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EXTINCTION PROFILE OF SOLUTIONS OF A SINGULAR DIFFUSION EQUATION

Shu-Yu Hsu

School of Mathematics
University of Minnesota
Minneapolis, MN 55455, USA

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ABSTRACT: We will prove that if u is the solution of the equation $u_t = \Delta \log u$, $u > 0$, in $B_R \times (0, T)$, $u(x, 0) = u_0(x)$ on B_R, $\partial(\log u)/\partial n = -\alpha$ on $\partial B_R \times (0, T)$, where $B_R = \{x \in R^n : |x| < R\}$, $\alpha > 0$ for $n = 1$, and $0 < \alpha < 4/R$ for $n = 2$, and $T = \int_\Omega u_0 dx/(\alpha|\partial B_R|)$, then there exists a constant $\lambda > 0$ such that the rescaled function $v(x, s) = u(x, t)/(T - t)$, $s = -\log\,(T - t)$, will converge uniformly on $\overline{B}_R$ to $2/(\lambda \cosh^2(x/\sqrt{\lambda}))$ as $s \to \infty$ for $n = 1$. For $n = 2$, if u_0 is radially symmetric, then $v(x, s)$ will converge uniformly on $\overline{B}_R$ to $8\lambda/(\lambda + |x|^2)^2$ for some constant $\lambda > 0$ as $s \to \infty$.

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0. INTRODUCTION

The singular diffusion equation

$$u_t = \Delta \log u \tag{0.1}$$

occurs in many natural phenomena such as the expansion of a thermalized electron cloud (Lonngren and Hirose [27]), the central limit approximation to Carleman's model of the Boltzmann equation (Carleman [5]), and the limiting density distribution of gases which obey the Boltzmann equation (Kurtz [24]). Recently Lions and Toscani [26] showed that it arises in the diffusive limit of finite velocity Boltzmann kinetic models. The solution of the above equation is also the singular limit of the solution of the famous porous medium equation

$$u_t = \Delta(u^m/m) \tag{0.2}$$

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as $m \to 0$ Estaban et al [9], Hui [22]. We refer the reader to the survey papers of Aronson [1] and Peletier [28] for extensive references on the equation (0.2).

Recently there are a lot of research on the equation (0.1) Davis et al [7], Hsu [16], Hsu [17], Hui [23], Esteban et al [8]. Existence of multiple solutions of (0.1) which extinct in finite time for $u_0 \in L^1(R^2) \cap L^p(R^2)$ for some constant $p > 1$ was proved by Hui [23] and for radially symmetric $u_0 \in L^1(R^2)$ by Esteban et al [8]. Regularity and some other properties of solution of (0.1) have been obtained by Dibenedetto and Diller [6] and Davis et al [7]. Existence and uniqueness of global solution of (0.1) satisfying

$$\liminf_{r\to\infty} \frac{\log u(x,t)}{\log r} \ge -2 \text{ uniformly in } [t_1,t_2] \quad \forall t_2 > t_1 > 0$$

and

$$u_t \le \frac{u}{t} \tag{0.3}$$

in $R^2 \times (0,\infty)$ under the very general condition

$$u_0 \notin L^1(R^2), u_0 \in L^p_{loc}(R^2), u_0 \ge 0, \qquad \text{for some } p > 1$$

was recently proved by Hsu [17]. Existence of global solution for u_0 satisfying the growth condition

$$\liminf_{|x|\to\infty} u_0(x)|x|^{-p} > 0$$

for some constant $p > -2$ was also proved by Guo [15]. Large time behaviour of global solutions of (0.1) in $R^2 \times (0,\infty)$ was obtained by Hsu in the papers Hsu [18], Hsu [19]. Asymptotic behaviour of finite mass solution of (0.1) with decay rate $|x|^{-4}$ at infinity in $R^2 \times (0,T)$ near the extinction time with radial symmetric initial value was obtained by in Hsu [16]. Existence of solutions of (0.1) in bounded domains with Neumann boundary condition was proved in the paper of Hui [23].

It was shown in Hsu [21] that the solution of the following Neumann problem

$$\begin{cases} u_t = \Delta(\log u) & \text{in } B_R \times (0,T) \\ u > 0 & \text{on } \overline{B}_R \times (0,T) \\ \dfrac{\partial}{\partial n}(\log u)(x,t) = -\alpha & \text{on } \partial B_R \times (0,T) \\ u(x,0) = u_0(x) & \text{in } B_R \end{cases}, \tag{0.4}$$

where $\alpha > 0$ is a constant and $B_R = \{x \in R^n : |x| < R\}$ will extinct in a finite time T given by

$$T = \frac{1}{\alpha\sigma_n(\partial B_R)} \int_{B_R} u_0 dx\,, \tag{0.5}$$

where $\sigma_n(\partial B_R)$ is the surface measure of ∂B_R. A natural question to ask is what is the behaviour of the solution of (0.4) as the extinction time T is reached. We will answer this question in this paper.

We will show that if u is the solution of (0.4) in $B_R \times (0, T)$, where T is given by (0.5) and

$$v(x, s) = \frac{u(x, t)}{T - t}, \tag{0.6}$$

where

$$s = -\log(T - t), \tag{0.7}$$

then for

$$\begin{cases} n = 1 \\ 0 < \alpha < \infty \end{cases} \tag{0.8}$$

the rescaled function v will converge uniformly on $\overline{B}_R$ to the unique solution of the equation

$$\begin{cases} \Delta \log \overline{v} + \overline{v} = 0 & \text{in } B_R \\ \partial(\log \overline{v})/\partial n = -\alpha & \text{on } \partial B_R \end{cases} \tag{0.9}$$

as $s \to \infty$. For

$$\begin{cases} n = 2 \\ 0 < \alpha < 4/R \end{cases} \tag{0.10}$$

the same convergence result will hold provided u_0 is radially symmetric in B_R. If we write

$$\overline{w} = \log \overline{v}, \tag{0.11}$$

then (0.9) reduces to the following famous equation

$$\begin{cases} \Delta \overline{w} + e^{\overline{w}} = 0 & \text{in } B_R \\ \partial \overline{w}/\partial n = -\alpha & \text{on } \partial B_R. \end{cases} \tag{0.12}$$

Behaviour of solutions of similar type of equations near extinction time have been studied by Galaktionov and Peletier [12], Galaktionov and Peletier [13], Galaktionov and Peletier [14], and Friedman and Herrero [10] etc.

The plan of the paper is as follows. In Section 1 we recall some existence and uniqueness results of (0.4). We will also use a modification of the dynamical system approach of Galaktionov and Peletier [12] to prove some uniform upper and lower bound estimates for the function v when the initial value of (0.4) is radially symmetric and monotone decreasing in Section 1. In Section 2 we will use symmetrization techniques to remove the restriction that the initial value has to be radially symmetric and monotone decreasing in the upper and lower bound estimates of v. Finally in Section 3 we will use energy method to prove the asymptotic behaviour of solution of (0.4) as t tends to the extinction time of the solution.

We begin with some definitions. For any $x_0 \in R^n, R > 0$, we let $B_R(x_0) = \{x \in R^n : |x - x_0| < R\}$, $B_R = B_R(0)$, and $Q_R^T = B_R \times (0,T)$. For any $T > 0$, we say that u is a solution of (0.1) in Q_R^T if $u \in C^\infty(\overline{B}_R \times (0,T))$ satisfies (0.1) in Q_R^T in the classical sense and

$$\inf_{B_R \times [t_1,t_2]} u > 0 \quad \forall 0 < t_1 < t_2 < T. \tag{0.13}$$

For any $u_0 \in L^1(B_R)$, $u_0 \geq 0$, $g \in L^\infty(\partial B_R \times [0,T))$, we say that a solution u of (0.1) in Q_R^T has initial value u_0 if $\|u(\cdot,t) - u_0(\cdot)\|_{L^1(K)} \to 0$ as $t \to 0$ for any compact set $K \subset B_R$ and we say that u is a solution of the following Neumann problem

$$\begin{cases} u_t = \Delta(\log u) & \text{in } Q_R^T \\ u > 0 & \text{on } \overline{B}_R \times (0,T) \\ \dfrac{\partial}{\partial n}(\log u)(x,t) = -g & \text{on } \partial B_R \times (0,T) \\ u(x,0) = u_0(x) & \text{in } B_R \end{cases}, \tag{0.14}$$

if u is a solution of (0.1) in Q_R^T with initial value u_0 and satisfies the following integral identity

$$\int_{t_1}^{t_2}\int_{B_R}(\log u\Delta\eta + u\eta_t)dxds = \int_{t_1}^{t_2}\int_{\partial B_R} g\eta d\sigma ds + \int_{B_R} u\eta dx\bigg|_{t_1}^{t_2}$$

for all $0 < t_1 < t_2 < T$, $\eta \in C^\infty(\overline{B}_R \times (0,T))$ satisfying $\partial\eta/\partial n = 0$ on $\partial B_R \times (0,T)$. For any set $E \subset R^n$, we let $|E|_n$ be the n-dimensional Lebesgue measure of the set E and ∂E be the boundary of the set E. For any set $E \subset \partial B_R \subset R^n$, we let $\sigma_n(E)$ be the measure of the set E with respect to the surface measure on ∂B_R.

1. SECTION ONE

In this section we will recall some existence and uniqueness results of Hsu [21], Hui [23]. We will also use a modification of the dynamical system approach of Galaktionov and Peletier [12] to prove some uniform upper and lower bounds of v for $n = 1$ or 2 when the initial value of the solution of (0.4) is radially symmetric and monotone decreasing in $r \geq 0$. We first start with a result from Hsu [21]:

Theorem 1.1. (Theorem 1.2 of Hsu [21]) *Let $g \in L^\infty(\partial B_R \times [0,\infty))$ and let $u_0 \in L^p(B_R)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$ on $\overline{B}_R$ where $p > n/2$ for $n \geq 3$ and $p > 1$ for $n = 1, 2$. Then there exists a solution u to the Neumann problem (0.14) in $B_R \times (0, T)$ which satisfies*

$$\int_{B_R} u(x,t)dx = \int_{B_R} u_0 dx - \int\int_{\partial B_R \times (0,t)} g d\sigma ds \quad \forall 0 \leq t < T \tag{1.1}$$

where T is given by

$$T = \sup\left\{ t_0 > 0 : \int_{B_R} u_0 dx > \int\int_{\partial B_R \times (0,t)} g d\sigma ds \quad \forall 0 < t \leq t_0 \right\}. \tag{1.2}$$

If $g \geq 0$, then the solution of (0.14) is unique.

By Theorem 1.1 and the same argument as the proof of Lemma 2.2 of Hui [23] we have

Corollary 1.2. *Let $0 \leq \alpha < \infty$ be a constant and let $u_0 \in L^p(B_R)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$ on $\overline{B}_R$ where $p > n/2$ for $n \geq 3$ and $p > 1$ for $n = 1, 2$. Then there exists a unique solution u to the Neumann problem (0.4) in Q_R^T satisfying (0.3) where T is given by (0.5) for $0 < \alpha < \infty$ and $T = \infty$ for $\alpha = \infty$. Moreover the solution satisfies*

$$\int_{B_R} u(x,t)dx = \int_{B_R} u_0 dx - \alpha\sigma_n(\partial B_R)t \quad \forall 0 \leq t < T. \tag{1.3}$$

Lemma 1.3. *Let $0 \leq \alpha < \infty$ be a constant. If $0 \leq u_0 \in L^p(B_R)$ for some constant $p > \max(1, n/2)$ is a radially symmetric and monotone decreasing function of $r \geq 0$, $u_0 \not\equiv 0$, and u is the unique solution of (0.4) in Q_R^T where T is given by (0.5) for $\alpha > 0$ and $T = \infty$ for $\alpha = 0$, then for each $0 \leq t < T$ $u(\cdot, t)$ is radially symmetric and monotone decreasing in $0 \leq r \leq R$.*

Proof. For any $k = 1, 2, \dots$, we let $u_{0,k} = min(u_0, k)$. Without loss of generality we may assume $u_{0,k} \not\equiv 0$. Let u_k be the unique solution of (0.4) in $Q_R^{T_k}$ with initial value $u_{0,k}$ given by Corollary 1.2 where T_k is given by (0.5) with u_0 being replaced by $u_{0,k}$ for $\alpha > 0$ and $T_k = \infty$ for $\alpha = 0$. By Theorem 1.5 of Hsu [16] or Theorem 1.1.5 of Hsu [20] u_k is a radially symmetric and monotone decreasing function of $r > 0$. Since by the proof of Theorem 1.2 of Hsu [21], u_k will converge uniformly to u on every compact subset of Q_R^T to the unique solution u of (0.4) in Q_R^T. Hence u is also radially symmetric and is a monotone decreasing function of $r > 0$.

We will now assume $0 < \alpha < \infty$ in this section. We will let u be the solution of (0.4) in Q_R^T where T is given by (0.5) and let v, s be given by (0.6), (0.7) for the rest of the paper. We first observe that v satisfies

$$v_s = \Delta \log v + v \quad \text{in } B_R \times (-\log T, \infty) \tag{1.4}$$

with $v(x, -\log T) = u(x,0)/T$ and

$$\int_{B_R} v(x,s)dx = \alpha\sigma_n(\partial B_R) \quad \forall s > -\log T. \tag{1.5}$$

Let w be given by

$$w = \log v. \tag{1.6}$$

Then w satisfies

$$e^w w_s = \Delta w + e^w \quad \text{in } B_R \times (-\log T, \infty). \tag{1.7}$$

For any $\lambda > 0$ we let

$$u_\lambda(x,t) = \begin{cases} \dfrac{2(T-t)}{\lambda\cosh^2(x/\sqrt{\lambda})} & \text{for } n = 1 \\ \dfrac{8\lambda(T-t)}{(\lambda + |x|^2)^2} & \text{for } n = 2 \end{cases}$$

and let

$$v_\lambda(x) = \begin{cases} \dfrac{2}{\lambda\cosh^2(x/\sqrt{\lambda})} & \text{for } n = 1 \\ \dfrac{8\lambda}{(\lambda + |x|^2)^2} & \text{for } n = 2. \end{cases} \tag{1.8}$$

Then u_λ satisfies (0.1) in $R^n \times (0,\infty)$ and v_λ satisfies

$$\Delta \log v + v = 0 \quad \text{in } R^n$$

for $n = 1$ or 2.

Lemma 1.4. *Suppose $n = 1$ or 2, $u_0 \in L^p(B_R)$ for some $p > 1$, $u_0 \geq 0$ on B_R, $u_0 \not\equiv 0$, and u_0 is a radially symmetric function. Then for any $s_1 > -\log T$, there exists a constant $\lambda_0 > 0$ such that for any $0 < \lambda \leq \lambda_0$, the graph of v_λ and $v(r, s_1)$ will intersect exactly once on $[0, R)$ and there exists $r_0 \in (\sqrt{\lambda}, R)$ such that the following holds,*

$$\textit{(i)} \qquad v_\lambda(r) > v(r, s_1) \qquad \forall 0 \leq r < r_0$$

$$\textit{(ii)} \qquad v_\lambda(r) < v(r, s_1) \qquad \forall r_0 < r \leq R.$$

Proof. We will use a modification of the proof in Hsu [20] to prove the lemma. Let $C_1 = \min_{0\leq r\leq R} v(r, s_1) > 0$, $C_2 = \max_{0\leq r\leq R} v(r, s_1) > 0$. Since $v_r(0, s_1) = 0$, we can choose $0 < \varepsilon < R$ such that $\sup_{0\leq r\leq \varepsilon} |v_r(r, s_1)| < C_1^{3/2}$. For $n = 1$ since

$$\lambda\cosh^2(\frac{r}{\sqrt{\lambda}}) \geq \frac{\lambda e^{2\varepsilon/\sqrt{\lambda}}}{4} \to \infty \quad \text{as } \lambda \to 0 \quad \forall \varepsilon \leq r \leq R,$$

there exists a constant $\lambda_1 > 0$ such that

$$\lambda\cosh^2(\frac{r}{\sqrt{\lambda}}) > \frac{2}{C_1} \quad \forall\varepsilon \le r \le R, 0 < \lambda \le \lambda_1.$$

Then for any $0 < \lambda \le \lambda_1$, we have

$$\begin{cases} v_\lambda(r) \ge \dfrac{2}{\lambda\cosh^2 1} & \forall 0 \le r \le \sqrt{\lambda} \\ v_\lambda(r) < C_1 & \forall r \ge \varepsilon \end{cases} \quad \text{for } n = 1 \tag{1.9}$$

and

$$\begin{cases} v_\lambda(r) = \dfrac{8\lambda}{(\lambda + r^2)^2} \ge \dfrac{2}{\lambda} & \forall 0 \le r \le \sqrt{\lambda} \\ v_\lambda(r) < 8\lambda r^{-4} & \forall r > 0 \end{cases} \quad \text{for } n = 2. \tag{1.10}$$

Let $\lambda_0 = \min(\varepsilon^2/2, \varepsilon^4 C_1/8, 1/(2C_2\cosh^2 1), 1/2C_2, \lambda_1)$. Then by (1.9) and (1.10) for any $0 < \lambda \le \lambda_0$ we have

$$\begin{cases} v_\lambda(r) > v(r, s_1) & \forall 0 \le r \le \sqrt{\lambda} \\ v_\lambda(r) < v(r, s_1) & \forall \varepsilon \le r \le R. \end{cases} \tag{1.11}$$

Let $g(r) = v_\lambda(r) - v(r, s_1)$. By (1.11), $g(r) > 0 > g(r')$ for all $0 \le r \le \sqrt{\lambda}$, $\varepsilon \le r' \le R$. By the intermediate value theorem, there exists $r_0 \in (\sqrt{\lambda}, \varepsilon)$ such that $g(r_0) = 0$. Hence g has a zero in $(\sqrt{\lambda}, \varepsilon)$ and no zero in $[0, \sqrt{\lambda}] \cup [\varepsilon, R)$. Suppose g has another zero at $r_1 \in (\sqrt{\lambda}, \varepsilon)$. Without loss of generality we may assume that $r_0 < r_1$. By the mean value theorem, there exists $r_2 \in (r_0, r_1)$ such that $g'(r_2) = 0$. Then

$$v_r(r_2, s_1) = v'_\lambda(r_2) = \begin{cases} -\dfrac{4\sinh(r_2/\sqrt{\lambda})}{\lambda^{3/2}\cosh^3(r_2/\sqrt{\lambda})} & \text{if } n = 1 \\ -\dfrac{32\lambda r_2}{(\lambda + r_2^2)^3} & \text{if } n = 2 \end{cases}$$

and

$$v_\lambda(r_2) \ge v_\lambda(r_1) = v(r_1, s_1) \ge C_1.$$

So

$$\begin{aligned} -C_1^{3/2} \le v_r(r_2, s_1) &= \begin{cases} -\sqrt{2}\sinh(r_2/\sqrt{\lambda}) \cdot \left(\dfrac{2}{\lambda\cosh^2(r_2/\sqrt{\lambda})}\right)^{3/2} & \text{if } n = 1 \\ -\dfrac{\sqrt{2}r_2}{\lambda^{1/2}} \cdot \left(\dfrac{8\lambda}{(\lambda + r_2^2)^2}\right)^{3/2} & \text{if } n = 2 \end{cases} \\ &\le \begin{cases} -\sqrt{2}\sinh(1) v_\lambda(r_2)^{3/2} & \text{if } n = 1 \\ -\sqrt{2} v_\lambda(r_2)^{3/2} & \text{if } n = 2 \end{cases} \\ &\le -\sqrt{2} v_\lambda(r_2)^{3/2} \le -\sqrt{2} C_1^{3/2}. \end{aligned}$$

This is a contradiction. Hence no such r_1 exists and g has exactly only one zero $r_0 \in (\sqrt{\lambda}, \varepsilon)$ on $[0, R)$. By (1.11) the lemma follows. □

Lemma 1.5. *For any $R > 0$, $\mu \in (0,\infty)$ if $n = 1$ and $\mu \in (0, 4/R)$ if $n = 2$, there exists a unique constant $\lambda(\mu) > 0$ such that*

$$\frac{\partial}{\partial r}(\log v_{\lambda(\mu)})(R) = -\mu. \tag{1.12}$$

Moreover $\lambda(\mu)$ is a strictly monotone decreasing function of μ mapping $(0,\infty)$ onto $(0,\infty)$ for $n = 1$ and mapping $(0, 4/R)$ onto $(0,\infty)$ for $n = 2$.

Proof. Let

$$f(\lambda) = -\frac{\partial}{\partial r}(\log v_\lambda)(R) = \begin{cases} \dfrac{2}{\sqrt{\lambda}}\tanh(R/\sqrt{\lambda}) & \text{if } n = 1 \\ \dfrac{4R}{\lambda + R^2} & \text{if } n = 2. \end{cases}$$

Then $f'(\lambda) < 0$ for all $0 < \lambda < \infty$, $\lim_{\lambda\to\infty} f(\lambda) = 0$, and $f(0^+) = \infty$ if $n = 1$, $f(0^+) = 4/R$ if $n = 2$. Hence for any $\mu \in (0,\infty)$ for $n = 1$ and $\mu \in (0, 4/R)$ for $n = 2$, there exists a unique constant $\lambda(\mu) > 0$ such that $f(\lambda(\mu)) = \mu$. Moreover f is a strictly monotone decreasing function which maps $(0,\infty)$ onto $(0,\infty)$ for $n = 1$ and maps $(0,\infty)$ onto $(0, 4/R)$ for $n = 2$. Hence $\lambda(\mu)$ is a strictly monotone decreasing function of μ mapping $(0,\infty)$ onto $(0,\infty)$ for $n = 1$ and mapping $(0, 4/R)$ onto $(0,\infty)$ for $n = 2$.

We will now assume $n = 1$ or 2, $0 \le u_0 \in L^p(B_R)$ for some constant $p > 1$, $u_0 \not\equiv 0$, and u_0 is radially symmetric and monotone decreasing in $r \ge 0$ for the rest of this section. Then by Lemma 1.4 both u and v are radially symmetric and monotone decreasing in $r \ge 0$. In order to obtain uniform bounds for v we will need to compare v with v_λ. We will do so by investigating the evolution of the intersection points of v and v_λ as time increases. Similar to Hsu [16] and Hsu [20], we let

$$\begin{cases} g_\lambda(x,s) = v_\lambda(x) - v(x,s) \\ h_\lambda(r,s) = g_\lambda(|x|, s). \end{cases}$$

Then $g_\lambda(x,t)$ satisfies

$$g_s = \Delta(a(x,s)g) + g \quad \text{ in } B_R \times (-\log T, \infty)$$

and $h_\lambda(r,t)$ satisfies

$$h_s = (a(r,t)h)_{rr} + \frac{n-1}{r}(a(r,t)h)_r + h \quad \text{ in } (0,R) \times (-\log T, \infty), n = 1, 2\,,$$

where

$$a(x,s) = \frac{\log\, v_\lambda(x) - \log\, v(x,s)}{v_\lambda(x) - v(x,s)} = \frac{1}{\xi(x,s)} > 0$$

for some $\xi(x,s)$ between $v_\lambda(x)$ and $v(x,s)$. Note that $v_\lambda(x)$ is a real analytic function of $x \in \overline{B}_R$ and by the discussion on P.241 of Samarskii and Galaktionov [29] and

Galaktionov and Peletier [12] for each $s > -\log T$, $v(x,s)$ is also a real analytic function of $x \in \overline{B}_R$. Similarly both $v_\lambda(r)$ and $v(r,s)$ are analytic function of $0 \le r \le R$. Hence for each $s > -\log T$, either $h_\lambda(r,s) \equiv 0$ on $[0,R]$ or all the zeros of $h_\lambda(r,s)$ in $[0,R]$ are isolated zeros and if $g_\lambda(x,s) \not\equiv 0$ and $g_\lambda(x,s)$ has a zero at $x = 0$, then $x = 0$ is an isolated zero of $g_\lambda(x,s)$.

We say that $0 < r_0 < R$ is a transversal point of $h_\lambda(r,s)$ if $h_\lambda(r_0,s) = 0$ and $\partial h_\lambda(r_0,s)/\partial r \ne 0$. We say that $0 < r_0 < R$ is a tangency point of $h_\lambda(r,s)$ if $h_\lambda(r_0,s) = 0$ and there exists a constant $0 < \delta < \min(r_0, R - r_0)$ such that either

$$h_\lambda(r,s) > 0 \quad \forall 0 < |r - r_0| \le \delta\,,$$

or

$$h_\lambda(r,s) < 0 \quad \forall 0 < |r - r_0| \le \delta$$

holds and we say that $0 < r_0 < R$ is an inflection point of $h_\lambda(r,s)$ if $h_\lambda(r_0,s) = 0$, $\partial h_\lambda(r_0,s)/\partial r = 0$, $\partial h_\lambda(r,s)/\partial r \ne 0$ for all $0 < |r - r_0| \le \delta$ for some constant $0 < \delta < \min(r_0, R - r_0)$ and $\partial h_\lambda(r,s)/\partial r$ has the same sign on $0 < |r - r_0| \le \delta$. Similarly we say that $x = 0$ is a tangency point of $g_\lambda(x,s)$ if $g_\lambda(0,s) = 0$ and there exists a constant $0 < \delta < R$ such that either

$$g_\lambda(x,s) > 0 \quad \forall 0 < |x| \le \delta\,, \tag{1.13}$$

or

$$g_\lambda(x,s) < 0 \quad \forall 0 < |x| \le \delta. \tag{1.14}$$

Since both $v_{\lambda,r}(r)$ and $v_r(r,s)$ are analytic function of $0 \le r \le R$, for each $s > -\log T$ if $h_\lambda(r,s) \not\equiv 0$ on $[0,R]$, then all the zeros of $h_{\lambda,r}(r,s)$ in $(0,R)$ are isolated zeros. Hence any zero of $h_{\lambda,r}(r,s)$ in $(0,R)$ is either a tangency point or a transversal point or an inflection point. By symmetry if $g_\lambda(\cdot,s) \not\equiv 0$ on B_R and $g_\lambda(x,s)$ has a zero at $x = 0$, it is a tangency point at $x = 0$.

We will now recall some results of Hsu [16], Hsu [20]. Although they are proved for the case $n = 2$ in Hsu [16], Hsu [20], by the same argument as Hsu [16], Hsu [20] these results still remain valid for the case $n = 1$.

Lemma 1.6. (cf. Lemma 2.4 of Hsu [16] or Lemma 1.2.4 of Hsu [20]) *If* $0 < r_1 < R$ *is a tangency point of* $h_\lambda(r,s_0)$*, then there exists a constant* $0 < \delta_1 < \min(r_1, R - r_1)$ *such that* $h_\lambda(r,s) \ne 0$ *for all* $|r - r_1| \le \delta_1$*,* $s_0 < s \le s_0 + \delta_1$*.*

Lemma 1.7. (cf. Lemma 2.5 of Hsu [16] or Lemma 1.2.5 of Hsu [20]) *If* $x = 0$ *is a tangency point of* $g_\lambda(x,s_0)$*, then there exists a constant* $0 < \delta_2 < R$ *such that*

$$g_\lambda(x,s) > 0 \quad \forall |x| \le \delta_2, s_0 < s \le s_0 + \delta_2 \tag{1.15}$$

if $g_\lambda(x,s_0)$ *satisfies (1.13) and*

$$g_\lambda(x,s)<0 \quad \forall |x|\le \delta_2, s_0<s\le s_0+\delta_2$$

if $g_\lambda(x,s_0)$ *satisfies (1.14).*

Lemma 1.8. (cf. Lemma 2.6 of Hsu [16] or Lemma 1.2.6 of Hsu [20]) *If* $0<r_0<R$ *is an inflection point of* $h_\lambda(r,s_0)$*, then there exist constants* $0<\delta_4<\delta_3<\min(r_0,R-r_0)$*, such that for each* $s_0<s\le s_0+\delta_4$*,* $h_\lambda(r,s)$ *has exactly one zero in* $|r-r_0|\le\delta_3$ *which is a transversal point and for each* $s_0-\delta_4\le s<s_0$*,* $h_\lambda(r,s)$ *has at least one zero in* $|r-r_0|\le\delta_3$ *which is either a transversal point or an inflection point.*

Lemma 1.9. (cf. Lemma 2.7 of Hsu [16] or Lemma 1.2.7 of Hsu [20]) *If* $0<r_0<R$ *is a transversal point of* $h_{\lambda_1}(r,s_0)$*, then there exist constants* $0<\delta_4<\delta_3<\min(r_0,R-r_0)$*,* $0<\delta_5<\lambda_1$*, such that for each* $s_0-\delta_4\le s\le s_0+\delta_4$ *and* $|\lambda-\lambda_1|\le\delta_5$*,* $h_\lambda(r,s)$ *has exactly one zero in* $|r-r_0|\le\delta_3$ *which is a transversal point.*

By an argument similar to the proof of Lemma 2.9 of Hsu [16] we have

Lemma 1.10. (cf. Lemma 2.9 of Hsu [16]) *For any* $s_0>-\log T$*, if* $h_\lambda(r,s_0)$ *has at least one zero on* $(0,R)$ *with exactly one zero* $r_0\in(0,R)$ *which is either a transversal point or inflection point such that either*

$$h_\lambda(r,s_0)\ge 0\ge h_\lambda(r',s_0) \quad \forall 0\le r\le r_0\le r'\le R \tag{1.16}$$

or

$$h_\lambda(r,s_0)\le 0\le h_\lambda(r',s_0) \quad \forall 0\le r\le r_0\le r'\le R \tag{1.17}$$

holds and $h_\lambda(R,s_0)\ne 0$*, then there exists a constant* $\delta>0$ *such that for any* $s_0<s\le s_0+\delta$*,* $h_\lambda(r,s)$ *has only one zero* $r(s)$ *on* $(0,R)$ *and* $r(s)$ *is a transversal point such that*

$$\begin{cases} h_\lambda(r,s)>0>h_\lambda(r',s) & \forall 0\le r<r(s)<r'\le R \quad \text{if (1.16) holds}\\ h_\lambda(r,s)<0<h_\lambda(r',s) & \forall 0\le r<r(s)<r'\le R \quad \text{if (1.17) holds.}\end{cases}$$

We are now ready to show that v is uniformly bounded above and below for $n=1,2$.

Lemma 1.11. *Suppose* $n=1$ *or* 2 *and* $0<\alpha<\infty$ *for* $n=1$ *or* $0<\alpha<4/R$ *for* $n=2$*,* $0\le u_0\in L^p(B_R)$ *for some constant* $p>1$*,* $u_0\not\equiv 0$*, and* u_0 *is a radially symmetric and monotone decreasing function of* $r\ge 0$*. Then for any* $s_1>-\log T$*, there exist constants* $C_1>0$*,* $C_2>0$*, such that*

$$C_1\le v(x,s)\le C_2 \quad \forall |x|\le R, s\ge s_1. \tag{1.18}$$

Proof. Let $s_1 > -\log T$ and let $\lambda_0 > 0$ be the constant given in Lemma 1.4. By Lemma 1.5 we can choose a constant $\mu \in (\alpha, \infty)$ for $n = 1$ and $\mu \in (\alpha, 4/R)$ for $n = 2$ such that $0 < \lambda(\mu) \le \lambda_0$ where $\lambda(\mu)$ is the unique constant satisfying (1.12). We now let

$$\begin{cases} K_\mu = \{s_2 \ge s_1 : \forall s_1 \le s \le s_2, h_{\lambda(\mu)}(r,s) \text{ has at most one transversal point or} \\ \qquad\qquad \text{inflection point in (0,R)}\} \\ \widetilde{K}_\mu = \{s_2 \ge s_1 : \forall s_1 \le s \le s_2, \exists r(s) \in (0,R] \text{ such that } h_{\lambda(\mu)}(r,s) \ge 0 \ge h_{\lambda(\mu)}(r',s) \\ \qquad\qquad \text{for all } 0 \le r \le r(s) \le r' \le R\} \\ s_\mu = \sup\limits_{s \in K(\mu)} s, \widetilde{s}_\mu = \sup\limits_{s \in \widetilde{K}(\mu)} s. \end{cases} \tag{1.19}$$

By Lemma 1.4 the graphs of $v_{\lambda(\mu)}$ and $v(r, s_1)$ intersect exactly once on $[0, R]$ and there exists $r_0 \in (\sqrt{\lambda(\mu)}, R)$ such that (i) and (ii) of Lemma 1.4 holds. By Lemma 1.10 $K_\mu \ne \phi$ and $\widetilde{K}_\mu \ne \phi$.

If there exists $s \ge s_1$ such that $h_{\lambda(\mu)}(r,s) \le 0$ for all $r \in [0, R]$, then

$$\begin{aligned} & v_{\lambda(\mu)}(r) \le v(r,s) \quad \forall r \in [0,R] \\ \Rightarrow \quad & \mu\sigma_n(\partial B_R) = \int_{B_R} v_{\lambda(\mu)}(x)dx \le \int_{B_R} v(x,s)dx = \alpha\sigma_n(\partial B_R) \\ \Rightarrow \quad & \mu \le \alpha. \end{aligned}$$

This contradicts the choice of μ. Hence

$$h_{\lambda(\mu)}(r,s) \not\le 0 \text{ on } [0,R] \quad \forall s \ge s_1 \quad \Rightarrow \quad h_{\lambda(\mu)}(r,s) \not\equiv 0 \text{ on } [0,R] \quad \forall s \ge s_1. \tag{1.20}$$

Observe that by the definition of $\widetilde{K}_\mu$ and K_μ we have $\widetilde{K}_\mu \subset K_\mu$. Thus $s_\mu \ge \widetilde{s}_\mu > s_1$. If $\widetilde{s}_\mu < \infty$, then by the definition of $\widetilde{s}_\mu$ there exists a sequence $\{s_i\}_{i=2}^\infty \subset \widetilde{K}_\mu$ such that $s_i \nearrow \widetilde{s}_\mu$ as $i \to \infty$. Then for each $i = 2, 3, \dots,$ there exists $r_i \in (0, R]$ satisfying

$$h_{\lambda(\mu)}(r, s_i) \ge 0 \ge h_{\lambda(\mu)}(r', s_i) \quad \forall 0 \le r \le r_i \le r' \le R.$$

Then r_i will have a subsequence r_{i_k} converging to some point $r_0 \in [0, R]$ as $k \to \infty$. Putting $i = i_k$ and letting $k \to \infty$ in the above inequality we get

$$h_{\lambda(\mu)}(r, \widetilde{s}_\mu) \ge 0 \ge h_{\lambda(\mu)}(r', \widetilde{s}_\mu) \quad \forall 0 \le r \le r_0 \le r' \le R. \tag{1.21}$$

By (1.20) $r_0 \ne 0$. We claim that $r_0 = R$. Suppose not. Then $r_0 \in (0, R)$. Since $h_{\lambda(\mu)}(r, \widetilde{s}_\mu) \not\equiv 0$ on $[0, R]$ and $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ is an analytic function of r, all the zeros of $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ are isolated zeros. Hence $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ can only have a finite number of zeros in $(0, r_0) \cup (r_0, R)$ and by (1.21) all the other zeros of $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ except r_0 are tangency points and r_0 is either a transversal or inflection point. Thus $\widetilde{s}_\mu \in K_\mu$.

Without loss of generality we may assume that $h_{\lambda(\mu)}(r,\widetilde{s}_\mu)$ has only one tangency point r_1 in $(0,R)$ and $r_1\in(r_0,R)$. Since r_0 is either a transversal or inflection point, by (1.21) there exists a constant $0<\delta_6<\min(r_0/3,(r_1-r_0)/3)$ such that

$$h_{\lambda(\mu)}(r,\widetilde{s}_\mu)>0>h_{\lambda(\mu)}(r',\widetilde{s}_\mu)\quad\forall r_0-\delta_6\le r<r_0<r'\le r_0+\delta_6.$$

By Lemma 1.6 there exists a constant $0<\delta_1<\min((r_1-r_0)/3,(R-r_1)/3)$ such that $h_{\lambda(\mu)}(r,s)\ne 0$ for all $|r-r_1|\le\delta_1$, $\widetilde{s}_\mu<s\le\widetilde{s}_\mu+\delta_1$. If $h_{\lambda(\mu)}(0,\widetilde{s}_\mu)=0$, then by symmetry 0 is a tangency point of $g_{\lambda(\mu)}(\cdot,\widetilde{s}_\mu)$. By (1.21) and Lemma 1.7 there exists a constant $0<\delta_2<r_0/3$ such that (1.15) holds with $s_0=\widetilde{s}_\mu$. If $h_{\lambda(\mu)}(0,\widetilde{s}_\mu)>0$, then by continuity there exists a constant $0<\delta_2<r_0/3$ such that (1.15) holds with $s_0=\widetilde{s}_\mu$. Hence there always exists a constant $0<\delta_2<r_0/3$ such that (1.15) holds with $s_0=\widetilde{s}_\mu$.

By Lemma 1.8 and Lemma 1.9 there exist constants $0<\delta_4<\delta_3<\min(r_0/3,(r_1-r_0)/3,\delta_6)$ such that for each $\widetilde{s}_\mu<s\le\widetilde{s}_\mu+\delta_4$ $h_{\lambda(\mu)}(r,s)$ has exactly one zero in $|r-r_0|\le\delta_3$ which is a transversal point.

If

$$h_{\lambda(\mu)}(R,\widetilde{s}_\mu)=0,$$

then by (1.21) we have

$$\begin{cases} h'_{\lambda(\mu)}(R,\widetilde{s}_\mu)=v'_{\lambda(\mu)}(R)-v'(R,\widetilde{s}_\mu)\ge 0\\ v_{\lambda(\mu)}(R)=v(R,\widetilde{s}_\mu)\end{cases}$$
$$\Rightarrow\quad \mu=-\frac{\partial}{\partial r}\log v_{\lambda(\mu)}(R)\le-\frac{\partial}{\partial r}\log v(R,\widetilde{s}_\mu)=\alpha.$$

Contradiction arises. Hence

$$h_{\lambda(\mu)}(R,\widetilde{s}_\mu)\ne 0\quad\Rightarrow\quad h_{\lambda(\mu)}(R,\widetilde{s}_\mu)<0\quad\text{(by (1.21))}.\tag{1.22}$$

By continuity there exists $0<\delta'\le\delta_1$ such that

$$h_{\lambda(\mu)}(r,\widetilde{s}_\mu)<0\quad\forall R-\delta'\le r\le R.$$

Let

$$I_0=[\delta_2,r_0-\delta_3],I_1=[r_0+\delta_3,r_1-\delta_1],I_2=[r_1+\delta_1,R].$$

By the above inequality and the fact that $h_{\lambda(\mu)}(r,\widetilde{s}_\mu)$ has no zeros in $\cup_{i=0}^2 I_i$ we have

$$\inf_{r\in I_i}(-1)^i h_{\lambda(\mu)}(r,\widetilde{s}_\mu)>0\quad\forall i=0,1,2.$$

Then by continuity there exists a constant $\delta_7>0$ such that

$$\inf_{r\in I_i}(-1)^i h_{\lambda(\mu)}(r,s)>0\quad\forall\widetilde{s}_\mu\le s\le\widetilde{s}_\mu+\delta_7, i=0,1,2.$$

Let $\delta_8 = \min(\delta_1, \delta_2, \delta_3, \delta_4, \delta_6, \delta_7)$. Then for any $\widetilde{s}_\mu < s \le \widetilde{s}_\mu + \delta_8$, $h_{\lambda(\mu)}(r, s)$ has exactly one transversal point ($r(s)$ say) in $(0, R)$ which satisfies

$$h_{\lambda(\mu)}(r, s) \ge 0 \ge h_{\lambda(\mu)}(r', s) \quad \forall 0 \le r \le r(s) \le r' \le R. \tag{1.23}$$

Thus $\widetilde{s}_\mu + \delta_8 \in \widetilde{K}_\mu$. This contradicts the choice of $\widetilde{s}_\mu$. Hence $r_0 = R$ and $\widetilde{K}_\mu = [s_1, \widetilde{s}_\mu]$. Thus

$$h_{\lambda(\mu)}(r, \widetilde{s}_\mu) \ge 0 \quad \forall r \in [0, R] \quad \Rightarrow \quad v(r, \widetilde{s}_\mu) \le v_{\lambda(\mu)}(r) \quad \forall r \in [0, R]. \tag{1.24}$$

Since $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ is an analytic function of $r \in [0, R]$ and $h_{\lambda(\mu)}(r, \widetilde{s}_\mu) \not\equiv 0$, $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ can only have a finite number of zeros in $[0, R]$. By (1.24) all the zeros of $h_{\lambda(\mu)}(r, \widetilde{s}_\mu)$ in $[0, R)$ are tangency zeros. Then by a similar argument as before for any $0 < \delta < R$, there exists a constant $\delta_9 > 0$ such that

$$h_{\lambda(\mu)}(r, s) > 0 \quad \forall 0 \le r \le R - \delta, \widetilde{s}_\mu < s \le \widetilde{s}_\mu + \delta_9. \tag{1.25}$$

We now claim that $K_\mu = \widetilde{K}_\mu$. Suppose not. Then there exists a sequence $\{s_i'\}_{i=1}^\infty \subset K(\mu) \setminus \widetilde{K}(\mu)$, $\widetilde{s}_\mu < s_i' \le \widetilde{s}_\mu + \delta_9$ for all $i \in \mathcal{Z}^+$, such that $s_i' \searrow \widetilde{s}_\mu$ as $i \to \infty$. Since $s_i' \notin \widetilde{K}(\mu)$, for any $i \in \mathcal{Z}^+$ $\nexists r(s_i') \in (0, R]$ such that

$$h_{\lambda(\mu)}(r, s_i') \ge 0 \ge h_{\lambda(\mu)}(r', s_i') \quad \forall 0 \le r \le r(s_i') \le r' \le R.$$

Hence for any $i \in \mathcal{Z}^+$ by (1.25) we have

$$h_{\lambda(\mu)}(r, s_i') \not\ge 0 \quad \forall 0 \le r \le R \quad \Rightarrow \quad \exists \rho_i \in (0, R] \text{ such that } h_{\lambda(\mu)}(\rho_i, s_i') < 0. \tag{1.26}$$

By (1.25), (1.26), and the Intermediate Value Theorem for any $i \in \mathcal{Z}^+$, $h_{\lambda(\mu)}(r, s_i')$ has at least one transversal point or inflection point in $(0, R)$. Since $s_i' \in K(\mu)$, $h_{\lambda(\mu)}(r, s_i')$ has exactly one transversal point or inflection point $r(s_i')$ in $(0, R)$. Hence for any $i \in \mathcal{Z}^+$, $h_{\lambda(\mu)}(r, s_i')$ satisfies either (1.16) or (1.17) with $r_0 = r(s_i')$ and $s_0 = s_i'$. By (1.25), $h_{\lambda(\mu)}(r, s_i')$ satisfies (1.16) for any $i \in \mathcal{Z}^+$. Hence $s_i' \in \widetilde{K}_\mu$ for all $i \in \mathcal{Z}^+$. Contradiction arises. Thus $K_\mu = \widetilde{K}_\mu = [0, \widetilde{s}_\mu]$ and $\widetilde{s}_\mu = s_\mu$.

We now choose a sequence $\{s_i''\}$, $s_\mu < s_i'' \le s_\mu + \delta_9$ for all $i \in \mathcal{Z}^+$, such that $s_i'' \searrow s_\mu$ as $i \to \infty$. Since $s_i'' \notin K_\mu$ for all $i \in \mathcal{Z}^+$, for any $i \in \mathcal{Z}^+$ there exists $s_\mu < \widetilde{s}_i < s_i''$ such that $h_{\lambda(\mu)}(r, \widetilde{s}_i)$ has at least two transversal or inflection points in $(0, R)$. For any $i \in \mathcal{Z}^+$, let $0 < \rho_{1,i} < \rho_{2,i} < R$ be two transversal or inflection points of $h_{\lambda(\mu)}(r, \widetilde{s}_i)$. By (1.26) $\rho_{1,i} \ge R - \delta$ $\forall i \in \mathcal{Z}^+$. Then by the Mean Value Theorem for any $i \in \mathcal{Z}^+$ there exists $\rho_i' \in (\rho_{1,i}, \rho_{2,i})$ such that

$$h'_{\lambda(\mu)}(\rho_i', \widetilde{s}_i) = 0. \tag{1.27}$$

Hence

$$R-\delta\le\rho_i'\le R\quad\forall i\in\mathcal{Z}^+. \tag{1.28}$$

Then $\{\rho_i'\}$ will have a subsequence $\{\rho_{i_k}'\}$ converging to some point $\rho_0\in(0,R]$ as $k\to\infty$. Putting $\rho_i'=\rho_{i_k}'$ and letting $k\to\infty$ in (1.28) we get

$$R-\delta\le\rho_0\le R. \tag{1.29}$$

Since $\delta>0$ is arbitrary. Letting $\delta\to 0$ in (1.29) we get $\rho_0=R$. Hence putting $i=i_k$ in (1.27) and letting $k\to\infty$ we get

$$h_{\lambda(\mu)}'(R,\widetilde{s}_\mu)=0\quad\Rightarrow\quad v_{\lambda(\mu)}'(R)=v'(R,\widetilde{s}_\mu). \tag{1.30}$$

Now by (1.21) and (1.24) we have

$$v_{\lambda(\mu)}(R)=v(R,\widetilde{s}_\mu). \tag{1.31}$$

Hence by (1.30) and (1.31) we have

$$\alpha=-\frac{\partial}{\partial r}\log\,v(R,s_\mu)=-\frac{\partial}{\partial r}\log\,v_\mu(R)=\mu$$

Contradiction arises. Hence $\widetilde{s}_\mu=s_\mu=\infty$. Thus

$$h_{\lambda(\mu)}(0,s)\ge 0\ge h_{\lambda(\mu)}(R,s)\quad\forall s\ge s_1$$

$$\Rightarrow\quad\begin{cases} v(r,s)\le v(0,s)\le v_{\lambda(\mu)}(0) & \forall r\in[0,R], s\ge s_1\\ v(r,s)\ge v(R,s)\ge v_{\lambda(\mu)}(R) & \forall 0\le r\le R, s\ge s_1\end{cases}$$

and (1.18) follows.

2. SECTION TWO

In this section we will use a modification of the symmetrization technique of Bandle [3] and Bandle [4] to prove the uniform upper and lower bound estimates of v for general initial value. We will first start with some definitions.(cf. Bandle [2], Bandle [3]) For any bounded domain $D\subset R^n$, we let D^* denote the closed ball centred at the origin with the same volume as D. Let u be the solution of (0.4) with $\alpha\ge 0$ and let T be given by (0.5) for $\alpha>0$ and $T=\infty$ for $\alpha=0$. For each $0\le t<T$ and $\mu\ge 0$, we let

$$\begin{cases} D(\mu,t)=\{x\in\overline{B}_R:u(x,t)\ge\mu\}\\ D(\mu)=\{x\in\overline{B}_R:u_0(x)\ge\mu\}\\ \Gamma(\mu,t)=\partial D(\mu,t)\\ a(\mu,t)=|D(\mu,t)|_n, A=\text{volume of }B_R\end{cases}$$

and let

$$\begin{cases} u^*(x,t) = \sup\{\mu : x \in D(\mu,t)\} \\ u_0^*(x) = \sup\{\mu : x \in D(\mu)\} \end{cases}$$

be the symmetrization of $u(\cdot,t)$ and u_0 respectively. For any $0 < a \le A$, we let $V(a)$ be the ball in R^n with center at the origin and $|V(a)|_n = a$, $V(0) = \phi$, $r(a)$ be the radius of the ball $V(a)$, and $s(a) = \sigma_n(\partial V(a))$. For each $0 \le t < T$, we let

$$\begin{cases} \mu_1(t) = \min_{\overline{B}_R} u(x,t) \\ \mu_2(t) = \max_{\overline{B}_R} u(x,t). \end{cases}$$

By the discussion in Bandle [3] for each $0 \le t < T$, $a(\mu,t)$ is a strictly decreasing function of μ in $[\mu_1(t), \mu_2(t)]$ and $a(\mu,t)$ is a continuous function of μ and t. Hence for each $0 \le t < T$, $a(\mu,t)$ has a unique inverse $\mu(a,t)$ which is a continuous function of a and t. Since $0 \le u_0 \in L^p(B_R)$, by Remark F on P.49 of Bandle [2], we have

$$\begin{cases} \displaystyle\int_{B_R} u_0^* dx = \int_{B_R} u_0 dx \\ \displaystyle\int_{B_R} (u_0^*)^p dx \le \int_{B_R} u_0^p dx < \infty. \end{cases}$$

By Corollary 1.2 there exists a unique solution $\widetilde{u}$ of (0.4) in Q_R^T with initial value u_0^* and satisfying (1.3) on $(0,T)$ where T is given by (0.5) for $\alpha > 0$ and $T = \infty$ for $\alpha = 0$. Then similarly for each $0 \le t < T$ and $\widetilde{\mu} \ge 0$, we let

$$\begin{cases} \widetilde{D}(\widetilde{\mu},t) = \{x \in \overline{B}_R : \widetilde{u}(x,t) \ge \widetilde{\mu}\} \\ \widetilde{u}^*(x,t) = \sup\{\widetilde{\mu} : x \in \widetilde{D}(\widetilde{\mu},t)\} \\ \widetilde{\Gamma}(\widetilde{\mu},t) = \partial\widetilde{D}(\widetilde{\mu},t) \\ \widetilde{a}(\widetilde{\mu},t) = |\widetilde{D}(\widetilde{\mu},t)|_n, \widetilde{\mu}_1(t) = \min_{\overline{B}_R} \widetilde{u}(x,t), \widetilde{\mu}_2(t) = \max_{\overline{B}_R} \widetilde{u}(x,t). \end{cases}$$

As before for each $0 \le t < T$, $\widetilde{a}(\widetilde{\mu},t)$ is a strictly decreasing function of $\widetilde{\mu}$ in $[\widetilde{\mu}_1(t), \widetilde{\mu}_2(t)]$ and $\widetilde{a}(\widetilde{\mu},t)$ is a continuous function of $\widetilde{\mu}$ and t. Hence for each $0 \le t < T$, $\widetilde{a}(\widetilde{\mu},t)$ has a unique inverse $\widetilde{\mu}(\widetilde{a},t)$ which is a continuous function of $\widetilde{a}$ and t. We note also that since u_0^* is radially symmetric and monotone decreasing in $r \ge 0$, by Lemma 1.3 $\widetilde{u}$ is also radially symmetric and monotone decreasing in $r \ge 0$. Hence $\widetilde{u}^* \equiv \widetilde{u}$.

By the same argument as proof of Proposition 1.2 of Bandle [3] we have

Lemma 2.1. *u^* is Lipschitz continuous on every compact subset of $\overline{B}_R \times (0,T)$.*

We also need the following weak version of maximum principle for parabolic equations.

Lemma 2.2. *Let* $D_{T_1} = [L_1, L_2] \times [0, T_1]$. *Suppose* $q \in C^{1,1}(D_{T_1})$ *is such that for all* $0 < t < T_1$ $q_{xx}(x,t)$ *exists for a.e.* $x \in [L_1, L_2]$, $q_{xx} \in L^\infty(D_{T_1})$, $b(x,t) \in C(D_{T_1})$, $b(x,t) \geq \delta > 0$ *on* D_{T_1} *for some positive constant* δ, *and* $c(x,t) \in L^\infty(D_{T_1})$. *If for all* $0 < t < T_1$ q *satisfies*

$$q_t \leq b(x,t)q_{xx} + c(x,t)q_x \qquad a.e.\ x \in [L_1, L_2],$$

then

$$\max_{D_{T_1}} q \leq \max_S q$$

where $S = \partial_p D_{T_1} = \{L_1, L_2\} \times [0, T_1] \cup [L_1, L_2] \times \{0\}$ *is the parabolic boundary of* D_{T_1}.

Proof. Without loss of generality we may assume $L_1 = 0$. Let $\widetilde{q}(x,t) = q(x,t) + \varepsilon x^\beta$ where $\beta = 2 + 2\delta^{-1}\|c\|_{L^\infty} L_2$. Then for all $0 < t < T_1$, $\widetilde{q}$ satisfies

$$\begin{aligned} b(x,t)\widetilde{q}_{xx} + c(x,t)\widetilde{q}_x &= b(x,t)q_{xx} + c(x,t)q_x + \varepsilon\beta x^{\beta-2}((\beta-1)b(x,t) + c(x,t)x) \\ &\geq \widetilde{q}_t + \varepsilon\beta x^{\beta-2}((\beta-1)\delta - \|c\|_{L^\infty} L_2) \\ &\geq \widetilde{q}_t + \varepsilon\delta\beta x^{\beta-2} \quad \text{for a.e. } x \in (0, L_2). \end{aligned} \tag{2.1}$$

Suppose $\widetilde{q}$ has a maximum at $(x_0, t_0) \in D_{T_1} \setminus S$. Then

$$\widetilde{q}_t(x_0, t_0) \geq 0 \quad \text{and} \quad \widetilde{q}_x(x_0, t_0) = 0. \tag{2.2}$$

We claim that there exists a sequence $\{x_i\} \subset (0, L_2)$, $x_i \to x_0$ as $i \to \infty$, such that $\widetilde{q}_{xx}(x_i, t_0)$ exists for all $i \in \mathcal{Z}^+$ and

$$\widetilde{q}_{xx}(x_i, t_0) \leq 0 \quad \forall i = 1, 2, \ldots. \tag{2.3}$$

Suppose the claim is not true. Then there exists a constant $0 < \delta' < \min(x_0, L_2 - x_0)$ such that

$$\widetilde{q}_{xx}(x, t_0) > 0 \quad \text{a.e. } x \in (x_0 - \delta', x_0 + \delta'). \tag{2.4}$$

Now we choose $x_1 \in (x_0 - \delta', x_0)$. By the Mean Value Theorem there exists $x_1' \in (x_1, x_0)$ such that

$$\widetilde{q}_x(x_1', t_0) = \frac{\widetilde{q}(x_0, t_0) - \widetilde{q}(x_1, t_0)}{x_0 - x_1} \geq 0.$$

Hence by (2.4) we have

$$\begin{aligned} & \widetilde{q}_x(x, t_0) = \widetilde{q}_x(x_1', t_0) + \textstyle\int_{x_1'}^x \widetilde{q}_{xx}(y, t_0)dy > \widetilde{q}_x(x_1', t_0) \geq 0 \quad \forall x \in (x_1', x_0 + \delta') \\ \Rightarrow \quad & \widetilde{q}(x, t_0) > \widetilde{q}(x_0, t_0) \quad \forall x \in (x_0, x_0 + \delta'). \end{aligned}$$

This contradicts the maximality of the point (x_0, t_0). Hence there exists a sequence $\{x_i\} \subset (0, L_2)$, $x_i \to x_0$ as $i \to \infty$, such that $\widetilde{q}_{xx}(x_i, t_0)$ exists for all $i \in \mathcal{Z}^+$ and (2.3) holds. Since $c(x,t) \in L^\infty(D_{T_1})$ and $q_x \in C(D_{T_1})$, putting $t = t_0$, $x = x_i$, in (2.1) and letting $i \to \infty$, by (2.2) and (2.3) we have

$$0 \ge \widetilde{q}_t(x_0,t_0) + \varepsilon\delta\beta x_0^{\beta-2} > \widetilde{q}_t(x_0,t_0).$$

This contradicts (2.1). Hence no such (x_0,t_0) exists and $\widetilde{q}$ attains its maximum on S. Thus

$$\begin{aligned} &\max\nolimits_{D_{T_1}} q \le \max\nolimits_{D_{T_1}} \widetilde{q} \le \max\nolimits_S \widetilde{q} \le \max\nolimits_S q + \varepsilon L_2^\beta \\ \Rightarrow \quad &\max\nolimits_{D_{T_1}} q \le \max\nolimits_S q \quad \text{as } \varepsilon \to 0 \end{aligned}$$

and the lemma follows. □

Theorem 2.3. *If $\widetilde{u}$ is the unique solution of (0.4) in Q_R^T with initial value u_0^* and satisfying (1.3), then*

$$\int_{B_\rho} u^*(x,t)dx \le \int_{B_\rho} \widetilde{u}(x,t)dx \quad \forall 0 \le \rho \le R, 0 \le t < T. \tag{2.5}$$

In particular

$$\max_{x\in B_R} u(x,t) \le \max_{x\in B_R} \widetilde{u}(x,t) = \widetilde{u}(0,t) \quad \forall 0 \le \rho \le R, 0 \le t < T. \tag{2.6}$$

Proof. For any $0 \le t < T$ and $a \in [0,A]$, let

$$H(a,t) = \int_{D(\mu(a,t),t)} u(x,t)dx = \int_0^a \mu(a',t)da'.$$

By an argument similar to the proof in Bandle [3] we have

$$\frac{\partial H}{\partial a}(a,t) = \mu(a,t) \quad \forall (a,t) \in (0,A) \times (0,T) \tag{2.7}$$

and for all $t \in (0,T)$ we have

$$\frac{\partial^2 H}{\partial a^2}(a,t) = \frac{\partial \mu}{\partial a}(a,t) = -\left(\int_{\partial V(a)} \frac{d\sigma_n}{|\nabla_x u^*|}\right)^{-1} = \frac{|\nabla_x u^*(r(a),t)|}{s(a)} \qquad \text{a.e. } a \in (0,A). \tag{2.8}$$

Now by the same argument as proof of Lemma 1.1 of Bandle [3] but with Lemma 2.1 replacing Proposition 1.2 there we have

$$\frac{\partial H}{\partial t}(a,t) = \int_{D(\mu(a,t),t)} u_t(x,t)dx \quad \forall (a,t) \in (0,A) \times (0,T). \tag{2.9}$$

Since u satisfies (0.4), by (2.7), (2.8), (2.9), and the results in Bandle [2] for all $0 < t < T$ we have

$$\begin{aligned}
\frac{\partial H}{\partial t}(a,t) &= \int_{D(\mu(a,t),t)} \Delta \log u(x,t)dx \\
&= \int_{\Gamma(\mu(a,t),t)} \frac{\partial}{\partial n} \log u(\sigma,t)d\sigma_n \\
&= -\int_{\Gamma(\mu(a,t),t)} \frac{|\nabla_x u|}{u} d\sigma_n \\
&= -\frac{1}{\mu(a,t)} \int_{\Gamma(\mu(a,t),t)} |\nabla_x u| d\sigma_n \\
&\leq -\frac{1}{\mu(a,t)} \int_{\partial V(a)} |\nabla_x u^*| d\sigma_n \\
&= -\frac{s(a)|\nabla_x u^*(r(a),t)|}{\mu(a,t)} \\
&= \frac{s(a)^2}{\mu(a,t)} \frac{\partial^2 H}{\partial a^2}(a,t) \qquad \text{a.e. } a \in (0,A)
\end{aligned} \tag{2.10}$$

with

$$\begin{cases} H(A,t) = \int_{B_R} u(x,t)dx = \int_{B_R} u_0 dx - \alpha|\partial B_R|t & \forall 0 \leq t < T \\ H(0,t) = 0 & \forall 0 \leq t < T. \end{cases} \tag{2.11}$$

We now let

$$\widetilde{H}(a,t) = \int_{\widetilde{D}(\widetilde{\mu}(a,t),t)} \widetilde{u}(x,t)dx.$$

Then

$$\frac{\partial \widetilde{H}}{\partial a}(a,t) = \widetilde{\mu}(a,t) \quad \forall (a,t) \in (0,A) \times (0,T).$$

Since $\widetilde{u}$ is radially symmetric and monotone decreasing in $r \geq 0$, by the same argument as before $\widetilde{H}$ satisfies

$$\begin{cases} \forall 0 < t < T, \quad \frac{\partial \widetilde{H}}{\partial t}(a,t) = \frac{s(a)^2}{\widetilde{\mu}(a,t)} \frac{\partial^2 \widetilde{H}}{\partial a^2}(a,t) & \text{a.e. } a \in (0,A) \\ \widetilde{H}(A,t) = \int_{B_R} \widetilde{u}(x,t)dx = \int_{B_R} u_0 dx - \alpha|\partial B_R|t & \forall 0 \leq t < T \\ \widetilde{H}(0,t) = 0 & \forall 0 \leq t < T. \end{cases} \tag{2.12}$$

Let $q = \widetilde{H} - H$. By (2.10), (2.11) and (2.12) q satisfies

$$\begin{cases} \forall 0 < t < T, \quad \dfrac{\partial q}{\partial t} \ge \dfrac{s(a)^2}{\widetilde{\mu}(a,t)} \dfrac{\partial^2 q}{\partial a^2} - \dfrac{s(a)^2}{\mu(a,t)\widetilde{\mu}(a,t)} \dfrac{\partial^2 H}{\partial a^2} \dfrac{\partial q}{\partial a} & \text{a.e. } a \in (0,A) \\ q(A,t) = q(0,t) = 0 & \forall 0 \le t < T \\ q(a,0) \ge 0 & \forall a \in [0,A]. \end{cases} \tag{2.13}$$

Let $D_\varepsilon = [\varepsilon, A-\varepsilon] \times [\varepsilon, T-\varepsilon]$, $S_\varepsilon = \partial_p D_\varepsilon = \{\varepsilon, A-\varepsilon\} \times [\varepsilon, T-\varepsilon] \cup [\varepsilon, A-\varepsilon] \times \{\varepsilon\}$, $D = [0,A] \times [0,T]$, and $S = \{0,A\} \times [0,T) \cup [0,A] \times \{0\}$. By applying Lemma 2.2 to $-q$ in D_ε we get

$$\begin{aligned} & \min\nolimits_{D_\varepsilon} q \ge \min\nolimits_{S_\varepsilon} q \\ \Rightarrow \quad & \inf\nolimits_D q \ge \inf\nolimits_S q = 0 && \text{as } \varepsilon \to 0 \\ \Rightarrow \quad & \textstyle\int_{D(\widetilde{\mu}(a,t),t)} u(x,t)dx \le \int_{\widetilde{D}(\widetilde{\mu}(a,t),t)} \widetilde{u}(x,t)dx && \forall (a,t) \in [0,A] \times [0,T). \end{aligned} \tag{2.14}$$

Since

$$\begin{cases} \displaystyle\int_{D(\mu(a,t),t)} u(x,t)dx = \int_{V(a)} u^*(x,t)dx \\ \displaystyle\int_{\widetilde{D}(\widetilde{\mu}(a,t),t)} \widetilde{u}(x,t)dx = \int_{V(a)} \widetilde{u}(x,t)dx, \end{cases} \tag{2.15}$$

by (2.14) and (2.15) we get (2.5). Dividing both sides of (2.5) by $|B_\rho|_n$ and letting $\rho \to 0$ we get (2.6) and the theorem follows. □

By the same argument as the proof of Theorem 2.3 we have the following corollary.

Corollary 2.4. *Suppose $\phi \in C^\infty((0,\infty))$ is a non-decreasing function such that ϕ' is a real analytic function on $(0,\infty)$. If u is a smooth solution of the following Neumann problem*

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } Q_R^T \\ u > 0 & \text{on } \overline{B}_R \times (0,T) \\ \dfrac{\partial}{\partial n}\phi(u)(x,t) = -\alpha & \text{on } \partial B_R \times (0,T) \\ u(x,0) = u_0(x) & \text{in } B_R \end{cases} \tag{2.16}$$

in Q_R^T and $\widetilde{u}$ is the smooth radially symmetric monotone decreasing solution of (2.16) in Q_R^T with initial value u_0^* and satisfying (1.3), then

$$\int_{B_\rho} u^*(x,t)dx \le \int_{B_\rho} \widetilde{u}(x,t)dx \quad \forall 0 \le \rho \le R, 0 \le t < T$$

where for each $0 \le t < T$, $u^*(\cdot,t)$ is the symmetrization of u. In particular

$$\max_{x\in B_R} u(x,t) \le \max_{x\in B_R} \widetilde{u}(x,t) = \widetilde{u}(0,t) \quad \forall 0 \le \rho \le R, 0 \le t < T.$$

We are now ready to remove the restriction of radial symmetry in Lemma 1.11.

Corollary 2.5. *Suppose* $0 < \alpha < \infty$ *for* $n = 1$ *or* $0 < \alpha < 4/R$ *for* $n = 2$, $0 \le u_0 \in L^p(B_R)$ *for some constant* $p > 1$ *and* $u_0 \not\equiv 0$. *Then for any* $s_1 > -log\, T$, *there exists a constant* $C_2 > 0$ *such that*

$$v(x,s) \le C_2 \quad \forall |x| \le R, s \ge s_1. \tag{2.17}$$

Proof. Let $\widetilde{u}$ be the solution of (0.4) with initial value u_0^* and let

$$\widetilde{v}(x,s) = \frac{\widetilde{u}(x,t)}{T-t}, s = -\log\,(T-t).$$

Then by Lemma 1.11 there exists a constant C_2 such that

$$\widetilde{v}(x,s) \le C_2 \quad \forall x \in \overline{B}_R, s \ge s_1.$$

Hence by Theorem 2.3 we have

$$\begin{aligned} & u(x,t) \le \widetilde{u}(0,t) && \forall x \in \overline{B}_R, 0 \le t < T \\ \Rightarrow \quad & v(x,s) \le \widetilde{v}(0,s) \le C_2 && \forall x \in \overline{B}_R, s \ge s_1 \end{aligned}$$

and the corollary follows. □

By the same argument as the proof of Theorem 2.12 of Hsu [16] or Theorem 1.2.12 of Hsu [20] we have:

Lemma 2.6. *Suppose* $\alpha > 0$ *and* $0 < R_1 < R$. *If* v_1, $v_2 \in C^\infty(\overline{B_R \setminus B_{R_1}} \times [s_1,\infty))$ *are the solutions of (1.4) in* $(B_R \setminus B_{R_1}) \times (s_1,\infty)$ *with normal dervative*

$$\frac{\partial(log\ v_i)}{\partial n}(x,s) = -\alpha \quad \forall |x| = R, s \ge s_1, i = 1,2$$

and

$$\begin{cases} v_2(x,s) \ge v_1(x,s) & \forall |x| = R_1, s \ge s_1 \\ v_2(x,s_1) \ge v_1(x,s_1) & \forall R_1 \le |x| \le R. \end{cases}$$

Then $v_2 \ge v_1$ *on* $\overline{B_R \setminus B_{R_1}} \times [s_1,\infty)$.

Lemma 2.7. *Suppose* $n = 1$ *or* 2 *and* $0 < \alpha < \infty$ *for* $n = 1$ *or* $0 < \alpha < 4/R$ *for* $n = 2$, $0 \le u_0 \in L^p(B_R)$ *for some constant* $p > 1$ *and* $u_0 \not\equiv 0$. *Then for any* $s_1 > -log\, T$, *there exists a constant* $C_1 > 0$ *such that*

$$v(x,s) \ge C_1 \quad \forall |x| \le R, s \ge s_1. \tag{2.18}$$

Proof. Let $s_1 > -\log T$. By Lemma 2.5 there exists a constant $C_2 > 0$ such that (2.17) holds. We choose $0 < R' < R$ such that

$$C_2|B_R \setminus B_{R'}|_n < \frac{\alpha}{4}\sigma_n(\partial B_R).$$

Suppose there exists $s_2 \geq s_1$ such that

$$v(x, s_2) \leq \frac{\alpha\sigma_n(\partial B_R)}{2|B_R|_n} \quad \forall |x| \leq R'. \tag{2.19}$$

Since v satisfies (1.5), by (2.19) we have

$$\alpha\sigma_n(\partial B_R) = \textstyle\int_{B_R} v(x, s_2)dx \leq C_2|B_R \setminus B_{R'}|_n + (\alpha/2)\sigma_n(\partial B_R) \cdot (|B_{R'}|_n/|B_R|_n)$$

$$\Rightarrow \quad (\alpha/2)\sigma_n(\partial B_R) \leq C_2|B_R \setminus B_{R'}|_n < (\alpha/4)\sigma_n(\partial B_R).$$

Contradiction arises. Hence (2.19) is false. Thus for any $s \geq s_1$ there exists $y_s \in \overline{B}_{R'}$ such that

$$v(y_s, s) > \frac{\alpha\sigma_n(\partial B_R)}{2|B_R|_n}. \tag{2.20}$$

Since u satisfies (0.3), v satisfies

$$v_s \leq \frac{T}{t}v \quad \forall |x| \leq R, s > -\log T.$$

Let $\delta = (s_1 + \log T)/2$. Then by the same argument as the proof of Theorem 2.4 of Hui [23], Lemma 6 of Vazquez [30], and Lemma 2.9 of Hui [23], there exist constants $\beta_1, \dots, \beta_k > 0$, $C_1', \dots, C_k' > 0$, $R_1 > 0$, such that for any $s \geq s_1$, $|x| \leq R_1$, there exists $i \in \{1, \dots, k\}$ such that

$$v(x, s - \delta) \geq C_i' v(y_s, s)^{\beta_i}.$$

This together with (2.20) implies that there exists a constant $C_3 > 0$ such that

$$v(x, s) \geq C_3 \quad \forall |x| \leq R_1, s \geq s_1. \tag{2.21}$$

Let $C_4 = \min_{\overline{B}_R} v(x, s_1) > 0$ and $C_5 = \min(C_3, C_4)$. We next let

$$v_1(x) = \begin{cases} C_5 e^{\alpha(R_1 - |x|)} & \text{for } n = 1 \\ C_5 \left(\dfrac{R_1}{|x|}\right)^{\alpha R} & \text{for } n = 2. \end{cases}$$

Then v_1 satisfies

$$\begin{cases} v_{1,s} = 0 = \Delta \log v_1 \leq \Delta \log v_1 + v_1 & \text{in } (B_R \setminus B_{R_1}) \times (s_1, \infty) \\ v_1(x) = C_5 \leq v(x, s) & \forall |x| = R_1, s \geq s_1 \\ \dfrac{\partial}{\partial n}(\log v_1)(x) = -\alpha & \forall |x| = R \\ v_1(x) \leq C_5 \leq v(x, s_1) & \forall R_1 \leq |x| \leq R. \end{cases}$$

Hence by Lemma 2.6 we have

$$v(x,s) \geq v_1(x) \qquad \forall R_1 \leq |x| \leq R, s \geq s_1$$

$$\Rightarrow \quad v(x,s) \geq \begin{cases} C_5 e^{\alpha(R_1-R)} & \text{for } n=1 \\ C_5\left(\dfrac{R_1}{R}\right)^{\alpha R} & \text{for } n=2 \end{cases} \qquad \forall R_1 \leq |x| \leq R, s \geq s_1. \tag{2.22}$$

By (2.21) and (2.22) we get (2.18) and the lemma follows.

3. SECTION THREE

In this section we will use energy method to prove the asymptotic behaviour of solution of (0.4) as t tends to the extinction time of the solution. We first start with a lemma.

Lemma 3.1. *Suppose $n = 1$ or 2 and $0 < \alpha < \infty$ for $n = 1$ or $0 < \alpha < 4/R$ for $n = 2$. Then there exists a unique smooth solution to the following O.D.E.*

$$\begin{cases} (\log \overline{v})'' + \dfrac{n-1}{r}(\log \overline{v})' + \overline{v} = 0 \quad \forall r \in (0,R) \\ \overline{v}'(0) = 0 \\ (\log \overline{v})'(R) = -\alpha. \end{cases} \tag{3.1}$$

Proof. We will use a shooting argument to prove the lemma. For any $\mu > 0$ let $\lambda_1 = 2/\mu$ for $n = 1$ and $\lambda_1 = 8/\mu$ for $n = 2$. Then $w_{\lambda_1} = \log v_{\lambda_1}$ is the unique solution of the following O.D.E.

$$\begin{cases} (\overline{w})'' + \dfrac{n-1}{r}(\overline{w})' + e^{\overline{w}} = 0 \quad \forall r \in (0,R) \\ \overline{w}(0) = \log \mu \\ \overline{w}'(0) = 0 \end{cases}$$

where v_{λ_1} is given by (1.8). Hence v_{λ_1} is the unique solution of the following O.D.E.

$$\begin{cases} (\log \overline{v})'' + \dfrac{n-1}{r}(\log \overline{v})' + \overline{v} = 0 \quad \forall r \in (0,R) \\ \overline{v}(0) = \mu \\ \overline{v}'(0) = 0 \end{cases} \tag{3.2}$$

where v_{λ_1} is given by (1.8). By Lemma 1.5 there exists a unique constant $\lambda(\alpha) > 0$ such that (1.12) holds with μ there being replaced by α. If v_{λ_1} also satisfies $(\log \overline{v})'(R) = -\alpha$, then $\lambda_1 = \lambda(\alpha)$ or $\mu = 2/\lambda(\alpha)$ for $n = 1$ and $\mu = 8/\lambda(\alpha)$ for $n = 2$. Hence the solution of (3.1) will satisfy (3.2) with $\mu = 2/\lambda(\alpha)$ for $n = 1$ and $\mu = 8/\lambda(\alpha)$ for $n = 2$. Thus the solution of (3.1) is uniquely given by $v_{\lambda(\alpha)}$. □

Lemma 3.2. *Suppose $n = 1$ and $0 < \alpha < \infty$. Then there exists a unique smooth solution to (0.9).*

Proof. Let $n = 1$. By Lemma 1.5 there exists a unique constant $\lambda(\alpha) > 0$ that satisfies (1.12) with μ there being replaced by α. Then $v_{\lambda(\alpha)}$ is a solution of (0.9).

For uniqueness we let $\overline{v}$ be a solution of (0.9). If $\overline{w}$ is given by (0.11). Multiplying (0.12) by $\overline{w}'$ and integrating over $x \in [-R, R]$ we get

$$\begin{aligned} & \tfrac{\overline{w}'^2(R)}{2} + e^{\overline{w}(R)} = \tfrac{\overline{w}'^2(-R)}{2} + e^{\overline{w}(-R)} \\ \Rightarrow \quad & e^{\overline{w}(R)} = e^{\overline{w}(-R)} \\ \Rightarrow \quad & \overline{w}(R) = \overline{w}(-R). \end{aligned}$$

By the results of Gidas et al [11] $\overline{w}$ is radially symmetric and monotone decreasing in $0 \le r \le R$. By symmetry

$$\overline{w}'(0) = 0 \quad \Rightarrow \quad (\log \overline{v})'(0) = 0 \quad \Rightarrow \quad \overline{v}'(0) = 0.$$

Hence $\overline{v}$ satisfies (3.1) with $n = 1$. By Lemma 3.1 the solution of (0.9) is unique and the lemma follows. □

Lemma 3.3. *For any $s_1 > -\log T$, the function w given by (1.6) satisfies*

$$\int_{s_1}^{\infty} \int_{B_R} e^w w_s^2 \, dx ds < \infty. \tag{3.3}$$

Proof. Since w satisfies (1.7), by (1.5) we have

$$\begin{aligned} \int_{B_R} e^w w_s^2 dx &= \int_{B_R} (\Delta w + e^w) w_s dx \\ &= \int_{\partial B_R} \frac{\partial w}{\partial n} w_s d\sigma_n - \int_{B_R} \nabla w \cdot \nabla w_s dx + \frac{\partial}{\partial s}\left(\int_{B_R} e^w dx\right) \\ &= -\alpha \int_{\partial B_R} w_s d\sigma_n - \frac{1}{2}\frac{\partial}{\partial s}\left(\int_{B_R} |\nabla w|^2 dx\right) + \frac{\partial}{\partial s}(\alpha \sigma_n(\partial B_R)) \\ &= -\frac{\partial}{\partial s}\left(\alpha \int_{\partial B_R} w d\sigma_n + \frac{1}{2}\int_{B_R} |\nabla w|^2 dx\right) \quad \forall s \ge s_1. \end{aligned}$$

Integrating over $s \in (s_1, s_2)$ we get

$$\begin{aligned} & \int_{s_1}^{s_2} \int_{B_R} e^w w_s^2 dx ds + \alpha \int_{\partial B_R} w(x, s_2) d\sigma_n + \frac{1}{2}\int_{B_R} |\nabla w(x, s_2)|^2 dx \\ = \; & \alpha \int_{\partial B_R} w(x, s_1) d\sigma_n + \frac{1}{2}\int_{B_R} |\nabla w(x, s_1)|^2 dx \quad \forall s_2 \ge s_1. \end{aligned} \tag{3.4}$$

Choose $-\log T < s' < s_1$. Then by Corollary 2.5 and Lemma 2.7 there exist constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1 \le v(x,s) \le C_2 \quad \forall |x| \le R, s \ge s'. \tag{3.5}$$

Hence there exists a constant $C_3 > 0$ such that

$$\left| \int_{\partial B_R} w(x,s) d\sigma_n \right| \le C_3 \quad \forall s \ge s_1. \tag{3.6}$$

and (1.4) is uniformly parabolic on $\overline{B}_R \times (s', \infty)$. By the Schauder's estimate Ladyzenskaya and Solonnikov [25], $|\nabla v|$ is uniformly bounded in $\overline{B}_R \times [s_1, \infty)$. Hence there exists a constant $C_4 > 0$ such that

$$\int_{B_R} |\nabla w(x, s_1)|^2 dx \le C_4. \tag{3.7}$$

Thus by (3.4), (3.5), (3.6) and (3.7) there exists a constant $C_5 > 0$ such that

$$\begin{aligned} & \textstyle\int_{s_1}^{s_2} \int_{B_R} e^w w_s^2 dx ds \le C_5 \quad \forall s_2 \ge s_1 \\ \Rightarrow \quad & \textstyle\int_{s_1}^{\infty} \int_{B_R} e^w w_s^2 dx ds \le C_5 \quad \text{as } s_2 \to \infty \end{aligned}$$

and the lemma follows. □

Theorem 3.4. *Suppose $n = 1$ or 2 and $0 < \alpha < \infty$ for $n = 1$ or $0 < \alpha < 4/R$ for $n = 2$, $0 \le u_0 \in L^p(B_R)$ for some constant $p > 1$ and $u_0 \not\equiv 0$. Suppose u_0 is radially symmetric if $n = 2$. Then v will converge uniformly on $\overline{B}_R$ to $v_{\lambda(\alpha)}$ as $s \to \infty$ for some constant $\lambda(\alpha) > 0$ satisfying (1.12) with μ there being replaced by α where $v_{\lambda(\alpha)}$ is given by (1.8).*

Proof. Let $s' > -\log T$. By Corollary 2.5 and Lemma 2.7 there exist constants $C_1 > 0$, $C_2 > 0$, such that (3.5) holds. Hence (1.4) is uniformly parabolic on $\overline{B}_R \times (s', \infty)$. By the Schauder's estimates (Ladyzenskaya and Solonnikov [25]) $\{v(\cdot, s)\}_{s > s'+1}$, $\{\nabla v(\cdot, s)\}_{s > s'+1}$, and $\{\partial_{x_i} \partial_{x_j} v(\cdot, s)\}_{s > s'+1}$ where $i, j = 1, 2$, are equi-Holder continuous on $\overline{B}_R$. Hence any sequence $\{v(\cdot, s_i)\}$, $s_i \to \infty$ as $i \to \infty$, will have a subsequence $\{v(\cdot, s_i')\}$ converging uniformly on $C^2(\overline{B}_R)$ to some function $\overline{v}$ as $s_i \to \infty$. Moreover $\overline{v}$ satisfies

$$\frac{\partial}{\partial n} (\log \overline{v})(x) = -\alpha \quad \forall |x| = R.$$

We claim that $\overline{v}$ satisfies (0.9). To prove the claim we let $\eta \in C_0^\infty(B_R)$. Multiplying (1.4) by η and integrating over $(s_i', s_i' + 1) \times B_R$ we have

$$\int_{s_i'}^{s_i'+1} \int_{B_R} v_s \eta dx ds = \int_{s_i'}^{s_i'+1} \int_{B_R} (\log v) \Delta \eta dx ds + \int_{s_i'}^{s_i'+1} \int_{B_R} v \eta dx ds \qquad \forall i = 1, 2, \dots. \tag{3.8}$$

Let w be given by (1.6). Then by (3.5) and Lemma 3.3 we have

$$\int_{s'}^{\infty}\int_{B_R} w_s^2 dxds < \infty \quad \Rightarrow \quad \int_{s'}^{\infty}\int_{B_R} v_s^2 dxds < \infty. \tag{3.9}$$

Hence

$$\left|\int_{s_i'}^{s_i'+1}\int_{B_R} v_s\eta dxds\right| \le C\left(\int_{s_i'}^{s_i'+1}\int_{B_R} v_s^2 dxds\right)^{1/2} \to 0 \quad \text{as } s_i' \to \infty. \tag{3.10}$$

and

$$\begin{aligned}
&\left|\int_{s_i'}^{s_i'+1}\int_{B_R}(\log v(x,s'))\Delta\eta dxds' - \int_{B_R}(\log v(x,s_i'))\Delta\eta dx\right| \\
=\ &\left|\int_{s_i'}^{s_i'+1}\left(\int_{B_R} w(x,s')\Delta\eta dx - \int_{B_R} w(x,s_i')\Delta\eta dx\right)ds'\right| \\
\le\ &\int_{s_i'}^{s_i'+1}\left|\int_{s_i'}^{s'}\int_{B_R} w_s(x,s)\Delta\eta dxds\right|ds' \\
\le\ &C\left(\int_{s_i'}^{s_i'+1}\int_{B_R} w_s^2 dxds\right)^{1/2} \to 0 \quad \text{as } s_i' \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{s_i'\to\infty}\int_{s_i'}^{s_i'+1}\int_{B_R}(\log v(x,s'))\Delta\eta dxds' &= \lim_{s_i'\to\infty}\int_{B_R}(\log v(x,s_i'))\Delta\eta dx \\
&= \int_{B_R}(\log \overline{v})\Delta\eta dx.
\end{aligned} \tag{3.11}$$

Similarly

$$\lim_{s_i'\to\infty}\int_{s_i'}^{s_i'+1}\int_{B_R} v\eta dxds = \int_{B_R}\overline{v}\eta dx. \tag{3.12}$$

Letting $s_i' \to \infty$ in (3.8), by (3.10), (3.11) and (3.12) we have

$$\int_{B_R}(\log \overline{v})\Delta\eta dx + \int_{B_R}\overline{v}\eta dx = 0 \quad \forall \eta \in C_0^{\infty}(B_R).$$

Hence $\overline{v}$ satisfies (0.9). By Lemma 3.2 there exists a unique solution of (0.9) when $n = 1$. When $n = 2$ and u_0 is radially symmetric, v will be radially symmetric. Thus $\overline{v}$ will also be radially symmetric for $n = 2$. Then by symmetry $\overline{v}'(0) = 0$ for $n = 2$. By Lemma 3.1 $\overline{v}$ is uniquely determined for $n = 2$. Hence $\overline{v} = v_{\lambda(\alpha)}$ for some unique constant $\lambda(\alpha) > 0$ satisfying (1.12) with μ there being replaced by α. Since the limit $\overline{v} = v_{\lambda(\alpha)}$ is independent of the choice of the sequence $\{v(\cdot, s_i)\}$, $v(\cdot, s)$ converges uniformly to $v_{\lambda(\alpha)}$ on $\overline{B}_R$ as $s \to \infty$ and the theorem follows. □

As a consequence of the proof of Theorem 3.4 we have

Corollary 3.5. *Suppose $n = 2$, $0 < \alpha < 4/R$, $0 \le u_0 \in L^p(B_R)$ for some constant $p > 1$ and $u_0 \not\equiv 0$. If the solution of (0.9) is unique, then v will converge uniformly on $\overline{B}_R$ to $v_{\lambda(\alpha)}$ as $s \to \infty$ for some constant $\lambda(\alpha) > 0$ satisfying (1.12) with μ there being replaced by α where $v_{\lambda(\alpha)}$ is given by (1.8).*

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