omega = \mathbb{T}^{\mathbb{N}}$, then g satisfies the estimate $\|\nabla g\|_r^2 \leq C(\|\Delta g\|_{r-1}^2 + \|g\|_0^2)$, where $r \geq 1$ and C depends on r.*

Proof. This is a well-known result, for a proof see for example, Evans [6]. □

Lemma 2.7. *If $\mathbf{u}$, a, b, p, and ρ are sufficiently smooth in the equation*

$$\partial \mathbf{u}/\partial t = a\nabla p + c\nabla\Delta\rho - b\mathbf{g}, \tag{2.5}$$

where $\nabla \cdot \mathbf{u} = \mathbf{0}$, c is a constant, $\mathbf{g}$ is a constant, $\Omega = \mathbb{T}^{\mathbb{N}}$, and $r > \frac{N}{2} + 2$, we obtain the following estimate for $\Delta\rho$:

$$\|\Delta\rho\|_r^2 \leq C(1 + |a|_{L^\infty}^2 + \|Da\|_{r-1}^2)\,\|\nabla\rho\|_r^2 + C\,\|\nabla p\|_r^2 + C\,\|b\|_r^2 .$$

Proof. First, we obtain an L^2 estimate for $\Delta\rho$. We apply the divergence operator to equation (2.5), obtaining:

$$\begin{aligned} \Delta^2\rho &= -\frac{1}{c}\nabla\cdot(a\nabla p) + \frac{1}{c}\frac{\partial(\nabla\cdot\mathbf{u})}{\partial \mathbf{t}} + \frac{1}{c}\nabla\cdot(b\mathbf{g}) \\ &= -\frac{1}{c}\nabla\cdot(a\nabla p) + \frac{1}{c}\nabla\cdot(b\mathbf{g})\,, \end{aligned} \tag{2.6}$$

where we used the fact that $\nabla \cdot \mathbf{u} = \mathbf{0}$. Integrating by parts with ρ yields:

$$\begin{aligned} (\Delta\rho, \Delta\rho) &= (\Delta^2\rho, \rho) = -\frac{1}{c}(\nabla\cdot(a\nabla p), \rho) + \frac{1}{c}(\nabla\cdot(b\mathbf{g}), \rho) \\ &= \frac{1}{c}(a\nabla p, \nabla\rho) - \frac{1}{c}(b\mathbf{g}, \nabla\rho) \\ &\leq C(1 + |a|_{L^\infty}^2)\,\|\nabla\rho\|_0^2 + C\,\|\nabla p\|_0^2 + C\,\|b\|_0^2 . \end{aligned} \tag{2.7}$$

Next, after applying D^α to the equation (2.6), we obtain the equation:

$$\Delta^2\rho_\alpha = -\frac{1}{c}\nabla\cdot(a\nabla p_\alpha) + \frac{1}{c}\nabla\cdot(b_\alpha\mathbf{g}) + \mathbf{F}_\alpha\,,$$

where $F_\alpha = -\frac{1}{c}[\nabla\cdot(a\nabla p)_\alpha - \nabla\cdot(a\nabla p_\alpha)]$. Integrating this equation by parts with ρ_α yields:

$$\begin{aligned} (\Delta\rho_\alpha, \Delta\rho_\alpha) &= (\Delta^2\rho_\alpha, \rho_\alpha) = -\frac{1}{c}(\nabla\cdot(a\nabla p_\alpha), \rho_\alpha) + \frac{1}{c}(\nabla\cdot(b_\alpha\mathbf{g}), \rho_\alpha) + (\mathbf{F}_\alpha, \rho_\alpha) \\ &= \frac{1}{c}(a\nabla p_\alpha, \nabla\rho_\alpha) - \frac{1}{c}(b_\alpha\mathbf{g}, \nabla\rho_\alpha) + (\mathbf{F}_\alpha, \rho_\alpha) \\ &\leq C(1 + |a|_{L^\infty}^2)\,\|\nabla\rho_\alpha\|_0^2 + C\,\|\nabla p_\alpha\|_0^2 + C\,\|b_\alpha\|_0^2 + |(F_\alpha, \rho_\alpha)|\,. \end{aligned} \tag{2.8}$$

Next, we use the commutator estimate from Lemma 2.2 to estimate $|(F_\alpha, \rho_\alpha)|$, obtaining:

$$\begin{aligned} |(F_\alpha, \rho_\alpha)| &= \left| \frac{1}{c}([\nabla \cdot (a\nabla p)_\alpha - \nabla \cdot (a\nabla p_\alpha)], \rho_\alpha) \right| \\ &= \left| \frac{1}{c}([(a\nabla p)_\alpha - (a\nabla p_\alpha)], \nabla\rho_\alpha) \right| \\ &\leq C\left\|Da\right\|_{k_1}\left\|\nabla p\right\|_{k-1}\left\|\nabla\rho\right\|_k , \end{aligned} \tag{2.9}$$

where $k = |\alpha|$, and $k_1 = \max(k-1, s_0)$. Subsitituting (2.9) into (2.8), and adding (2.8) over $|\alpha| = k \leq r$, including the L^2 estimate (2.7), we obtain

$$\begin{aligned} \left\|\Delta\rho\right\|_r^2 &\leq C(1+|a|_{L^\infty}^2)\left\|\nabla\rho\right\|_r^2 + C\left\|\nabla p\right\|_r^2 + C\left\|b\right\|_r^2 + C\left\|Da\right\|_{r_1}\left\|\nabla p\right\|_{r-1}\left\|\nabla\rho\right\|_r \\ &\leq C(1+|a|_{L^\infty}^2 + \left\|Da\right\|_{r_1}^2)\left\|\nabla\rho\right\|_r^2 + C\left\|\nabla p\right\|_r^2 + C\left\|b\right\|_r^2 , \end{aligned}$$

where $r_1 = \max(r-1, s_0) = r-1$ for $r > \dfrac{N}{2} + 2$. This completes the proof. □

Lemma 2.8. *If* $\mathbf{u}$, a, b, p, *and* ρ *are sufficiently smooth in the equation*

$$\partial\mathbf{u}/\partial t = a\nabla p + c\nabla\Delta\rho - b\mathbf{g}, \tag{2.10}$$

where $\nabla\cdot\mathbf{u} = \mathbf{0}$, c *is a constant,* $\mathbf{g}$ *is a constant, and* $\Omega = \mathbb{T}^{\mathbb{N}}$, *then for* $r > \dfrac{N}{2} + 3$, $\nabla\rho$ *satisfies the following estimates:*

$$\begin{aligned} \left\|\nabla\rho\right\|_r^2 \leq C\left\|\rho\right\|_0^2 + C(1+|a|_{L^\infty}^2 + \left\|Da\right\|_{r-2}^2)\left\|\rho\right\|_r^2 \\ + C\left\|p\right\|_r^2 + C\left\|b\right\|_{r-1}^2 \end{aligned}$$

and

$$\begin{aligned} \left\|\nabla\rho\right\|_{r+1}^2 \leq C\left\|\rho\right\|_0^2 + C(1+|a|_{L^\infty}^2 + \left\|Da\right\|_{r-1}^2)[\left\|\rho\right\|_0^2 + (1+|a|_{L^\infty}^2 \\ + \left\|Da\right\|_{r-2}^2)\left\|\rho\right\|_r^2 + \left\|p\right\|_r^2 + \left\|b\right\|_{r-1}^2] + C\left\|\nabla p\right\|_r^2 + C\left\|b\right\|_r^2 \end{aligned}$$

and

$$\begin{aligned} \left\|\nabla\rho\right\|_{r+2}^2 \leq C\left\|\rho\right\|_0^2 + C(1+|a|_{L^\infty}^2 + \left\|Da\right\|_r^2)[\left\|\rho\right\|_0^2 + (1+|a|_{L^\infty}^2 \\ + \left\|Da\right\|_{r-1}^2)[\left\|\rho\right\|_0^2 + (1+|a|_{L^\infty}^2 + \left\|Da\right\|_{r-2}^2)\left\|\rho\right\|_r^2 + \left\|p\right\|_r^2 + \left\|b\right\|_{r-1}^2] \\ + \left\|\nabla p\right\|_r^2 + \left\|b\right\|_r^2] + C\left\|\nabla p\right\|_{r+1}^2 + C\left\|b\right\|_{r+1}^2 , \end{aligned}$$

where C *depends on* r.

Proof. From Lemma 2.7 applied to equation (2.10), we have the estimate

$$\|\Delta\rho\|_s^2 \le C(1+|a|_{L^\infty}^2+\|Da\|_{s-1}^2)\,\|\nabla\rho\|_s^2 + C\,\|\nabla p\|_s^2 + C\,\|b\|_s^2\,, \tag{2.11}$$

where $s > \dfrac{N}{2}+2$. Letting $s=r-1$ in (2.11), and using the fact that $\|\nabla\rho\|_{r-1}^2 \le C\,\|\rho\|_r^2$, and $\|\nabla p\|_{r-1}^2 \le C\,\|p\|_r^2$, yields the estimate:

$$\|\Delta\rho\|_{r-1}^2 \le C(1+|a|_{L^\infty}^2+\|Da\|_{r-2}^2)\,\|\rho\|_r^2 + C\,\|p\|_r^2 + C\,\|b\|_{r-1}^2\,. \tag{2.12}$$

Using Lemma 2.6, we obtain from (2.12) the estimate

$$\|\nabla\rho\|_r^2 \le C(\|\Delta\rho\|_{r-1}^2+\|\rho\|_0^2) \le C\,\|\rho\|_0^2 + C(1+|a|_{L^\infty}^2+\|Da\|_{r-2}^2)\,\|\rho\|_r^2$$

$$+C\,\|p\|_r^2 + C\,\|b\|_{r-1}^2\,. \tag{2.13}$$

Letting $s=r$ in (2.11) yields the estimate

$$\|\Delta\rho\|_r^2 \le C(1+|a|_{L^\infty}^2+\|Da\|_{r-1}^2)\,\|\nabla\rho\|_r^2 + C\,\|\nabla p\|_r^2 + C\,\|b\|_r^2\,. \tag{2.14}$$

Using Lemma 2.6, we obtain from (2.14) the estimate

$$\|\nabla\rho\|_{r+1}^2 \le C(\|\Delta\rho\|_r^2+\|\rho\|_0^2)$$

$$\le C\,\|\rho\|_0^2 + C(1+|a|_{L^\infty}^2+\|Da\|_{r-1}^2)\,\|\nabla\rho\|_r^2$$

$$+C\,\|\nabla p\|_r^2 + C\,\|b\|_r^2$$

$$\le C\,\|\rho\|_0^2 + C(1+|a|_{L^\infty}^2+\|Da\|_{r-1}^2)[\|\rho\|_0^2$$

$$+(1+|a|_{L^\infty}^2+\|Da\|_{r-2}^2)\,\|\rho\|_r^2$$

$$+\|p\|_r^2+\|b\|_{r-1}^2] + C\,\|\nabla p\|_r^2 + C\,\|b\|_r^2\,, \tag{2.15}$$

where we inserted the estimate (2.13) for $\|\nabla\rho\|_r^2$ into the right-side of the above inequality. Next, letting $s=r+1$ in (2.11) yields the estimate

$$\|\Delta\rho\|_{r+1}^2 \le C(1+|a|_{L^\infty}^2+\|Da\|_r^2)\,\|\nabla\rho\|_{r+1}^2$$

$$+C\,\|\nabla p\|_{r+1}^2 + C\,\|b\|_{r+1}^2$$

$$\le C(1+|a|_{L^\infty}^2+\|Da\|_r^2)[\|\rho\|_0^2+(1+|a|_{L^\infty}^2$$

$$+\|Da\|_{r-1}^2)[\|\rho\|_0^2+(1+|a|_{L^\infty}^2+\|Da\|_{r-2}^2)\,\|\rho\|_r^2$$

$$+\|p\|_r^2+\|b\|_{r-1}^2]+\|\nabla p\|_r^2+\|b\|_r^2] + C\,\|\nabla p\|_{r+1}^2 + C\,\|b\|_{r+1}^2\,, \tag{2.16}$$

where we inserted the estimate (2.15) for $\|\nabla\rho\|_{r+1}^2$ into the right-side of the above inequality. Finally, using Lemma 2.6, we obtain from (2.16) the estimate:

$$\|\nabla\rho\|_{r+2}^2 \leq C(\|\Delta\rho\|_{r+1}^2 + \|\rho\|_0^2) \leq C\,\|\rho\|_0^2 + C(1+|a|_{L^\infty}^2 + \|Da\|_r^2)[\|\rho\|_0^2$$

$$+(1+|a|_{L^\infty}^2 + \|Da\|_{r-1}^2)[\|\rho\|_0^2 + (1+|a|_{L^\infty}^2 + \|Da\|_{r-2}^2)\,\|\rho\|_r^2$$

$$+\,\|p\|_r^2 + \|b\|_{r-1}^2] + \|\nabla p\|_r^2 + \|b\|_r^2] + C\,\|\nabla p\|_{r+1}^2 + C\,\|b\|_{r+1}^2\,. \tag{2.17}$$

The estimates (2.13), (2.15), 2.17) give the desired result. □

3. EXISTENCE THEOREM

In this subsection, we prove the local-in-time existence of a unique classical solution to the initial-value problem for equations (1.1), (1.2), (1.4), (1.5) with periodic boundary conditions. We assume that p, c_v, and κ are sufficiently smooth functions of the thermodynamic state variables (ρ,θ) in an open set $G \subset \mathbb{R}^2$. The constant capillary coefficient c may be positive or negative. We fix convex, bounded open sets G_0 and G_1 such that $\bar{G}_0 \subset G_1$ and $\bar{G}_1 \subset G$, and we require that the initial data satisfy $(\rho_0(\mathbf{x}),\boldsymbol{\theta_0}(\mathbf{x})) \in \mathbf{G_0}$, for all $\mathbf{x}\in\Omega$, where $\Omega = \mathbb{T}^{\mathbb{N}}$, $N=2$ or 3.

Theorem 3.1. *Suppose* $s > \dfrac{N}{2}+3$ *and* $0 < L_0 < \infty$. *Then there is a* $T_{**} > 0$ *such that, if the initial data* ρ_0, θ_0, $\mathbf{v_0}$ *satisfy* $(\rho_0(\mathbf{x}),\boldsymbol{\theta_0}(\mathbf{x})) \in \mathbf{\bar{G}_0}$ *for all* $\mathbf{x}\in\Omega$ *and* $\rho_0 \in H^s(\Omega)$, $\theta_0 \in H^{s+1}(\Omega)$, $\mathbf{v_0}\in\mathbf{H^{s+1}}(\Omega)$, $\nabla\cdot\mathbf{v_0} = \mathbf{0}$, *and* $\max\left(\|\rho_0\|_s, \|\theta_0\|_{s+1}, \|\mathbf{v_0}\|_{s+1}\right) \leq L_0$, *then the initial-value problem for (1.1), (1.2), (1.4), (1.5) has a unique, classical solution with*

$$\rho \in C([0,T_{**}],C^5)\cap L^\infty([0,T_{**}],H^{s+2})\cap L^2([0,T_{**}],H^{s+3}),$$

$$\theta \in C([0,T_{**}],C^4)\cap L^\infty([0,T_{**}],H^{s+1})\cap L^2([0,T_{**}],H^{s+2}),$$

$$\mathbf{v} \in \mathbf{C}([\mathbf{0},\mathbf{T}_{**}],\mathbf{C^3})\cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^s})\cap \mathbf{L^2}([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^{s+1}}),$$

$$\frac{\partial\rho}{\partial t} \in C([0,T_{**}],C^3)\cap L^\infty([0,T_{**}],H^{s})\cap L^2([0,T_{**}],H^{s+1}),$$

$$\frac{\partial\theta}{\partial t} \in C([0,T_{**}],C^2)\cap L^\infty([0,T_{**}],H^{s-1})\cap L^2([0,T_{**}],H^{s}),$$

$$\frac{\partial\mathbf{v}}{\partial\mathbf{t}} \in C([0,T_{**}],C^2)\cap L^\infty([0,T_{**}],H^{s-1})\cap L^2([0,T_{**}],H^{s}).$$

Moreover, $(\rho(\mathbf{x},\mathbf{t}),\theta(\mathbf{x},\mathbf{t}))\in\mathbf{\bar{G}_1}$ *for* $\mathbf{x}\in\Omega=\mathbb{T}^{\mathbb{N}}$, $0\leq t\leq T_{**}$.

Proof. We will construct the solution of the initial-value problem for (1.1), (1.2), (1.4), (1.5) with $\Omega = \mathbb{T}^{\mathbb{N}}$ through an iteration scheme. Set $\rho^0(\mathbf{x},\mathbf{t}) = \rho_{\mathbf{0}}(\mathbf{x})$, $\theta^0(\mathbf{x},\mathbf{t}) = \theta_{\mathbf{0}}(\mathbf{x})$, $\mathbf{v^0}(\mathbf{x},\mathbf{t}) = \mathbf{v_0}(\mathbf{x})$, where $\nabla \cdot \mathbf{v_0} = \mathbf{0}$. For $k = 0,1,2,\ldots$, define ρ^{k+1}, θ^{k+1}, p^{k+1}, and $\mathbf{v^{k+1}}$ inductively in two steps:

Step 1: Construct ρ^{k+1} and θ^{k+1} from the previous iterates ρ^k, θ^k, and $\mathbf{v^k}$ by solving

$$\frac{D^k\rho^{k+1}}{Dt} = 0 \quad , \tag{3.1}$$

$$\frac{D^k\theta^{k+1}}{Dt} = \frac{1}{\rho^k c_v^k}\left(\nabla\kappa^k \cdot \nabla\theta^k + \kappa^k \Delta\theta^{k+1}\right), \tag{3.2}$$

with initial data $\rho^{k+1}(\mathbf{x},\mathbf{0}) = \rho_{\mathbf{0}}(\mathbf{x})$ and $\theta^{k+1}(\mathbf{x},\mathbf{0}) = \theta_{\mathbf{0}}(\mathbf{x})$. Here, $D^k/Dt = \partial/\partial t + \mathbf{v^k}\cdot\nabla$, and the coefficients are defined by $c_v^k = c_v(\rho^k,\theta^k)$ and $\kappa^k = \kappa(\rho^k,\theta^k)$.

Set $p^{k+1} = p(\rho^{k+1},\theta^{k+1})$, from the equation of state for p.

Step 2: Construct $\mathbf{v^{k+1}}$ from ρ^{k+1}, p^{k+1}, and from the previous iterate ρ^k by solving

$$\rho^k\frac{\partial\mathbf{v^{k+1}}}{\partial\mathbf{t}} + \nabla p^{k+1} = c\rho^k\nabla\Delta\rho^{k+1} - \rho^{k+1}\mathbf{g}, \tag{3.3}$$

$$\nabla\cdot\mathbf{v^{k+1}} = 0, \tag{3.4}$$

with initial data $\mathbf{v^{k+1}}(\mathbf{x},\mathbf{0}) = \mathbf{v_0}(\mathbf{x})$.

Existence of a sufficiently smooth solution to equations (3.1)–(3.4) for fixed k follows from the proof provided in the appendix. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique, classical solution of (1.1), (1.2), (1.4), (1.5).

Recall that the nonlinear functions in (1.1), (1.2), (1.4), (1.5) depend only on (ρ,θ) and are sufficiently smooth in the open set G. Recall $(\rho^k(\mathbf{x},\mathbf{0}),\theta^{\mathbf{k}}(\mathbf{x},\mathbf{0})) = (\rho_{\mathbf{0}}(\mathbf{x}),\theta_{\mathbf{0}}(\mathbf{x})) \in \bar{\mathbf{G}}_{\mathbf{0}}$, and $\bar{G}_0 \subset G_1$ and $\bar{G}_1 \subset G$. We fix $\delta = \hat{\delta}(G_0,G_1)$ so that $0 < \delta < \operatorname{dist}(\bar{G}_0,\partial G_1)$; therefore if $|\rho-\rho_0|_{L^\infty} < \delta$ and $|\theta-\theta_0|_{L^\infty} < \delta$, then $(\rho(\mathbf{x}),\theta(\mathbf{x})) \in \mathbf{G_1}$ for all $\mathbf{x} \in \mathbb{T}^{\mathbb{N}}$. The values of ρ and c_v, κ are strictly positive, bounded, and bounded away from zero for $(\rho,\theta) \in G_1$. We fix $c_1 = \hat{c}_1(G_0,G_1) > 0$ so that the values of ρ^{-1} and $\kappa/\rho c_v$ are greater than c_1 for all $(\rho,\theta) \in \bar{G}_1$. To ensure that the iteration scheme is well-defined, we restrict the kth iterate (ρ^k,θ^k) to $[0,T_k]$, where $T_k > 0$ is the maximum time T for which $\max_{0\le t\le T}\left|\rho^k-\rho_0\right|_{L^\infty} \le \delta$ and $\max_{0\le t\le T}\left|\theta^k-\theta_0\right|_{L^\infty} \le \delta$. Clearly, $T_0 = \infty$ and $T_k \ge T_{k+1}$. We will show that there exists $T_* > 0$ such that $T_k \ge T_*$ for all k. Next, we proceed with the proof of uniform boundedness of the approximating sequence in a high Sobolev norm.

Proposition 3.1. *Assume that the hypotheses of Theorem 3.1 hold. Then there is a $T_* > 0$ and constants L_1, L_2, L_3, L_4, L_5, L_6, and L_7 large enough, such that for $k = 1,2,3\ldots$ the following estimates hold:*

(a) $||\rho^k||_{s,T_*} \leq L_1, \quad |||\theta^k|||_{s+1,T_*} \leq L_1,$

(b) $\max_{0\leq t\leq T_*} |\rho^k - \rho_0|_{L^\infty} \leq \delta, \quad \max_{0\leq t\leq T_*} |\theta^k - \theta_0|_{L^\infty} \leq \delta,$

(c) $||\nabla\rho^k||_{s,T_*} \leq L_2, \quad ||\nabla\rho^k||_{s+1,T_*} \leq L_3, \quad \int_0^{T_*} ||\nabla\rho^k||_{s+2}^2 \, dt \leq L_4^2,$
$|||\rho^k|||_{s+2,T_*} \leq L_5,$

(d) $|||\mathbf{v^k}|||_{s,T_*} \leq L_6,$

(e) $|||\partial\rho^k/\partial t|||_{s,T_*} \leq L_7, \quad |||\partial\theta^k/\partial t|||_{s-1,T_*} \leq L_7, \quad |||\partial\mathbf{v^k}/\partial t|||_{s-1,T_*} \leq L_7.$

Proof. The proof is by induction on k. We show only the inductive step. We will derive estimates for ρ^{k+1}, θ^{k+1}, and $\mathbf{v^{k+1}}$, and then use these estimates to prescribe T_*, L_1, L_2, L_3, L_4, L_5, L_6, and L_7 a priori, independent of k, so that if ρ^k, θ^k, and $\mathbf{v^k}$ satisfy the estimates in (a)-(e), then ρ^{k+1}, θ^{k+1}, and $\mathbf{v^{k+1}}$ also satisfy the same estimates. In the estimates below, all constants may depend on s, G_0, G_1, and the equation of state for p. We use C to denote a generic constant whose value may change from one instance to the next, but is independent of L_1, L_2, L_3, L_4, L_5, L_6, and L_7.

Estimates for $\boldsymbol{\rho^{k+1}}$ and $\boldsymbol{\theta^{k+1}}$: Applying Lemma 2.3, we obtain from equation (3.1) the following estimate for ρ^{k+1}

$$\left\|\rho^{k+1}\right\|_{s,T}^2 \leq \; Ce^{\alpha_k(T)} \left\|\rho_0\right\|_s^2 \leq Ce^{C_0 T} \left\|\rho_0\right\|_s^2 \tag{3.5}$$

for any $T \leq T_k$. Here, we used the induction hypothesis to make the estimate $\alpha_k(T) = \int_0^T C(1 + \left\|D\mathbf{v^k}\right\|_{s-1})dt \leq C_0 T$, where $C_0 = \hat{C}_0(L_6)$. Therefore, we have $||\rho^{k+1}||_{s,T} \leq L_1$ provided that we choose L_1 large enough so that $C\left\|\rho_0\right\|_s^2 < L_1^2$, and provided that we choose T small enough so that $Ce^{C_0 T}\left\|\rho_0\right\|_s^2 < L_1^2$.

Applying Lemma 2.4 to equation (3.2), we derive the following estimate for θ^{k+1}

$$\begin{aligned} &\left\|\theta^{k+1}\right\|_{s+1,T}^2 + \int_0^T \left\|D\theta^{k+1}\right\|_{s+1}^2 dt \\ &\leq \; Ce^{\alpha_k(T)} \left\|\theta_0\right\|_{s+1}^2 + Ce^{\alpha_k(T)} \int_0^T \left\|(\rho^k c_v^k)^{-1}\nabla\kappa^k \cdot \nabla\theta^k\right\|_s^2 dt \end{aligned} \tag{3.6}$$

for any $T \leq T_k$, where $\alpha_k(T) = \int_0^T C(1 + \left|\nabla\cdot\mathbf{v^k}\right|_{L^\infty} + \left\|\mathbf{v^k}\right\|_s^2 + \left\|D((\rho^k c_v^k)^{-1}\kappa^k)\right\|_s^2)dt$. Here, we have used $c_1 < \kappa^k/\rho^k c_v^k$, valid since $(\rho^k, \theta^k) \in \bar{G}_1$ for $(\mathbf{x}, \mathbf{t}) \in \mathbb{T}^{\mathbf{N}} \times [\mathbf{0}, \mathbf{T_k}]$. Using the induction hypothesis, we can estimate $\alpha_k(T) \leq C_1 T$, where $C_1 = \hat{C}_1(L_1, L_2, L_6)$, and we can estimate $\int_0^T C\left\|(\rho^k c_v^k)^{-1}\nabla\kappa^k \cdot \nabla\theta^k\right\|_s^2 dt \leq C_2 T$, where $C_2 = \hat{C}_2(L_1, L_2)$. Using these estimates in the right side of (3.6), we find

$$\begin{aligned} &|||\theta^{k+1}|||_{s+1,T}^2 \\ &\quad = \; \left\|\theta^{k+1}\right\|_{s+1,T}^2 + \int_0^T \left\|D\theta^{k+1}\right\|_{s+1}^2 dt \leq e^{C_3 T}(C\left\|\theta_0\right\|_{s+1}^2 + C_3 T), \end{aligned} \tag{3.7}$$

where $C_3 = \hat{C}_3(L_1, L_2, L_6)$. Therefore, $\left|\left|\left|\theta^{k+1}\right|\right|\right|_{s+1,T} \leq L_1$ provided that we choose L_1 large enough so that $C\left\|\theta_0\right\|_{s+1}^2 < L_1^2$ and provided that we choose T small enough so that $e^{C_3 T}(C\left\|\theta_0\right\|_{s+1}^2 + C_3 T) < L_1^2$.

This proves part (a) of the proposition. We next establish part (b). Directly from the evolution equation (3.1) for ρ^{k+1}, we obtain $\left|\rho^{k+1} - \rho_0\right| \leq \int_0^t \left|\rho_t^{k+1}\right|_{L^\infty} d\tau \leq C\int_0^T \left\|\mathbf{v^k} \cdot \nabla\rho^{\mathbf{k+1}}\right\|_{s-1} d\tau \leq C\int_0^T \left\|\mathbf{v^k}\right\|_{s-1} \left\|\nabla\rho^{k+1}\right\|_{s-1} d\tau \leq C_4 T$, where $C_4 = \hat{C}_4(L_1, L_6)$, where we used the induction hypothesis in the estimate of $\mathbf{v^k}$, and where we have used the results just proven in part (a) for ρ^{k+1}. It follows that

$$\max_{0\leq t\leq T} \left|\rho^{k+1} - \rho_0\right|_{L^\infty} \leq \delta,$$

provided that we choose T small enough so that $C_4 T \leq \delta$.

From the evolution equation (3.2) for θ^{k+1} we obtain

$$\left|\theta^{k+1} - \theta_0\right| \leq \int_0^t \left|\theta_t^{k+1}\right|_{L^\infty} d\tau \leq C\int_0^T \left\|\mathbf{v^k} \cdot \nabla\theta^{\mathbf{k+1}}\right\|_{s-1} d\tau$$

$$+C\int_0^T \left(\left\|(\rho^k c_v^k)^{-1}\nabla\kappa^k \cdot \nabla\theta^k\right\|_{s-1} + \left\|(\rho^k c_v^k)^{-1}\kappa^k\Delta\theta^{k+1}\right\|_{s-1}\right) d\tau$$

$$\leq C_5 T\,, \tag{3.8}$$

where $C_5 = \hat{C}_5(L_1, L_6)$, and where we have used the results just proven in part (a) for θ^{k+1}. It follows that $\max_{0\leq t\leq T} \left|\theta^{k+1} - \theta_0\right|_{L^\infty} \leq C_5 T \leq \delta$, provided that we choose T small enough so that $C_5 T \leq \delta$.

This completes the proof of part (b) of the proposition. We next establish part (c).

Estimates for $\nabla\rho^{\mathbf{k+1}}$: Applying Lemma 2.8, we obtain from equations (3.3), (3.4) the following estimates for $\nabla\rho^{k+1}$:

$$\left\|\nabla\rho^{k+1}\right\|_s^2 \leq C\left\|\rho^{k+1}\right\|_0^2 + C(1 + \left|(\rho^k)^{-1}\right|_{L^\infty}^2 + \left\|D((\rho^k)^{-1})\right\|_{s-2}^2)\left\|\rho^{k+1}\right\|_s^2$$

$$+C\left\|p^{k+1}\right\|_s^2 + C\left\|(\rho^k)^{-1}\rho^{k+1}\right\|_{s-1}^2$$

$$\leq C_6, \tag{3.9}$$

and

$$\left\|\nabla\rho^{k+1}\right\|_{s+1}^2 \leq C\left\|\rho^{k+1}\right\|_0^2 + C(1 + \left|(\rho^k)^{-1}\right|_{L^\infty}^2 + \left\|D((\rho^k)^{-1})\right\|_{s-1}^2)[\left\|\rho^{k+1}\right\|_0^2$$

$$+(1 + \left|(\rho^k)^{-1}\right|_{L^\infty}^2 + \left\|D((\rho^k)^{-1})\right\|_{s-2}^2)\left\|\rho^{k+1}\right\|_s^2 + \left\|p^{k+1}\right\|_s^2 + \left\|(\rho^k)^{-1}\rho^{k+1}\right\|_{s-1}^2]$$

$$+C\left\|\nabla p^{k+1}\right\|_s^2 + C\left\|(\rho^k)^{-1}\rho^{k+1}\right\|_s^2$$

$$\leq C_7, \tag{3.10}$$

and

$$\int_0^T \left\|\nabla\rho^{k+1}\right\|_{s+2}^2 d\tau \le C\int_0^T (\left\|\rho^{k+1}\right\|_0^2 + C(1+\left|(\rho^k)^{-1}\right|_{L^\infty}^2$$

$$+\left\|D((\rho^k)^{-1})\right\|_s^2)[\left\|\rho^{k+1}\right\|_0^2 + (1+\left|(\rho^k)^{-1}\right|_{L^\infty}^2 + \left\|D((\rho^k)^{-1})\right\|_{s-1}^2)[\left\|\rho^{k+1}\right\|_0^2$$

$$+(1+\left|(\rho^k)^{-1}\right|_{L^\infty}^2 + \left\|D((\rho^k)^{-1})\right\|_{s-2}^2)\left\|\rho^{k+1}\right\|_s^2 + \left\|p^{k+1}\right\|_s^2 + \left\|(\rho^k)^{-1}\rho^{k+1}\right\|_{s-1}^2]$$

$$+\left\|\nabla p^{k+1}\right\|_s^2 + \left\|(\rho^k)^{-1}\rho^{k+1}\right\|_s^2] + C\left\|\nabla p^{k+1}\right\|_{s+1}^2 + C\left\|(\rho^k)^{-1}\rho^{k+1}\right\|_{s+1}^2)d\tau$$

$$\le C_8, \tag{3.11}$$

where C depends on s, and where $C_6 = \hat{C}_6(L_1)$, $C_7 = \hat{C}_7(L_1, L_2)$, and $C_8 = \hat{C}_8(L_1, L_2, L_3)$. Here, we used the equation of state for p, we used the estimates from part (a) for $\left\|\rho^{k+1}\right\|_{s,T}$ and $|||\theta^{k+1}|||_{s+1,T}$, and we used the induction hypothesis. We also used the Sobolev inequality $|f|_{L^\infty} \le C\left\|f\right\|_{s-3}$ for $s > \frac{N}{2}+3$. It follows from (3.9), (3.10), (3.11) and from the estimates in part (a) that $|||\rho^{k+1}|||_{s+2,T}^2 = \left\|\rho^{k+1}\right\|_{s+2,T}^2 + \int_0^T \left\|\nabla\rho^{k+1}\right\|_{s+2}^2 dt \le C_9$, where $C_9 = \hat{C}_9(L_1, L_2, L_3)$. Therefore, $\left\|\nabla\rho^{k+1}\right\|_{s,T} \le L_2$, $\left\|\nabla\rho^{k+1}\right\|_{s+1,T} \le L_3$, $\int_0^T \left\|\nabla\rho^{k+1}\right\|_{s+2}^2 dt \le L_4^2$, and $|||\rho^{k+1}|||_{s+2,T} \le L_5$, provided we choose L_2, L_3, L_4, L_5 large enough so that $L_2^2 \ge C_6$, $L_3^2 \ge C_7$, $L_4^2 \ge C_8$, and $L_5^2 \ge C_9$.

This completes the proof of part (c). We next consider part (d).

Estimate for $\mathbf{v^{k+1}}$: Applying Lemma 2.5 to equations (3.3), (3.4), we obtain for $\mathbf{v^{k+1}}$ the estimate

$$\left\|\mathbf{v^{k+1}}\right\|_s^2 \le Ce^{\alpha_k(t)}\left\|\mathbf{v_0}\right\|_s^2 + Ce^{\alpha_k(t)}\int_0^t e^{-\alpha_k(\tau)}\left(\left\|(\rho^k)^{-1}\rho^{k+1}\right\|_s^2 + \left\|p^{k+1}\right\|_s^2\right)d\tau$$

$$\le Ce^{C_{10}T}(\left\|\mathbf{v_0}\right\|_s^2 + C_{11}T)\,, \tag{3.12}$$

where $\alpha_k(T) = \int_0^T C\left(1+\left\|D((\rho^k)^{-1})\right\|_{s-1}^2\right)d\tau \le C_{10}T$, with $C_{10} = \hat{C}_{10}(L_1)$ and $C_{11} = \hat{C}_{11}(L_1)$. Here, we used the induction hypothesis for ρ^k, we used the equation of state for p, and we used the results just obtained from part (a) for ρ^{k+1} and θ^{k+1}. Using the results from part (c) for ρ^{k+1}, we similarly obtain:

$$\int_0^T \left\|D\mathbf{v^{k+1}}\right\|_s^2 dt \le C\int_0^T \left\|\mathbf{v^{k+1}}\right\|_{s+1}^2 dt \le C\int_0^T [e^{\beta_k(t)}\left\|\mathbf{v_0}\right\|_{s+1}^2$$

$$+e^{\beta_k(t)}\int_0^t e^{-\beta_k(\tau)}\left\|(\rho^k)^{-1}\rho^{k+1}\right\|_{s+1}^2 + \left\|p^{k+1}\right\|_{s+1}^2 d\tau]dt$$

$$\le Ce^{C_{12}T}(\left\|\mathbf{v_0}\right\|_{s+1}^2 T + C_{13}T)\,, \tag{3.13}$$

where $\beta_k(T) = \int_0^T C\left(1 + \left\|D((\rho^k)^{-1})\right\|_s^2\right) d\tau \leq C_{12}T$, with $C_{12} = \hat{C}_{12}(L_1, L_2)$ and $C_{13} = \hat{C}_{13}(L_1, L_2)$. Therefore, we have $\left|\left|\left|\mathbf{v^{k+1}}\right|\right|\right|_{s,T}^2 = \left\|\mathbf{v^{k+1}}\right\|_{s,T}^2 + \int_0^T \left\|D\mathbf{v^{k+1}}\right\|_s^2 dt \leq L_6^2$ provided that we choose L_6 large enough so that $C\left\|\mathbf{v_0}\right\|_s^2 < L_6^2$ and provided that we choose T small enough so that $Ce^{C_{10}T}(\left\|\mathbf{v_0}\right\|_s^2 + C_{11}T) + Ce^{C_{12}T}(\left\|\mathbf{v_0}\right\|_{s+1}^2 T + C_{13}T) < L_6^2$. This completes the proof of part (d). We next consider part (e).

Estimates for $\rho_{\mathbf{t}}^{\mathbf{k+1}}$, $\theta_{\mathbf{t}}^{\mathbf{k+1}}$, and $\mathbf{v_t^{k+1}}$: Directly from the evolution equation (3.1) for ρ^{k+1}, from the estimates already derived in parts (a) and (c), and from the induction hypothesis, we obtain the estimate $\left|\left|\left|\rho_t^{k+1}\right|\right|\right|_{s,T} \leq \left|\left|\left|\mathbf{v^k} \cdot \nabla\rho^{\mathbf{k+1}}\right|\right|\right|_{s,T} \leq C_{14}$, where $C_{14} = \hat{C}_{14}(L_5, L_6)$. Therefore, $\left|\left|\left|\rho_t^{k+1}\right|\right|\right|_{s,T} \leq L_7$, provided we choose L_7 large enough so that $L_7 \geq C_{14}$. Using equation (3.2), we can directly estimate $\left|\left|\left|\theta_t^{k+1}\right|\right|\right|_{s-1,T} \leq C_{15}$, where $C_{15} = \hat{C}_{15}(L_1, L_5, L_6)$. Therefore, $\left|\left|\left|\theta_t^{k+1}\right|\right|\right|_{s-1,T} \leq L_7$, provided we choose L_7 large enough so that $L_7 \geq C_{15}$. The estimate for $\left|\left|\left|\mathbf{v_t^{k+1}}\right|\right|\right|_{s-1,T}$ is also straightforward. From equation (3.3), we deduce that $\left|\left|\left|\mathbf{v_t^{k+1}}\right|\right|\right|_{s-1,T} \leq C_{16}$, where $C_{16} = \hat{C}_{16}(L_1, L_5)$. Thus $\left|\left|\left|\mathbf{v_t^{k+1}}\right|\right|\right|_{s-1,T} \leq L_7$ provided we choose L_7 large enough so that $L_7 \geq C_{16}$.

Summarizing, if we fix suitable values of $T = T_*$, L_1, L_2, L_3, L_4, L_5, L_6, and L_7 a priori and independent of k, then ρ^k, θ^k, and $\mathbf{v^k}$ satisfy the estimates in (a)-(e) for all $k \geq 1$. This completes the proof of Proposition 3.1. □

Next, we give the proof of contraction in low norm.

Proposition 3.2. *Given $0 < \zeta < 1$, there is $T_{**} \in (0, T_*]$ such that*

$$\left|\left|\left|\rho^{k+1} - \rho^k\right|\right|\right|_{2,T_{**}}^2 + \left|\left|\left|\theta^{k+1} - \theta^k\right|\right|\right|_{2,T_{**}}^2 + \left|\left|\left|\mathbf{v^{k+1}} - \mathbf{v^k}\right|\right|\right|_{2,T_{**}}^2$$

$$\leq \zeta\left(\left|\left|\left|\rho^k - \rho^{k-1}\right|\right|\right|_{2,T_{**}}^2 + \left|\left|\left|\theta^k - \theta^{k-1}\right|\right|\right|_{2,T_{**}}^2 + \left|\left|\left|\mathbf{v^k} - \mathbf{v^{k-1}}\right|\right|\right|_{2,T_{**}}^2\right) \tag{3.14}$$

for $k = 2, 3, 4\ldots$

Proof. Subtracting the equations (3.1)–(3.4) for ρ^k, θ^k, and $\mathbf{v^k}$ from the equations (3.1)–(3.4) for ρ^{k+1}, θ^{k+1}, and $\mathbf{v^{k+1}}$ yields equations which we write in the form

$$\frac{D^k(\rho^{k+1} - \rho^k)}{Dt} = F_1, \tag{3.15}$$

$$\frac{D^k(\theta^{k+1} - \theta^k)}{Dt} = (\rho^k c_v^k)^{-1}\kappa^k\Delta(\theta^{k+1} - \theta^k) + F_2, \tag{3.16}$$

$$\rho^k\frac{\partial(\mathbf{v^{k+1}} - \mathbf{v^k})}{\partial\mathbf{t}} + \nabla(p^{k+1} - p^k) = c\rho^k\nabla\Delta(\rho^{k+1} - \rho^k) - (\rho^{k+1} - \rho^k)\mathbf{g} + \mathbf{F_3}, \tag{3.17}$$

$$\nabla \cdot (\mathbf{v^{k+1}} - \mathbf{v^k}) = 0, \tag{3.18}$$

where

$$F_1 = -(\mathbf{v^k} - \mathbf{v^{k-1}}) \cdot \nabla \rho^{\mathbf{k}},$$

$$F_2 = -(\mathbf{v^k} - \mathbf{v^{k-1}}) \cdot \nabla \theta^{\mathbf{k}} + ((\rho^{\mathbf{k}}\mathbf{c_v^k})^{-\mathbf{1}}\kappa^{\mathbf{k}} - (\rho^{\mathbf{k-1}}\mathbf{c_v^{k-1}})^{-\mathbf{1}}\kappa^{\mathbf{k-1}})\Delta\theta^{\mathbf{k}}$$

$$+ (\rho^k c_v^k)^{-1}\nabla\kappa^k \cdot \nabla\theta^k - (\rho^{k-1}c_v^{k-1})^{-1}\nabla\kappa^{k-1}\cdot\nabla\theta^{k-1},$$

$$\mathbf{F_3} = -(\rho^k - \rho^{k-1})\left(\frac{\partial \mathbf{v^k}}{\partial \mathbf{t}} - c\nabla\Delta\rho^k\right),$$

and where $(\rho^{k+1} - \rho^k)(\mathbf{x}, \mathbf{0}) = \mathbf{0}$, $(\theta^{k+1} - \theta^k)(\mathbf{x}, \mathbf{0}) = \mathbf{0}$, and $(\mathbf{v^{k+1}} - \mathbf{v^k})(\mathbf{x}, \mathbf{0}) = \mathbf{0}$. Now, we obtain estimates for $\rho^{k+1} - \rho^k$, $\theta^{k+1} - \theta^k$, and $\mathbf{v^{k+1}} - \mathbf{v^k}$. We assume throughout the following derivations that the time T that appears in the estimates satisfies $T \leq T_*$ from Proposition 3.1.

Estimate for $\rho^{\mathbf{k+1}} - \rho^{\mathbf{k}}$: Applying Lemma 2.3 with $r = 2$ and $r = 3$ to (3.15) yields the following estimates for $\rho^{k+1} - \rho^k$:

$$\left\|\rho^{k+1} - \rho^k\right\|_{2,T}^2 \leq Ce^{\alpha_k(T)} \int_0^T \|F_1\|_2^2\, dt$$

$$\leq\ C_1 e^{C_2 T} T \left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_{2,T}^2, \tag{3.19}$$

where $C_1 = \hat{C}_1(L_1)$, and where $\alpha_k(T) = \int_0^T C(1 + \left\|D\mathbf{v^k}\right\|_2)d\tau \leq C_2 T$, with $C_2 = \hat{C}_2(L_6)$. And we obtain the estimate:

$$\int_0^T \left\|D(\rho^{k+1} - \rho^k\right\|_2^2)dt \leq C\left\|\rho^{k+1} - \rho^k\right\|_{3,T}^2 T \leq (Ce^{\beta_k(T)}\int_0^T \|F_1\|_3^2\, dt)T$$

$$\leq\ C_3 e^{C_4 T} T \int_0^T \left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_3^2 dt$$

$$\leq\ C_5 e^{C_4 T} T \int_0^T \left\|D(\mathbf{v^k} - \mathbf{v^{k-1}})\right\|_2^2 + \left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_0^2 dt$$

$$\leq\ C_5 e^{C_4 T} T \left(\int_0^T \left\|D(\mathbf{v^k} - \mathbf{v^{k-1}})\right\|_2^2 dt + T\left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_{2,T}^2\right), \tag{3.20}$$

where $C_3 = \hat{C}_3(L_1)$, $C_5 = \hat{C}_5(L_1)$, and where $\beta_k(T) = \int_0^T C(1 + \left\|D\mathbf{v^k}\right\|_2)d\tau \leq C_4 T$, with $C_4 = \hat{C}_4(L_6)$.

From (3.19), (3.20) we obtain the estimate

$$\left|\left|\left|\rho^{k+1} - \rho^k\right|\right|\right|_{2,T}^2 = \left\|\rho^{k+1} - \rho^k\right\|_{2,T}^2 + \int_0^T \left\|D(\rho^{k+1} - \rho^k\right\|_2^2 dt$$

$$\leq C_6 e^{C_6 T} T \left|\left|\left|\mathbf{v^k} - \mathbf{v^{k-1}}\right|\right|\right|_{2,T}^2, \tag{3.21}$$

where $C_6 = \hat{C}_6(L_1, L_6)$.

Estimate for $\theta^{\mathbf{k+1}} - \theta^{\mathbf{k}}$: Applying Lemma 2.4 with $r = 2$ to equation (3.16) yields the following estimate for $\theta^{k+1} - \theta^k$:

$$\left\|\theta^{k+1} - \theta^k\right\|_{2,T}^2 + \int_0^T \left\|D(\theta^{k+1} - \theta^k)\right\|_2^2 dt \le e^{\alpha_k(T)} \int_0^T C \left\|F_2\right\|_1^2 dt\,, \tag{3.22}$$

where $\alpha_k(T) = C \int_0^T (1 + \left|\nabla \cdot \mathbf{v^k}\right|_{L^\infty} + \left\|\mathbf{v^k}\right\|_2^2 + \left\|D((\rho^k c_v^k)^{-1}\kappa^k)\right\|_2^2) dt \le C_7 T$, and where C depends on c_1, and $C_7 = \hat{C}_7(L_1, L_6)$. And we estimate

$$\left\|F_2\right\|_1^2 \le C_8(\left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_2^2 + \left\|\theta^k - \theta^{k-1}\right\|_2^2 + \left\|\rho^k - \rho^{k-1}\right\|_2^2)\,, \tag{3.23}$$

where $C_8 = \hat{C}_8(L_1)$. Here, we used standard Sobolev space inequalities. Substituting (3.23) into the right side of (3.22), and using the fact that $\alpha_k(T) \le C_7 T$, we obtain the estimate

$$\begin{aligned}\left|\left|\left|\theta^{k+1} - \theta^k\right|\right|\right|_{2,T}^2 &= \left\|\theta^{k+1} - \theta^k\right\|_{2,T}^2 + \int_0^T \left\|D(\theta^{k+1} - \theta^k)\right\|_2^2 dt \\ &\le C_9 e^{C_9 T} T \left(\left\|\mathbf{v^k} - \mathbf{v^{k-1}}\right\|_{2,T}^2 + \left\|\theta^k - \theta^{k-1}\right\|_{2,T}^2 + \left\|\rho^k - \rho^{k-1}\right\|_{2,T}^2\right)\,,\end{aligned} \tag{3.24}$$

where $C_9 = \hat{C}_9(L_1, L_6)$.

Estimate for $\mathbf{v^{k+1}} - \mathbf{v^k}$: After applying Lemma 2.5 to (3.17), (3.18) using $r = 2$, we obtain

$$\begin{aligned}&\left\|\mathbf{v^{k+1}} - \mathbf{v^k}\right\|_2^2 \\ &\le C e^{\alpha_k(t)} \int_0^t e^{-\alpha_k(\tau)} \left(\left\|(\rho^k)^{-1}\mathbf{F_3}\right\|_2^2 + \left\|(\rho^k)^{-1}(\rho^{k+1} - \rho^k)\right\|_2^2 + \left\|p^{k+1} - p^k\right\|_2^2\right) d\tau \\ &\le C e^{\alpha_k(t)} \int_0^t \left\|(\rho^k)^{-1}\right\|_2^2 \left\|\rho^k - \rho^{k-1}\right\|_2^2 \left(\left\|\frac{\partial \mathbf{v^k}}{\partial \mathbf{t}}\right\|_2^2 + \left\|\nabla\Delta\rho^k\right\|_2^2\right) d\tau \\ &+ C e^{\alpha_k(t)} \int_0^t (\left\|(\rho^k)^{-1}\right\|_2^2 \left\|\rho^{k+1} - \rho^k)\right\|_2^2 + \left\|p^{k+1} - p^k\right\|_2^2) d\tau \\ &\le C_{10} e^{C_{11} T} \int_0^T \left\|\rho^k - \rho^{k-1}\right\|_2^2 d\tau \\ &+ C_{10} e^{C_{11} T} \int_0^T (\left\|\rho^{k+1} - \rho^k\right\|_2^2 + \left\|\theta^{k+1} - \theta^k\right\|_2^2) d\tau \\ &\le C_{10} e^{C_{11} T} T \left\|\rho^k - \rho^{k-1}\right\|_{2,T}^2 + C_{10} e^{C_{11} T} T (\left\|\rho^{k+1} - \rho^k\right\|_{2,T}^2 + \left\|\theta^{k+1} - \theta^k\right\|_{2,T}^2)\end{aligned} \tag{3.25}$$

with $C_{10} = \hat{C}_{10}(L_1, L_7)$, where we have used the fact that $s_0 = \left[\frac{N}{2}\right] + 1 = 2$, for $N = 2, 3$, and $r_1 = \max\{r - 1, s_0\} = 2$ for $r = 2$, and where $\alpha_k(T) = \int_0^T C(1 + \left\|D((\rho^k)^{-1})\right\|_2^2)d\tau \leq C_{11}T$, with $C_{11} = \hat{C}_{11}(L_1)$. We also used the fact that $s > \frac{N}{2} + 3$ implies $s \geq 5$ for $N = 2, 3$. And we used the equation of state to estimate $\left\|p^{k+1} - p^k\right\|_2^2$.

Similarly, we obtain the estimate:

$$\int_0^T \left\|D(\mathbf{v^{k+1}} - \mathbf{v^k})\right\|_2^2 dt \leq C \int_0^T \left\|\mathbf{v^{k+1}} - \mathbf{v^k}\right\|_3^2 dt \leq CT \left\|\mathbf{v^{k+1}} - \mathbf{v^k}\right\|_{3,T}^2$$

$$\leq C_{12}e^{C_{13}T}T \int_0^T \left\|\rho^k - \rho^{k-1}\right\|_3^2 dt$$

$$+C_{12}e^{C_{13}T}T \int_0^T (\left\|\rho^{k+1} - \rho^k\right\|_3^2 + \left\|\theta^{k+1} - \theta^k\right\|_3^2)dt$$

$$\leq C_{14}e^{C_{13}T}T \int_0^T \left(\left\|D(\rho^k - \rho^{k-1})\right\|_2^2 + \left\|\rho^k - \rho^{k-1}\right\|_0^2\right) dt$$

$$+C_{14}e^{C_{13}T}T \int_0^T (\left\|D(\rho^{k+1} - \rho^k)\right\|_2^2$$

$$+ \left\|\rho^{k+1} - \rho^k\right\|_0^2 + \left\|D(\theta^{k+1} - \theta^k)\right\|_2^2 + \left\|\theta^{k+1} - \theta^k\right\|_0^2)dt$$

$$\leq C_{14}e^{C_{13}T}T \left(\int_0^T \left\|D(\rho^k - \rho^{k-1})\right\|_2^2 dt + T \left\|\rho^k - \rho^{k-1}\right\|_{2,T}^2\right)$$

$$+C_{14}e^{C_{13}T}T \int_0^T \left\|D(\rho^{k+1} - \rho^k)\right\|_2^2 dt + C_{14}e^{C_{13}T}T^2 \left\|\rho^{k+1} - \rho^k\right\|_{2,T}^2$$

$$+C_{14}e^{C_{13}T}T \int_0^T \left\|D(\theta^{k+1} - \theta^k)\right\|_2^2 dt + C_{14}e^{C_{13}T}T^2 \left\|\theta^{k+1} - \theta^k\right\|_{2,T}^2 \tag{3.26}$$

with $C_{12} = \hat{C}_{12}(L_1, L_2, L_7)$, $C_{13} = \hat{C}_{13}(L_1)$, and $C_{14} = \hat{C}_{14}(L_1, L_2, L_7)$.

Therefore, from (3.25), (3.26) we obtain the estimate

$$\left|\left|\left|\mathbf{v^{k+1}} - \mathbf{v^k}\right|\right|\right|_{2,T}^2 \leq C_{15}e^{C_{15}T}T \left|\left|\left|\rho^k - \rho^{k-1}\right|\right|\right|_{2,T}^2$$

$$+C_{15}e^{C_{15}T}T(\left|\left|\left|\rho^{k+1} - \rho^k\right|\right|\right|_{2,T}^2 + \left|\left|\left|\theta^{k+1} - \theta^k\right|\right|\right|_{2,T}^2) \tag{3.27}$$

with $C_{15} = \hat{C}_{15}(L_1, L_2, L_7)$.

Next, we multiply the estimates (3.21) and (3.24) by $C_{15}e^{C_{15}T}T + 1$, and add the resulting inequality to (3.27) so that we can absorb into the left side the term

$C_{15}e^{C_{15}T}T(|||\rho^{k+1}-\rho^k|||^2_{2,T}+|||\theta^{k+1}-\theta^k|||^2_{2,T})$ coming from the right side of (3.27), so that we finally obtain

$$|||\rho^{k+1}-\rho^k|||^2_{2,T}+|||\theta^{k+1}-\theta^k|||^2_{2,T}+|||\mathbf{v^{k+1}}-\mathbf{v^k}|||^2_{2,T}$$

$$\leq\ C_{16}e^{C_{16}T}T\left(|||\rho^{k}-\rho^{k-1}|||^2_{2,T}+|||\theta^{k}-\theta^{k-1}|||^2_{2,T}+|||\mathbf{v^{k}}-\mathbf{v^{k-1}}|||^2_{2,T}\right),$$

where $C_{16}=\hat{C}_{16}(L_1,L_2,L_6,L_7)$.

We can ensure that $C_{16}e^{C_{16}T}T\leq\zeta<1$ by choosing $T=T_{**}$ small enough, and this completes the proof of Proposition 3.2. □

Using Propositions 3.1 and 3.2, we now complete the proof of Theorem 3.1. From Proposition 3.2, we conclude that there exist $\rho\in C([0,T_{**}],H^2(\Omega))$, $\theta\in C([0,T_{**}],H^2(\Omega))$, and $\mathbf{v}\in\mathbf{C}([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^2}(\Omega))$ so that

$$|||\rho^k-\rho|||_{2,T_{**}}\to 0,\quad |||\theta^k-\theta|||_{2,T_{**}}\to 0,$$

and $|||\mathbf{v^k}-\mathbf{v}|||_{2,T_{**}}\to 0,$ as $k\to\infty$. Using the interpolation inequality $\|f\|_{r'}\leq C\|f\|_2^{\alpha}\|f\|_r^{1-\alpha}$, with $\alpha=(r-r')/(r-2)$, where $r'<r$, and using Proposition 3.1, we can conclude that $|||\rho^k-\rho|||_{s'+2,T_{**}}\to 0$, $|||\theta^k-\theta|||_{s'+1,T_{**}}\to 0$, and $|||\mathbf{v^k}-\mathbf{v}|||_{s',T_{**}}\to 0,$ as $k\to\infty$ for any $s'<s$. For $s'>\dfrac{N}{2}+3$, Sobolev's lemma implies that $\rho^k\to\rho$ in $C([0,T_{**}],C^5)$, $\theta^k\to\theta$ in $C([0,T_{**}],C^4)$, and $\mathbf{v^k}\to\mathbf{v}$ in $C([0,T_{**}],C^3)$. From the linear system of equations (3.1)–(3.4) it follows that $|||\rho_t^k-\rho_t|||_{s',T_{**}}\to 0$, $|||\theta_t^k-\theta_t|||_{s'-1,T_{**}}\to 0$, $|||\mathbf{v_t^k}-\mathbf{v_t}|||_{s'-1,T_{**}}\to 0$, as $k\to\infty$, so that $\rho_t^k\to\rho_t$ in $C([0,T_{**}],C^3)$, $\theta_t^k\to\theta_t$ in $C([0,T_{**}],C^2)$, and $\mathbf{v_t^k}\to\mathbf{v_t}$ in $C([0,T_{**}],C^2)$, and ρ, θ, $\mathbf{v}$, is a classical solution of the system of equations (1.1), (1.2), (1.4), (1.5). The additional facts that $\mathbf{v}\in\mathbf{L}^\infty([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^s})\cap\mathbf{L^2}([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^{s+1}})$, $\mathbf{v_t}\in\mathbf{L}^\infty([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^{s-1}})\cap\mathbf{L^2}([\mathbf{0},\mathbf{T}_{**}],\mathbf{H^s})$, $\rho\in L^\infty([0,T_{**}],H^{s+2})\cap L^2([0,T_{**}],H^{s+3})$, $\rho_t\in L^\infty([0,T_{**}],H^s)\cap L^2([0,T_{**}],H^{s+1})$, $\theta\in L^\infty([0,T_{**}],H^{s+1})\cap L^2([0,T_{**}],H^{s+2})$, and $\theta_t\in L^\infty([0,T_{**}],H^{s-1})\cap L^2([0,T_{**}],H^s)$ can be deduced using boundedness in high norm and a standard compactness argument. The uniqueness of the solution follows by a standard proof, using estimates similar to the proof of Proposition 3.2.□

4. REMARKS ON THE REGULARITY OF THE SOLUTION

The regularity of the solution just obtained may be bootstrapped as follows. First, from Theorem 3.1, given initial data $\rho_0\in L^\infty([0,T],H^s)$, $\theta_0\in L^\infty([0,T],H^{s+1})$, and $\mathbf{v_0}\in\mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$, there exists a solution $\rho\in C([0,T],C^5)\cap L^\infty([0,T],H^{s+2})\cap L^2([0,T],H^{s+3})$, $\theta\in C([0,T],C^4)\cap L^\infty([0,T],H^{s+1})\cap L^2([0,T],H^{s+2})$,
$\mathbf{v}\in\mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{C^3})\cap\mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^s})\cap\mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$ to equations (1.1), (1.2), (1.4),

(1.5). From the a priori estimate for the velocity in Lemma 2.5, we see that the velocity $\mathbf{v} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{C^4}) \cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$, since $\rho \in L^\infty([0,T],H^{s+2})$, $\theta \in L^\infty([0,T],H^{s+1})$, $\mathbf{v_0} \in \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$. But then, by applying to equation (1.2) the a priori estimate in Lemma 2.4 for the temperature, we now have $\theta \in C([0,T],C^5) \cap L^\infty([0,T],H^{s+2}) \cap L^2([0,T],H^{s+3})$, if $\theta_0 \in L^\infty([0,T],H^{s+2})$. Applying Lemma 2.8 to equations (1.4), (1.5), we see now that $\rho \in C([0,T],C^6) \cap L^\infty([0,T],H^{s+3}) \cap L^2([0,T],H^{s+4})$. It follows by the a priori estimate from Lemma 2.5 that $\mathbf{v} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{C^5}) \cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+2}})$, which leads to $\theta \in C([0,T],C^6) \cap L^\infty([0,T],H^{s+3}) \cap L^2([0,T],H^{s+4})$, and which in turn leads to $\rho \in C([0,T],C^7) \cap L^\infty([0,T],H^{s+4}) \cap L^2([0,T],H^{s+5})$, provided $\theta_0 \in L^\infty([0,T],H^{s+3})$ and $\mathbf{v_0} \in \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+2}})$. By repeatedly differentiating the equations (1.1), (1.2), (1.5) with respect to time, the increased regularity of the right-hand side of these equations yields increased regularity with respect to time. This process may then be repeated, and the regularity of the solution ρ, θ, $\mathbf{v}$ thus obtained depends only on the regularity of the initial data for the velocity and the temperature, given that $\rho_0 \in L^\infty([0,T],H^s)$.

Finally, we remark that if the term $\mathbf{v}\cdot\nabla\theta$ is omitted from the temperature evolution equation (1.2), then the bootstrapping process may be carried out for the density and temperature, provided the initial data for the temperature is sufficiently smooth. This bootstrapping can be done for the density and temperature because the a priori estimates in Lemma 2.4 and Lemma 2.8 will be independent of the velocity. Therefore, given that $\rho_0 \in L^\infty([0,T],H^s)$ and given that $\mathbf{v_0} \in \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$, the regularity of the temperature and density may be increased by the bootstrapping procedure, provided that the temperature initial data is sufficiently smooth.

5. CONCLUSION

We have proven the local existence of a unique, classical solution to (1.1), (1.2), (1.4), (1.5) under periodic boundary conditions. To our knowledge, the existence of solutions to this particular system of equations, which was systematically derived in a more general form in Denny and Pego [2], has not been studied before. The presence of the capillary stress term $c\nabla\Delta\rho$ in the momentum equation permits us to derive an a priori estimate in Lemma 2.8 for the density, which we may then use in a bootstrapping argument to increase the regularity of the solution for the density, temperature, and velocity, provided the initial data for the temperature and velocity are sufficiently smooth.

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Appendix A. LINEAR EXISTENCE THEORY FOR THE INVISCID FLUID

In this appendix, we consider the proof of the existence of a unique solution to the linear equations (3.1)–(3.4) from Section 3 under periodic boundary conditions. The existence of a unique solution in $C([0,T],C^3)\cap L^\infty([0,T],H^s)$ to the linear equation (3.1), and the existence of a unique solution in $C([0,T],C^4)\cap L^\infty([0,T],H^{s+1})\cap L^2([0,T],H^{s+2})$ to the linear equation (3.2) are well-known (see, for example, Embid [4], Embid [5], Majda [8]). Consequently, we will proceed to present a proof of the existence of a unique solution to equations (3.3), (3.4). From parts (a), (c) of Proposition 3.1 the functions ρ^k, ρ^{k+1}, and p^{k+1} appearing in equation (3.3) have the regularity required in the statement of the following lemma. Therefore, the existence of a unique solution to equations (3.3), (3.4) follows from Lemma A.1 below.

Lemma A.1. *Given*

$$a,\mathbf{h}\in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^0})\cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^s})\cap \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}}),$$

$$f\in C([0,T],H^0)\cap L^\infty([0,T],H^s)\cap L^2([0,T],H^{s+1}),$$

$$\nabla g\in C([0,T],H^0)\cap L^\infty([0,T],H^{s-1})\cap L^2([0,T],H^s),$$

with $s > \frac{N}{2} + 3$, $\Omega = \mathbb{T}^{\mathbb{N}}$, *there is a unique classical solution of*

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = a\nabla f + \nabla g + \mathbf{h},$$

$$\nabla \cdot \mathbf{u} = \mathbf{0}, \tag{A.1}$$

$$\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{u_0} \in \mathbf{H^{s+1}}, \qquad \nabla \cdot \mathbf{u_0} = \mathbf{0},$$

with

$$\mathbf{u} \ \in \mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbf{C^3}) \cap \mathbf{L}^\infty([\mathbf{0}, \mathbf{T}], \mathbf{H^s}) \cap \mathbf{L^2}([\mathbf{0}, \mathbf{T}], \mathbf{H^{s+1}}),$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} \in C([0,T], C^2) \cap L^\infty([0,T], H^{s-1}) \cap L^2([0,T], H^s).$$

Proof. Since we are solving the initial-value problem under periodic boundary conditions, we will construct the solution using Galerkin's method. First, consider the standard orthonormal basis in L^2 of trigonometric functions $\{\phi_i\}_{i=1}^\infty$, where ϕ_i is of the form $\cos(2\pi \mathbf{n_i} \cdot \mathbf{x})$ or $\sin(2\pi \mathbf{n_i} \cdot \mathbf{x})$ with $\mathbf{n}_i \in \mathbb{Z}_+^{\mathbb{N}}$. We remark that $\{\phi_i\}_{i=1}^\infty$ is orthogonal in H^r for any $r \geq 0$ because each ϕ_i satisfies

$$(u, \phi_i)_{H^r} = \sum_{|\alpha| \leq r} (2\pi \mathbf{n_i})^{\mathbf{2}\alpha} (\mathbf{u}, \phi_\mathbf{i})_{\mathbf{L^2}}\,, \tag{A.2}$$

where $u \in H^r$. (The proof is by integration by parts.) Let P_k denote the orthogonal projection of L^2 onto the finite dimensional subspace $V_k = \text{span}\{\phi_1, \phi_2, ..., \phi_k\}$. From equation (A.2) it follows that $P_k|_{H^r}$ is the orthogonal projection of H^r onto V_k, for any $r \geq 0$.

Let V_k^N denote the space of N-dimensional vectors whose components are in V_k. Let P_k^N denote the projection of N-dimensional vectors $(v_1, v_2, ..., v_N)^T$, whose components are in L^2, onto the finite-dimensional subspace V_k^N, so that $P_k^N(v_1, v_2, ..., v_N)^T = (P_k v_1, P_k v_2, ..., P_k v_N)^T$.

Next, recall that every vector field $\mathbf{v} \in \mathbf{L^2}$ admits a unique orthogonal decomposition in terms of a solenoidal vector field $\mathbf{w}$ and a potential $\nabla\psi$, so that $\mathbf{v} = \mathbf{w} + \nabla\psi$. Here, $\mathbf{w} = \mathbf{P}\mathbf{v}$, where P is the projection onto the solenoidal vector field. If $\mathbf{v} \in \mathbf{H^r}$, with $r \geq 1$, then $\mathbf{w}$ satisfies $\nabla \cdot \mathbf{w} = \mathbf{0}$.

Let W_k^N denote the closed subspace of solenoidal vectors in V_k^N. Let Q_k^N denote the projection of N-dimensional vectors whose components are in L^2 onto the subspace W_k^N, i.e., $Q_k^N = P \circ P_k^N$.

We define the finite-dimensional approximation $\mathbf{u^k} \in \mathbf{W_k^N}$ as the solution of the equation

$$\begin{aligned} \frac{\partial \mathbf{u^k}}{\partial \mathbf{t}} &= Q_k^N \mathbf{F} \\ \mathbf{u^k(t)} &\in W_k^N \\ \mathbf{u^k(x{,}0)} &= Q_k^N \mathbf{u_0}\,, \end{aligned} \tag{A.3}$$

where $\mathbf{F} = \mathbf{a}\nabla\mathbf{f} + \nabla\mathbf{g} + \mathbf{h}$. We remark that from (A.1), it follows that $\nabla \cdot \mathbf{F} = \mathbf{0}$, and so $P\mathbf{F} = \mathbf{F}$.

Since $\mathbf{u^k(t)} \in \mathbf{V_k^N}$, we can write $\mathbf{u^k(t)} = \sum_{\mathbf{l=1}}^{\mathbf{k}} \mathbf{c_l(t)}\phi_\mathbf{l}$, for some $\mathbf{c_l(t)}$, $l = 1, 2, ...k$. We transform (A.3) into an equivalent linear system of ordinary differential equations for the coefficients $\mathbf{c_l(t)}$ by taking the L^2 inner product of each component of (A.3) with ϕ_i for $i = 1, ..., k$, obtaining

$$\frac{dc_{i,j}}{dt} = (F_j, \phi_i) \tag{A.4}$$

$$c_{i,j}(0) = ((u_0)_j, \phi_i), \qquad \text{for} \qquad j = 1, ..., N \quad i = 1, ..., k\,,$$

where $(u_0)_j$ is the jth component of $Q_k^N \mathbf{u_0}$, where $c_{i,j}$ is the jth component of $\mathbf{c_i}$, and where F_j is the jth component of $Q_k^N \mathbf{F}$, $j = 1, 2, ..., N$. Since $\mathbf{F} \in C([0,T], H^0)$ and $\phi_i \in C^\infty(\mathbb{T}^\mathbb{N})$, (A.4) has a unique solution $\{\mathbf{c_i(t)}\}_{\mathbf{i=1}}^{\mathbf{k}}$ in $C^1([0,T])$. It follows that $\mathbf{u^k(t)} \in \mathbf{C^1([0,T], H^r)}$ for any $r \geq 0$.

Next, we derive an energy estimate for $\mathbf{u^k(t)}$ in high Sobolev norm. Taking the inner product in H^r of (A.3) with $\mathbf{u^k}$, where $r = s$ or $r = s+1$, and using the fact that $P_k|_{H^r}$ is the orthogonal projection of H^r onto V_k, and also using the orthogonality of the decomposition into solenoidal and gradient subspaces, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\mathbf{u^k}\right\|_r^2 &= (\mathbf{F}, \mathbf{u^k})_{\mathbf{H^r}} \\ \mathbf{u^k(0)} &= \mathbf{Q_k^N u_0}\,, \end{aligned} \tag{A.5}$$

where $\mathbf{F} = \mathbf{a}\nabla\mathbf{f} + \nabla\mathbf{g} + \mathbf{h}$. From Lemma 2.5, and using the fact that $\left\|Q_k^N \mathbf{u_0}\right\|_r \leq \left\|\mathbf{u_0}\right\|_r$, we derive the estimate

$$\left\|\mathbf{u^k}\right\|_r^2 \leq C e^{\alpha(t)}(\left\|\mathbf{u_0}\right\|_r^2 + \int_0^t (\left\|f\right\|_r^2 + \left\|\mathbf{h}\right\|_r^2) d\tau)\,, \tag{A.6}$$

where $\alpha(t) = C\int_0^t (1 + \left\|Da\right\|_{r-1}^2) d\tau$, where $r = s$ or $r = s+1$, and where C depends on r. Since $f \in C([0,T], H^0) \cap L^\infty([0,T], H^s) \cap L^2([0,T], H^{s+1})$, and since $a, \mathbf{h} \in \mathbf{C([0,T], H^0)} \cap \mathbf{L^\infty([0,T], H^s)} \cap \mathbf{L^2([0,T], H^{s+1})}$, and since $\mathbf{u_0} \in \mathbf{H^{s+1}}$, we conclude that $\mathbf{u^k} \in \mathbf{C([0,T], H^0)} \cap \mathbf{L^\infty([0,T], H^s)} \cap \mathbf{L^2([0,T], H^{s+1})}$.

Next, we derive an estimate for $\partial \mathbf{u^k}/\partial \mathbf{t}$ in low Sobolev norm. From the equation (A.3) we immediately obtain

$$\left\| \frac{\partial \mathbf{u^k}}{\partial \mathbf{t}} \right\|_0 \leq \left\| Q_k^N \mathbf{F} \right\|_0 \leq \|\mathbf{F}\|_{0,T}\,, \tag{A.7}$$

and since $\mathbf{F} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^0}) \cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^{s-1}}) \cap \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^s})$, we conclude that $\left\|\partial \mathbf{u^k}/\partial \mathbf{t}\right\|_0$ is bounded.

From (A.6), (A.7) we know that $\{\mathbf{u^k}\}_{\mathbf{k=1}}^\infty$ is bounded in $L^\infty([0,T],H^s) \cap L^2([0,T],H^{s+1})$, and it is also bounded and equicontinuous in $C([0,T],H^0)$. From the Arzela-Ascoli theorem together with the weak-$*$ compactness of bounded sets in $L^\infty([0,T],H^s)$, and the weak compactness of bounded sets in $L^2([0,T],H^{s+1})$, it follows that there is a subsequence $\{\mathbf{u^{k_j}}\}_{\mathbf{j=1}}^\infty$ of $\{\mathbf{u^k}\}_{\mathbf{k=1}}^\infty$ and $\mathbf{u} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^0}) \cap \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^s}) \cap \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$ such that as $j \to \infty$

$$\mathbf{u^{k_j}} \to \mathbf{u} \quad \text{strongly} \quad \text{in} \quad \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^0})$$

$$\mathbf{u^{k_j}} \to \mathbf{u} \quad \text{weak-}* \quad \text{in} \quad \mathbf{L}^\infty([\mathbf{0},\mathbf{T}],\mathbf{H^s})$$

$$\mathbf{u^{k_j}} \to \mathbf{u} \quad \text{weakly} \quad \text{in} \quad \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}}).$$

And because each $\mathbf{u^{k_j}}$ also satisfies (A.6), we have

$$\|\mathbf{u}\|_r^2 \leq Ce^{\alpha(t)}(\|\mathbf{u_0}\|_r^2 + \int_0^t (\|f\|_r^2 + \|\mathbf{h}\|_r^2) d\tau) \tag{A.8}$$

for $\alpha(t)$ as defined in (A.6), and for $r = s$ or $r = s+1$. From (A.3), we know that $\mathbf{u^{k_j}}$ satisfies

$$\mathbf{u^{k_j}}(\mathbf{t}) = \mathbf{Q^N_{k_j}} \mathbf{u_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{Q^N_{k_j}} \mathbf{F} \mathbf{d}\tau \tag{A.9}$$

in H^0. Because $\mathbf{F} \in C([0,T],H^0)$, we deduce that $P_{k_j}^N \mathbf{F}$ is uniformly bounded on $[0,T]$ in H^0 norm, and converges pointwise to $\mathbf{F}$ as $j \to \infty$ in H^0 norm. Then it follows that $Q_{k_j}^N \mathbf{F}$ is uniformly bounded on $[0,T]$ in H^0 norm, and converges pointwise to $P\mathbf{F}$ as $j \to \infty$ in H^0 norm. (Recall that $Q_{k_j}^N \mathbf{F} = \mathbf{P} \circ \mathbf{P^N_{k_j}} \mathbf{F}$, and that $P\mathbf{F}$ is the projection of $\mathbf{F}$ onto the solenoidal vector field.) Hence, by the Lebesgue dominated convergence theorem, $\mathbf{u}$ satisfies

$$\mathbf{u}(\mathbf{t}) = \mathbf{u_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{P}\mathbf{F} \mathbf{d}\tau \tag{A.10}$$

in H^0. (Recall that $P\mathbf{u_0} = \mathbf{u_0}$, since $\nabla \cdot \mathbf{u_0} = \mathbf{0}$.) From (A.10), we deduce that $\mathbf{u} \in \mathbf{C^1}([\mathbf{0},\mathbf{T}],\mathbf{H^0})$ and solves

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = P\mathbf{F}$$

$$\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{u_0}. \tag{A.11}$$

Recall that from (A.1) we know that $\nabla \cdot \mathbf{F} = \mathbf{0}$ and so $P\mathbf{F} = \mathbf{F}$. Also, $\nabla \cdot \mathbf{u} = \mathbf{0}$ because $\nabla \cdot \mathbf{u^{k_j}} = \mathbf{0}$ for all j. Therefore, $\mathbf{u}$ solves

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{F}$$

$$\nabla \cdot \mathbf{u} = \mathbf{0}$$

$$\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{u_0}. \tag{A.12}$$

Since $\mathbf{u} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^0}) \cap \mathbf{L^\infty}([\mathbf{0},\mathbf{T}],\mathbf{H^s}) \cap \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^{s+1}})$, we have $\mathbf{u} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{H^{s'}})$ for any $s' < s$, where $s > \frac{N}{2} + 3$, and so it follows that $\mathbf{u} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{C^3})$. And, since the right side of (A.12) belongs to $C([0,T],C^2) \cap L^\infty([0,T],H^{s-1}) \cap L^2([0,T],H^s)$, we conclude that $\partial \mathbf{u}/\partial \mathbf{t} \in \mathbf{C}([\mathbf{0},\mathbf{T}],\mathbf{C^2}) \cap \mathbf{L^\infty}([\mathbf{0},\mathbf{T}],\mathbf{H^{s-1}}) \cap \mathbf{L^2}([\mathbf{0},\mathbf{T}],\mathbf{H^s})$.

To show uniqueness, suppose $\mathbf{u_1}, \mathbf{u_2}$ are two solutions of (A.12). Then $\mathbf{u_1} - \mathbf{u_2}$ satisfies $\partial(\mathbf{u_1} - \mathbf{u_2})/\partial \mathbf{t} = \mathbf{0}$, where $(\mathbf{u_1} - \mathbf{u_2})(\mathbf{x},\mathbf{0}) = \mathbf{0}$. Using Lemma 2.3, it follows that $\mathbf{u_1} - \mathbf{u_2} = \mathbf{0}$. $\square$