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ALGEBRAIC POLYNOMIALS WITH DEPENDENT RANDOM COEFFICIENTS

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ABSTRACT: This paper provides an asymptotic estimate for the expected number of real zeros of a random algebraic polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$. The coefficients a_i ($i = 0, 1, 2, \cdots, n-1$) are assumed to be dependent normal random variables with correlation coefficients of any two terms i and j, $i \neq j$, assumed $\rho_{ij} = -\rho^{|i-j|}$, where ρ is any positive constant in the interval $(0, 1/3)$. Previous results are mainly for independent distributed coefficients or for a positive correlation coefficient in $(0, 1/2)$.

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1. INTRODUCTION

The expected number of real zeros of a random algebraic polynomial

$$P_n(x) = \sum_{i=0}^{n-1} a_i x^i, \tag{1.1}$$

where the coefficients $a_0, a_1, \ldots, a_{n-1}$ are a sequence of random variables is well studied. Let $N_n(a,b)$ be the number of real zeros of $P_n(x)$ in the interval (a,b) and $EN_n(a,b)$ its expected value. For the independent normal standard distributed coefficients from the pioneer work of Kac [3] it is known that for n sufficiently large $EN_n(-\infty,\infty) \sim (2/\pi)\log n$. In fact this asymptotic formula remains invariant for the case when the coefficients belong to the domain of attraction of normal law with mean zero. However, if the mean of coefficients assumes any non zero constant the above asymptotic value reduces by half to $EN_n(-\infty,\infty) \sim (1/\pi)\log n$. For a more general class of distributions Logan and Shepp [4] and [5] obtained the asymptotic

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value for $EN_n(-\infty,\infty)$ for both Cauchy distributed coefficients and those of the domain of attraction of stable law. In an interesting development recently, Wilkins [12] used a new method to show that the error term for deriving the above asymptotic formula is $O(1)$. This is, indeed, a significant improvement to previous results. The earlier results concerning different types of random polynomials and assumptions on the distribution of the coefficients constitute the greater part of the books by Bharucha-Ried and Sambandham [1] and Farahmand [2].

In comparison there are few results for the case of dependent coefficients. One reason is that the analysis for this case is complicated. That which is available is restricted to the case of positive dependence. These works were pioneered and developed comprehensively by Sambandham [8], [10] and [11] for algebraic case, and Sambandham [9] and Renganathan and Sambandham [7] for the case of random trigonometric polynomials. So far there is no study for the case when the coefficients have negative correlation coefficients. Motivated by the recent development on the negative dependent random variables, see for example Newman [6], here we study the case when the coefficients $\{a_j\}_{j=0}^n$ in (1.1) are a sequence of dependent normal random variables with mean zero and variance one. We assume the joint density function be

$$\sqrt{\frac{|M|}{(2\pi)^n}}\exp\left\{-\frac{\overline{\mathbf{a}}'M\overline{\mathbf{a}}}{2}\right\},$$

where M^{-1} is the covariance matrix with

$$\rho_{ij}=\begin{cases}-\rho^{|i-j|} & j\neq i\\ 1 & j=i \qquad i,j=0,1,2,\cdots,n-1,\end{cases}$$

and $\overline{\mathbf{a}}'$ is the transpose of the column vector $\overline{\mathbf{a}}=(a_0,a_1,a_2,\cdots a_{n-1})$. We show that our result remains valid for any constant $0<\rho<1/3$. This upper limit of $\rho<1/3$ is the uppermost bound of interest. For instance, since

$$\begin{aligned}\operatorname{var}\left(\sum_{i=0}^{n-1}a_i\right) &= n-2\left(\frac{\rho(n-1)}{1-\rho}-\frac{\rho^2-\rho^{n+1}}{(1-\rho)^2}\right)\\ &= \left(1-\frac{2\rho}{1-\rho}\right)n+\frac{2(\rho-\rho^{n+1})}{(1-\rho)^2},\end{aligned}$$

for all sufficiently large n, the necessary condition for making the above variance positive, and therefore the arguments meaningful, is $\rho<1/3$. It is interesting to note that there is no analogue for the case of $\rho_{ij}=-\rho$, $\rho>0$. A similar evaluation will yield

$$\operatorname{var}\left(\sum_{i=0}^{n-1}a_i\right)=n-\rho n(n-1)$$

which requires $\rho < (n-1)^{-1} \to 0$ as $n \to \infty$. In this work we therefore consider the case of $\rho_{ij} = -\rho^{|i-j|}$ and prove:

Theorem 1. *If the random variables $a_i, i = 0, 1, 2, \cdots, n-1$ satisfy the above assumptions, then for all sufficiently large n, the expected number of real zeros of $P_n(x)$ in (1.1) is*

$$EN_n(-\infty, \infty) \sim \left(\frac{2}{\pi}\right) \log n.$$

2. PRELIMINARY ANALYSIS

Let

$$A^2 = \mathrm{var}(P_n(x)), \qquad B^2 = \mathrm{var}(P_n'(x))$$

$$C = \mathrm{cov}(P_n(x)P_n'(x)) \quad \text{and} \quad \Delta^2 = A^2B^2 - C^2. \tag{2.1}$$

Then from Kac [3], see also Farahmand [2], we have the Kac-Rice formula for the expected number of real zeros as

$$EN_n(a,b) = \frac{1}{\pi}\int_a^b \frac{\Delta}{A^2}\,dx. \tag{2.2}$$

From the above assumptions for the distributions of the coefficients of $P_n(x)$, we can easily show

$$A^2 = \sum_{i=0}^{n-1} x^{2i} - \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|} x^{i+j}, \tag{2.3}$$

$$B^2 = \sum_{i=0}^{n-1} i^2 x^{2i-2} - \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|} ij x^{i+j-2} \tag{2.4}$$

and

$$C = \sum_{i=0}^{n-1} i x^{2i-1} - \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|} i x^{i+j-1}. \tag{2.5}$$

Now similarly to the method of Sambandham [8], we obtain the value of the above identities by consequent differentiation of the previous one multiplied by x, to obtain

$$\begin{aligned} A^2 &= \frac{1-x^{2n}}{1-x^2} - \frac{\rho(\rho x - (\rho x)^n)}{(\rho - x)(1-\rho x)} + \frac{\rho(x^2 - x^{2n})}{(\rho - x)(1-x^2)} \\ &\quad - \frac{\rho x(1-x^{2n-2})}{(1-\rho x)(1-x^2)} + \frac{\rho^2 x^n(\rho^{n-1} - x^{n-1})}{(1-\rho x)(\rho - x)}, \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
B^2 \quad &= \frac{(1+x^2)(1-x^{2n})-n^2x^{2n-2}(1-x^2)^2-2nx^{2n}(1-x^2)}{(1-x^2)^3} \\
&-\frac{\rho(\rho-n\rho^n x^{n-1}+(n-1)\rho^{n+1}x^n)}{(\rho-x)^2(1-\rho x)^2}+\frac{\rho(x-nx^{2n-1}+(n-1)x^{2n+1})}{(\rho-x)^2(1-x^2)^2} \\
&+\frac{\rho[(1-x^2)(1-n^2x^{2n-2}...)+2x(x-x^{2n+1}...)]}{(\rho-x)(1-x^2)^3} \\
&-\frac{\rho(x-x^{2n-1}+(n-1)x^{2n-1}-(n-1)x^{2n-3})}{(1-\rho x)^2(1-x^2)^2} \\
&-\frac{\rho x[(1-x^2)(1-nx^{2n-2}...)+2x(x-x^{2n-1}...)]}{(1-\rho x)(1-x^2)^3} \\
&+\frac{[(1-\rho x)(n\rho^2x^{n-1})+\rho^3x^n][\rho^{n-1}+(n-2)x^{n-1}-(n-1)\rho x^{n-2}]}{(1-\rho x)^2(\rho-x)^2} \qquad (2.7)
\end{aligned}
$$

and

$$
\begin{aligned}
C = \quad &\frac{x-x^{2n+1}-nx^{2n-1}+nx^{2n+1}}{(1-x^2)^2} \\
&-\frac{\rho(\rho-n\rho^n x^{n-1}+(n-1)\rho^{n+1}x^n)}{(\rho-x)(1-\rho x)^2} \\
&+\frac{\rho(x-x^{2n+1}-nx^{2n-1}+nx^{2n+1})}{(\rho-x)(1-x^2)^2} \\
&-\frac{\rho x(x-x^{2n-1}+(n-1)x^{2n-1}-(n-1)x^{2n-3})}{(1-\rho x)(1-x^2)^2} \\
&+\frac{\rho^2x^n(\rho^{n-1}-x^{n-1}+(n-1)x^{n-1}-(n-1)\rho x^{n-2})}{(1-\rho x)(\rho-x)^2}. \qquad (2.8)
\end{aligned}
$$

The above evaluations for A^2, B^2 and C in (2.6)-(2.8) are not needed as we only use their estimates as $n \to \infty$. However, we give their actual values for their comparison with Sambandham [8].

3. PROOF OF THE THEOREM

We first note that changing x to $1/x$ leaves the distribution of the coefficients of $P_n(x)$ in (1.1) invariant. Hence the expected number of real zeros in the interval $(-1,1)$ is asymptotically the same as in $(-\infty,-1)\cup(1,\infty)$. Therefore we only give the result for $EN_n(-1,1)$. To this end we first assume $x \in (0,1)$ and we divide this interval into two subintervals $(0, 1-\epsilon)$ and $(1-\epsilon, 1)$ where for

$$a = 1 - \log\log n^{10}/\log n, \quad \text{we let} \quad \epsilon = n^{-a}. \tag{3.1}$$

For $0 \leq x \leq 1-\epsilon$, and all sufficiently large n, we can easily show that

$$x^n \leq n^{-10}.$$

Therefore, from the above values of (2.6)-(2.8) we can derive

$$A^2 = \frac{1-3\rho x}{(1-\rho x)(1-x^2)}\{1+O(n^{-20})\}, \tag{3.2}$$

$$B^2 = \frac{(1+x^2)(1-\rho x)\{2\rho(1+x^2) - x(1+3\rho^2)\} - \rho(1-x^2)^2(1+\rho x)}{(\rho - x)(1-\rho x)^2(1-x^2)^3} \times\{1+O(n^{-18})\}, \tag{3.3}$$

and

$$C = \frac{x - 3\rho x^2 + 3\rho^2 x^3 - \rho}{(1-x^2)^2(1-\rho x)^2}\{1+O(n^{-9})\}. \tag{3.4}$$

Therefore from (2.1), (3.2)-(3.4) we can easily see that

$$\frac{\Delta}{A^2} = \frac{\sqrt{A^2B^2 - C^2}}{A^2} \sim \sqrt{\frac{k(\rho, x)}{(\rho - x)(1-x^2)^2}} = \sqrt{\frac{\mid k(\rho, x) \mid}{\mid (\rho - x) \mid (1-x^2)^2}}, \tag{3.5}$$

where

$$k(\rho, x) = \frac{(1+x^2)(1-\rho x)\{2\rho(1+x^2) - x(1+3\rho^2)\} - \rho(1-x^2)^2(1+\rho x)}{(1-\rho x)(1-3\rho x)} - \frac{(\rho - x)(x - 3\rho x^2 + 3\rho^2 x^3 - \rho)^2}{(1-\rho x)^2(1-3\rho x)^2}. \tag{3.6}$$

It is clear that for evaluating Δ/A^2 and $k(\rho, x)$ we need to use a different method in the neighbourhood of $x = \rho$ as outside this neighbourhood. To this end we obtain EN_n in intervals $[0, \rho - 1/n]$, $[\rho - 1/n, \rho + 1/n]$, $[\rho + 1/n, 1/2]$, $[1/2, 1-\eta]$, $[1-\eta, 1-\epsilon]$ and $[1-\epsilon, 1]$, where

$$\eta = \exp\{-(\log n)^{1/3}\} \tag{3.7}$$

and ϵ is defined in (3.1).

When $0 \leq x \leq \rho - 1/n$, from (3.5) and (3.6), $k(\rho, x)$ is bounded and $(1-x^2)$ is bounded away from zero. Therefore,

$$\frac{\Delta}{A^2} < \frac{L_0}{(\rho - x)^{1/2}},$$

where L's here and in the following are absolute constants. Therefore from (2.2) we have

$$EN_n(0, \rho - 1/n) = O(1). \tag{3.8}$$

For $\rho - 1/n \leq x \leq \rho + 1/n$, from (2.1), (2.4) and (3.2) we have

$$\begin{aligned} \frac{\Delta}{A^2} &< \left(\frac{B^2}{A^2}\right)^{1/2} < \left[\frac{\sum_{i=0}^{n-1} i^2 x^{2i-2}}{(1-3\rho x)/[(1-\rho x)(1-x^2)]\{1+O(n^{-20})\}}\right]^{1/2} \\ &< n\left(\frac{1-\rho x}{1-3\rho x}\right)^{1/2}. \end{aligned}$$

Since $[(1-\rho x)/(1-3\rho x)]^{1/2}$ is bounded, from (2.2) we obtain

$$EN_n(\rho - 1/n, \rho + 1/n) = O(1). \tag{3.9}$$

Also for $\rho + 1/n \leq x \leq 1/2$, we note that $\mid k(\rho, x) \mid$ and $(1-x^2)$ in (3.5) and (3.6) are both bounded away from zero and therefore

$$\frac{\Delta}{A^2} < \frac{L_1}{(x-\rho)^{1/2}}.$$

Hence from (2.2)

$$EN_n(\rho + 1/n, 1/2) < L_1 \int_{\rho+1/n}^{1/2} \frac{d_x}{(x-\rho)^{1/2}} = O(1). \tag{3.10}$$

When $1/2 \leq x \leq 1 - \eta$, from (3.5) and (3.6) and since $k(\rho, x)/(\rho - x)$ is bounded away from zero, we can show

$$\frac{\Delta}{A^2} < \frac{L_2}{1-x^2},$$

and therefore, from (2.2),

$$EN_n(1/2, 1-\eta) < L_2 \int_{1/2}^{1-\eta} \frac{d_x}{1-x^2} = O(\log n)^{1/3}. \tag{3.11}$$

When $1 - \eta \leq x \leq 1 - \epsilon$ from (3.5) and (3.6) we can see

$$\frac{\Delta}{A^2} \sim \frac{k_1(\rho, x)}{1-x^2}, \tag{3.12}$$

where

$$k_1(\rho, x) = \sqrt{\frac{k(\rho, x)}{\rho - x}}.$$

In this interval, we can show that

$$k_1^{**}(\rho,\eta,\epsilon) \le k_1(\rho,x) \le k_1^{*}(\rho,\eta,\epsilon),$$

where $k_1^{**}(\rho,\eta,\epsilon)$ and $k_1^{*}(\rho,\eta,\epsilon)$ are $k_1(\rho,x)$ when x is subsituted by $1-\epsilon$ and $1-\eta$, respectively. Therefore from (2.2), (3.1) and (3.7), we have

$$\frac{ak_1^{**}(\rho,\eta,\epsilon)}{2\pi}\log n \ +O(\log n\ )^{1/3} \le EN_n(1-\eta,1-\epsilon)$$

$$\le \frac{ak_1^{*}(\rho,\eta,\epsilon)}{2\pi}\log n \ +O(\log n\ )^{1/3}.$$

Now since $a\to 1$, then $k_1^{**}(\rho,\eta,\epsilon)\to 1$ and $k_1^{*}(\rho,\eta,\epsilon)\to 1$, as $n\to\infty$ we obtain

$$EN_n(1-\eta,1-\epsilon) = \frac{1}{2\pi}\log n + O(\log n)^{1/3}. \tag{3.13}$$

Finally for $1-\epsilon\le x\le 1$, from (2.6)-(2.8) and with a little algebra, we have

$$\left(\frac{1-3\rho x}{1-\rho x}\right)\left(\frac{1-x^{2n}}{1-x^2}\right) + O(\rho^n) \le A^2 \le \sum_{i=0}^{n-1} x^{2i},$$

and

$$B^2 \le \sum_{i=0}^{n-1} i^2 x^{2i-2}.$$

Also it is easy to show

$$\sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\ne i}}^{n-1} \rho^{|i-j|} i x^{i+j-1} = \frac{2\rho x^{2n-1}}{(\rho-x)(1-x^2)}n + \frac{[\rho x^2(\rho-x)-\rho x(1-\rho x)](1-x^{2n})}{(1-\rho x)(\rho-x)(1-x^2)^2}$$

$$+\frac{\rho^2(\rho-x)+\rho x^{2n}(1-\rho x)}{(1-\rho x)^2(\rho-x)^2} + O(\rho^n). \tag{3.14}$$

Since the first term that appears in (3.14) is negative we obtain

$$\sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\ne i}}^{n-1} \rho^{|i-j|} i x^{i+j-1} \le L_3\frac{1-x^{2n}}{(1-x^2)^2}.$$

Now using a similar method to Farahmand [2], page 23, we can show that for large n

$$\frac{\Delta}{A^2} \le L_4\sqrt{\frac{n}{1-x}}, \tag{3.15}$$

and therefore

$$EN_n(1-\epsilon,1) = O(\log n)^{1/2}. \tag{3.16}$$

Therefore, from (3.8)-(3.13) and (3.16), we have

$$EN_n(0,1) = \frac{1}{2\pi}\log n + O(\log n)^{1/2}. \tag{3.17}$$

For $x \in [-1,0]$ in order to avoid duplication, in the following we only highlight the generalization necessary for the above arguments. To this end for $-1+\epsilon \le x \le 0$, and for all sufficiently large n, Δ/A^2 is similar to (3.5). Also when $-1+\eta \le x \le 0$, from (3.5) and (3.6), similar to (3.11), we have

$$EN_n(-1+\eta,0) = O(\log n)^{1/3}. \tag{3.18}$$

Again when $-1+\epsilon \le x \le -1+\eta$, similar to (3.13), we can obtain

$$EN_n(-1+\epsilon,-1+\eta) = \frac{1}{2\pi}\log n + O(\log n)^{1/3}. \tag{3.19}$$

Now let $-1 \le x \le -1+\epsilon$. Since $\mathrm{var}(\sum_{i=0}^{n-1} a_i \mid x \mid^i) \ge 0$,

$$\mid \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|}x^{i+j} \mid \le \sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|} \mid x \mid^{i+j} \le \sum_{i=0}^{n-1} x^{2i},$$

and therefore from (2.3), we have

$$A^2 \le 2\sum_{i=0}^{n-1} x^{2i}. \tag{3.20}$$

Similarly to (3.20), we can also show that

$$B^2 \le 2\sum_{i=0}^{n-1} i^2x^{2i-2},$$

which yields

$$\Delta^2 < 4[(\sum_{i=0}^{n-1} x^{2i})(\sum_{i=0}^{n-1} i^2x^{2i-2}) - (\sum_{i=0}^{n-1} ix^{2i-1})^2] + 3(\sum_{i=0}^{n-1} ix^{2i-1})^2$$

$$+2(\sum_{i=0}^{n-1} ix^{2i-1})(\sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|}ix^{i+j-1}). \tag{3.21}$$

Also since, in this range of x,

$$\sum_{i=0}^{n-1}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \rho^{|i-j|}x^{i+j} = \frac{2\rho^2x(1-x^{2n}) - 2\rho x^2(1-x^{2n-2})}{(1-x^2)(\rho-x)(1-\rho x)} + o(\rho^n) < 0,$$

from (2.3), we have

$$A^2 > \sum_{i=0}^{n-1} x^{2i}.$$

Therefore, similarly to (3.15), from (3.21), we can show that

$$\frac{\Delta}{A^2} \le L_5\sqrt{\frac{n}{1-x^2}} \le L_6\sqrt{\frac{n}{1+x}},$$

and therefore

$$EN_n(-1,-1+\epsilon) = O(\log n)^{1/2}. \tag{3.22}$$

Thus, from (3.18), (3.19) and (3.22), we have

$$EN_n(-1,0) = \frac{1}{2\pi}\log n + O(\log n)^{1/2}. \tag{3.23}$$

Hence finally, from (3.17) and (3.23), we have

$$EN_n(-\infty,\infty) = \frac{2}{\pi}\log n + O(\log n)^{1/2},$$

which is the proof of the theorem. □

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REFERENCES

[1] A.T. Bharucha-Reid and M. Sambandham, *Random Polynomials*, Academic Press, N.Y., 1986.

[2] K. Farahmand, *Topics in Random Polynomials*, Addison Wesley Longman, London, 1998.

[3] M. Kac, On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.*, **49** (1943), 314–320.

[4] B.F. Logan and L.A. Shepp, Real zeros of random polynomials, *Proc. London Math. Soc.*, **18** (1968), 29–35.

[5] B.F. Logan and L.A. Shepp, Real zeros of random polynomials, II, *Proc. London Math. Soc.*, **18** (1968), 308–314.

[6] C. Newman, Asymptotic independence and limit theorems for positively and negatively dependent random variables, *Inequalities in Statistics and Probability. IMS Lecture Notes-Monograhp Series*, **5** (1984), 127–140.

[7] N. Renganathan and M. Sambandham, On the average number of real zeros of a random trigonometric polynomial with dependent coefficients-II, *Indian J. Pure Appl. Math.*, **15** (1984), 951–956.

[8] M. Sambandham, On a random algebraic polynomial, *J. Indian Math. Soc.*, **41** (1977), 83–97.

[9] M. Sambandham, On the number of real zeros of a random trigonometric polynomial, *Trans. Amer. Math. Soc.*, **238** (1978), 57–70.

[10] M. Sambandham, On the upper bound of the number of real roots of a random algebraic equation, *J. Indian Math. Soc.*, **42** (1978), 15–26.

[11] M. Sambandham, On the average number of real zeros of a class of random algebraic curves, *Pacific J. Math.*, **81** (1979), 207–215.

[12] J.E. Wilkins, An asymptotic expansion for the expected number of real zeros of a random polynomial, *Proc. Amer. Math. Soc.*, **103** (1988), 1249–1258.