Communications in Applied Analysis **9** (2005), no. 1, 105–115

SOME SECOND ORDER DIFFERENCE EQUATIONS IN HILBERT SPACES

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Communicated by D. Bainov

ABSTRACT: An existence result for some second order difference equations is given. These equations are governed by maximal monotone operators in Hilbert spaces and they are the discrete analogs of some abstract evolution equations. The main tool we use is the theory of maximal monotone operators in Hilbert spaces.

AMS (MOS) Subject Classification. 39A12, 39A70, 47H05

1. INTRODUCTION

We are concerned with the existence and uniqueness of the solution to the difference equation

$$u_{i+1} - (1+\theta_i)\, u_i + \theta_i u_{i-1} \in c_i A u_i + f_i,\ i \geq 1, \tag{1.1}$$

subject to the boundary conditions

$$u_0 = a, \quad (u_i)_{i\geq 1} \in l^2_{a_i}(H). \tag{1.2}$$

Here $A : D(A) \subseteq H \to H$ is a (nonlinear) maximal monotone operator (see Barbu [9]) of domain $D(A)$, acting in a real Hilbert space H (endowed with the scalar product $(.,.)$ and the norm $||.||$). The element $a \in H$ is fixed, $(c_i)_{i\geq 1}$, $(\theta_i)_{i\geq 1}$ are sequences of real numbers, $c_i > 0,\ 0 < \theta_i < 1,\ (\forall)\ i \geq 1$ and $(f_i)_{i\geq 1}$ is a given sequence in H. We have denoted by $(a_i)_{i\geq 1}$ the sequence given by

$$a_0 = 1, \quad a_i = 1/\theta_1\theta_2...\theta_i,\ i \geq 1 \tag{1.3}$$

Received April 12. 2003

and by $l_{a_i}^2(H)$ the Hilbert space $l^2(H)$ with the weight sequence a_i. Then the scalar product in $l_{a_i}^2(H)$ is

$$<< (u_i)_{i\geq 1}\ ,(v_i)_{i\geq 1} >>= \sum_{i=1}^{\infty} a_i(u_i\ ,v_i), \tag{1.4}$$

whenever the series in the right hand side converges.

Observe that since $(a_i)_{i\geq 1}$ is a nondecreasing sequence $(1 = a_0 \leq a_1 \leq ... \leq a_i \leq a_{i+1} \leq ...)$, we have $l_{a_i}^2(H) \subseteq l^2(H)$ algebraically and topologically.

In Morosanu [13] the existence and uniqueness of the solution for the equation (1.1) with the boundary conditions

$$u_0 = a,\ \sup_{i\geq 0} ||u_i|| < \infty \tag{1.5}$$

is established in the case $\theta_i \equiv 1$, $f_i \equiv 0$. The asymptotic behavior of the solution is analyzed in Mitidieri and Moroanu [12]. The case of the Banach spaces is studied in Reich et al [14], [15]. In Apreutesei [5] this problem was investigated for $\theta_i \geq 1$. Equation (1.1) with the operator A strongly monotone is studied in Apreutesei [6]. In the present work we deal with the case $0 < \theta_i < 1$.

For $f_i \equiv 0$, problem (1.1), (1.5) is the discrete analog of the boundary value problem

$$\begin{cases} u''(t) \in Au(t), & t \in (0,\infty) \\ u(0) = a, & \sup_{t\geq 0} ||u(t)|| < \infty, \end{cases} \tag{1.6}$$

which is the subject of Barbu [7], [8]. The equation of problem (1.6) with the condition $u'(0) \in \partial j(u(0) - a)$ instead of $u(0) = a$ (where $j : H \to (-\infty, +\infty]$ is a convex, lower-semicontinuous and proper function and ∂j is its subdifferential) was investigated in Brezis [11].

A generalization of equation (1.6) is

$$pu'' + ru' \in Au + f,\ t \in (0,T) \quad (T \leq \infty) \tag{1.7}$$

with different boundary conditions. In both cases ($T < \infty$ and $T = \infty$), p and r are in $W^{1,\infty}(0,T)$. Papers concerned with this equation are due to Véron [16], Aftabizadeh and Pavel [1], [2], Apreutesei [3], [4].

Our goal is to give a discretization to equation (1.7). Thus we have an equation of the form

$$p_i(u_{i+1} - 2u_i + u_{i-1}) + r_i(u_{i+1} - u_i) \in k_i A u_i + g_i,\ i \geq 1,$$

with $p_i \geq C > 0$ and $k_i > 0,\ i \geq 1$. Denoting by $\theta_i = \dfrac{p_i}{p_i + r_i}$, $c_i = \dfrac{k_i}{p_i + r_i}$ and $f_i = \dfrac{g_i}{p_i + r_i}$, we find the equation (1.1).

Therefore (1.1) is of interest because it is the discrete analog of the equation (1.7). The weight sequence $(a_i)_{i\geq 1}$ has a similar role like the weight function $\widetilde{r}$ in Aftabizadeh et al [1], [2].

Recall an existence result for the auxiliary finite difference equation

$$\begin{cases} u_{i+1} - (1+\theta_i)\, u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \ i = \overline{1, N} \\ u_0 = a, \quad u_{N+1} = b, \end{cases} \tag{1.8}$$

in the case when A is also strongly monotone (Apreutsei [6]).

Theorem 1.1. *Let $A : D(A) \subseteq H \to H$ be a maximal monotone and strongly monotone operator in the real Hilbert space H, with $0 \in D(A)$. Consider the sequences $c_i > 0$, $\theta_i \in (0,1)$ and $f_i \in H$, for all $i = \overline{1, N}$. Then, for all $a, b \in H$, problem* (1.8) *admits a unique solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$.*

In the second section we establish with the aid of Theorem 1.1 the maximal monotonicity of some auxiliary operator. Section 3 deals with the existence and uniqueness of the solution to problem $(1.1)-(1.2)$. In the last section we give an example.

2. AN AUXILIARY RESULT

In order to study the existence of the solution for problem (1.1)-(1.2), the main step is to prove that the operator B below (given by $(2.3)-(2.4)$) is maximal monotone in $l^2_{a_i}(H)$, where

$$a_0 = 1, \quad a_i = 1/\theta_1\theta_2 ... \theta_i, \ i \geq 1. \tag{2.1}$$

Denote $\varphi_i = a_{i-1}(u_i - u_{i-1}),\ i \geq 1$. Since $\theta_i = a_{i-1}/a_i$, we have

$$u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} = \frac{1}{a_i}\left[a_i(u_{i+1} - u_i) - a_{i-1}(u_i - u_{i-1})\right],$$

therefore problem $(1.1) - (1.2)$ can be written under the form

$$\begin{cases} \dfrac{1}{a_i}(\varphi_{i+1} - \varphi_i) \in c_i A u_i + f_i, \ i \geq 1 \\ u_0 = a, \quad (u_i)_{i\geq 1} \in l^2_{a_i}(H). \end{cases} \tag{2.2}$$

Consider now the auxiliary operator

$$B\left((u_i)_{i\geq 1}\right) = \left(-u_{i+1} + (1+\theta_i)\, u_i - \theta_i u_{i-1}\right)_{i\geq 1}$$

$$= -\left(\frac{1}{a_i}(\varphi_{i+1} - \varphi_i)\right)_{i\geq 1}, \tag{2.3}$$

$$D(B) = \left\{ (u_i)_{i \geq 1} \in l^2_{a_i}(H),\ u_0 = a \right\}. \tag{2.4}$$

This operator is not maximal monotone in $l^2(H)$, but we have:

Proposition 2.1. *The operator B is maximal monotone in $l^2_{a_i}(H)$.*

Proof. If $(u_i)_i$, $(v_i)_i \in D(B)$, i.e. $u_0 = v_0 = a$, then denoting $\varphi_i = a_{i-1}(u_i - u_{i-1})$ and $\psi_i = a_{i-1}(v_i - v_{i-1})$, we have

$$<< B((u_i)_i) - B((v_i)_i), (u_i - v_i)_i >>$$

$$= -\sum_{i=1}^{\infty} < \varphi_{i+1} - \varphi_i - \psi_{i+1} + \psi_i, u_i - v_i >$$

$$= \sum_{i=1}^{\infty} < \varphi_{i+1} - \varphi_i - \psi_{i+1} + \psi_i, u_{i+1} - u_i - v_{i+1} + v_i >$$

$$- \sum_{i=1}^{\infty} < \varphi_{i+1} - \varphi_i - \psi_{i+1} + \psi_i, u_{i+1} - v_{i+1} > .$$

A simple computation implies

$$<< B((u_i)_i) - B((v_i)_i), (u_i - v_i)_i >>= \sum_{i=1}^{\infty} a_i ||u_{i+1} - u_i - v_{i+1} + v_i||^2$$

$$+ \sum_{i=1}^{\infty} [< \varphi_i - \psi_i, u_i - v_i > - < \varphi_{i+1} - \psi_{i+1}, u_{i+1} - v_{i+1} >]$$

$$= \sum_{i=1}^{\infty} a_i ||u_{i+1} - u_i - v_{i+1} + v_i||^2 + ||u_1 - v_1||^2 \geq 0. \tag{2.5}$$

So B is monotone in $l^2_{a_i}(H)$.

We show now that B is maximal monotone, i.e. for every $(g_i)_i \in l^2_{a_i}(H)$, there exists $(u_i)_i \in D(B)$ such that

$$B((u_i)_i) + (u_i)_i = -(g_i)_i. \tag{2.6}$$

With the aid of (2.3), this can be written as

$$\begin{cases} u_{i+1} - (2 + \theta_i) u_i + \theta_i u_{i-1} = g_i,\ i \geq 1 \\ u_0 = a,\ (u_i)_i \in l^2_{a_i}(H). \end{cases} \tag{2.7}$$

Consider the finite difference scheme

$$\begin{cases} u^N_{i+1} - (2 + \theta_i) u^N_i + \theta_i u^N_{i-1} = g_i,\ i = \overline{1, N} \\ u^N_0 = a, \quad u^N_{N+1} = 0. \end{cases} \tag{2.8}$$

It corresponds to problem (1.8) in the particular case when A is the maximal monotone and strongly monotone operator I (the identity operator). Applying Theorem 1.1, one deduces that (2.8) has a unique solution $\left(u_i^N\right)_{i=\overline{1,N}}$ in $D\left(A\right)^N$.

We first prove that $a_i||u_i^N||^2$ is bounded for all $i=\overline{1,N}$ and all $N \in \mathbb{N}^*$.

To this end, we multiply (2.8) by $a_i u_i^N$ and we sum from $i=1$ to $i=N$. One finds

$$\sum_{i=1}^N a_i||u_i^N||^2 \le \sum_{i=1}^N a_i\left(u_{i+1}^N - u_i^N, u_i^N\right)$$

$$-\sum_{i=1}^N a_i\theta_i\left(u_i^N - u_{i-1}^N, u_i^N\right) - \sum_{i=1}^N a_i\left(g_i, u_i^N\right).$$

Since $a_i\theta_i = a_{i-1}$, this implies that

$$\sum_{i=1}^N a_i||u_i^N||^2 \le \sum_{i=1}^N [a_i\left(u_{i+1}^N - u_i^N, u_i^N\right) - a_{i-1}\left(u_i^N - u_{i-1}^N, u_{i-1}^N\right)]$$

$$-\sum_{i=1}^N a_{i-1}||u_i^N - u_{i-1}^N||^2 - \sum_{i=1}^N a_i\left(g_i, u_i^N\right),$$

hence

$$\sum_{i=1}^N a_i||u_i^N||^2 + \sum_{i=1}^N a_{i-1}||u_i^N - u_{i-1}^N||^2 \le ||a||^2 + ||a||.||u_1^N||$$

$$+\left(\sum_{i=1}^N a_i||g_i||^2\right)^{1/2}\left(\sum_{i=1}^N a_i||u_i^N||^2\right)^{1/2}. \tag{2.9}$$

Taking into account the hypothesis $\left(g_i\right)_i \in l_{a_i}^2\left(H\right)$ and the inequality

$$\sqrt{a_i}||u_i^N|| \le \left(\sum_{i=1}^N a_i||u_i^N||^2\right)^{1/2},$$

where $i=\overline{1,N}$, this leads to

$$\sum_{i=1}^N a_i||u_i^N||^2 \le k_1, \tag{2.10}$$

where $k_1 > 0$ is constant. Therefore

$$||u_i^N|| \le k_2 \tag{2.11}$$

and by (2.9) we have also

$$\sum_{i=1}^N a_{i-1}||u_i^N - u_{i-1}^N||^2 \le k_3. \tag{2.12}$$

Here k_2 and k_3 are positive constants.

Now we prove that u_i^N is convergent as $N \to \infty$ uniformly on every finite set of indexes i to a sequence u_i which verifies problem (2.7). To do this, let $L \in \mathbb{N}$ be given and $M, N \in \mathbb{N}$ arbitrary, such that $L < M < N$. Denote $h_i = u_i^N - u_i^M$, $i = \overline{1, M}$. Subtracting equation (2.8) for N and for M, multiplying the difference by $a_i h_i$ and summing from $i = 1$ to $i = k$, where $k \in \{1, ..., M\}$, we get

$$\sum_{i=1}^{k} a_i ||h_i||^2 = \sum_{i=1}^{k} a_i (h_{i+1} - h_i, h_i) - \sum_{i=1}^{k} a_i \theta_i (h_i - h_{i-1}, h_i)$$

$$= \sum_{i=1}^{k} [a_i (h_{i+1} - h_i, h_i) - a_{i-1}(h_i - h_{i-1}, h_{i-1})] - \sum_{i=1}^{k} a_{i-1} ||h_i - h_{i-1}||^2.$$

Since $h_0 = 0$, this yields

$$\sum_{i=1}^{k} a_i ||h_i||^2 + \sum_{i=1}^{k} a_{i-1} ||h_i - h_{i-1}||^2 = a_k (h_{k+1} - h_k, h_k), \ k = \overline{1, M}. \tag{2.13}$$

But

$$||h_k|| = || \sum_{i=1}^{k} (h_i - h_{i-1}) || \leq \sqrt{k} \left(\sum_{i=1}^{k} a_{i-1} ||h_i - h_{i-1}||^2 \right)^{1/2},$$

therefore (2.13) implies that

$$||h_k||^2 \leq k a_k (h_{k+1} - h_k, h_k), \quad k = \overline{1, M}. \tag{2.14}$$

With the aid of the inequalities $2(h_{k+1} - h_k, h_k) \leq ||h_{k+1}||^2 - ||h_k||^2$ and $a_k \leq a_{k+1}$, we deduce

$$\sum_{k=L}^{M} \frac{1}{k} ||h_k||^2 \leq \frac{a_{M+1}}{2} ||h_{M+1}||^2. \tag{2.15}$$

By (2.13) we also deduce that the sequence $(h_k)_{k=\overline{1,M}}$ is nondecreasing, so (2.15) gives us

$$||h_i||^2 \left(\sum_{k=L}^{M} \frac{1}{k} \right) \leq \frac{a_{M+1}}{2} ||h_{M+1}||^2, \ i = \overline{1, L}, \tag{2.16}$$

which implies

$$||u_i^N - u_i^M||^2 \leq \frac{a_{M+1} ||u_{M+1}^N||^2}{2 \left(\sum_{k=L}^{M} \frac{1}{k} \right)}. \tag{2.17}$$

Using (2.11) we deduce the existence of the limit $\lim_{N \to \infty} u_i^N = u_i$, uniformly with respect to i on finite sets of natural numbers. Letting $N \to \infty$ in (2.8), it follows that $(u_i)_{i \geq 1}$ verifies the problem (2.7). Consequently, B is maximal monotone in $l_{a_i}^2(H)$.

3. THE MAIN RESULT

One states now an existence result for problem (1.1), (1.2). We work under the following hypotheses:

1) A is a maximal monotone operator in the Hilbert space H, with the domain $D(A)$, $0 \in D(A)$;

2) $(c_i)_{i\geq 1}$ and $(\theta_i)_{i\geq 1}$ are given sequences, $0 < c \leq c_i$, $\theta_i > 0$, $(\forall)$ $i \geq 1$;

3) $k = \sup\limits_{i\geq 1} \theta_i \in (0,1)$;

4) $(f_i)_{i\geq 1} \in l^2_{a_i}(H)$, where a_i is the sequence given by (2.1).

Theorem 3.1. *Under hypotheses 1)-4), if* $a \in H$, *then problem* (1.1), (1.2) *has a unique solution* $(u_i)_{i\geq 1} \in l^2_{a_i}(H)$, *with* $u_i \in D(A)$, $(\forall)$ $i \geq 1$.

Proof. Let $\mathcal{A}$ be the operator defined by $(\mathcal{A}u)_{i\geq 1} = (c_i v_i)_{i\geq 1}$, where $u = (u_i)_{i\geq 1}$ and $v_i \in Au_i$. Denote by A_λ and $\mathcal{A}_\lambda u$ the Yosida approximation of A and the sequence $(c_i A_\lambda u_i)_{i\geq 1}$, respectively. Since B is maximal monotone in $l^2_{a_i}(H)$ (see Proposition 2.1) and $\mathcal{A}_\lambda$ is maximal monotone and everywhere defined in $l^2_{a_i}(H)$, then $B + \mathcal{A}_\lambda$ is maximal monotone, that is

$$R(B + \mathcal{A}_\lambda + \omega I) = l^2_{a_i}(H), \quad (\forall)\, \lambda,\ \omega > 0, \tag{3.1}$$

i.e. $(\forall)$ λ, $\omega > 0$ and $(f_i)_{i\geq 1} \in l^2_{a_i}(H)$, there is a sequence $\left(u_i^{\lambda\omega}\right)$ such that

$$\begin{cases} u_{i+1}^{\lambda\omega} - (1+\theta_i)\, u_i^{\lambda\omega} + \theta_i u_{i-1}^{\lambda\omega} = c_i A_\lambda u_i^{\lambda\omega} + \omega u_i^{\lambda\omega} + f_i, \quad i \geq 1 \\ u_0^{\lambda\omega} = a, \qquad \left(u_i^{\lambda\omega}\right)_{i\geq 1} \in l^2_{a_i}(H). \end{cases} \tag{3.2}$$

One first proves the boundedness of $\left(u_i^{\lambda\omega}\right)$, $\left(A_\lambda u_i^{\lambda\omega}\right)$ and of $\left(u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\right)$ in $l^2_{a_i}(H)$ with respect to ω and λ.

A multiplication of (3.1) by $a_i u_i^{\lambda\omega}$, followed by a summation from $i = 1$ to $i = \infty$, leads to a relation which is analogous to (2.9) :

$$\omega \sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega}||^2 + \sum_{i=1}^{\infty} a_{i-1} ||u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}||^2 \leq ||a||^2 + ||a||.||u_1^{\lambda\omega}||$$

$$+ \left(\sum_{i=1}^{\infty} a_i ||f_i||^2\right)^{1/2} \left(\sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega}||^2\right)^{1/2}. \tag{3.3}$$

Similarly with the proof of Proposition 2.1, using the assumption 4), we have

$$\omega \sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega}||^2 + \sum_{i=1}^{\infty} a_{i-1} ||u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}||^2 \leq K_1 \left(\sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega}||^2\right)^{1/2} + K_2, \tag{3.4}$$

where K_1 and K_2 are independent of λ and ω.

Now we sum up the inequalities $a_i\left(u_i^{\lambda\omega}, u_{i+1}^{\lambda\omega} - u_i^{\lambda\omega}\right) \le \frac{a_i}{2}\left(||u_{i+1}^{\lambda\omega}||^2 - ||u_i^{\lambda\omega}||^2\right)$ for $i = 1$ to $i = \infty$ to find

$$\frac{1}{2}\sum_{i=1}^{\infty}(a_i - a_{i-1})\,||u_i^{\lambda\omega}||^2 \le \sum_{i=1}^{\infty} a_i\left(u_i^{\lambda\omega}, u_i^{\lambda\omega} - u_{i+1}^{\lambda\omega}\right). \tag{3.5}$$

With the aid of hypotheses 3) and of the equality $a_i\theta_i = a_{i-1}$, we arrive at

$$\frac{1-k}{2}\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega}||^2 \le \left(\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega}||^2\right)^{1/2}\left(\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega} - u_{i+1}^{\lambda\omega}||^2\right)^{1/2},$$

so

$$\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega}||^2 \le K_3\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega} - u_{i+1}^{\lambda\omega}||^2. \tag{3.6}$$

Combining (3.6) with (3.4) , we get

$$\sum_{i=1}^{\infty} a_{i-1}||u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}||^2 \le K_4 \tag{3.7}$$

and by (3.6) we have

$$\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega}||^2 \le K_5. \tag{3.8}$$

This implies that $u_i^{\lambda\omega}$ and $A_\lambda u_i^{\lambda\omega}$ are bounded in H with respect to λ and ω.

Now we can show that $\left(u_i^{\lambda\omega}\right)$ and $\left(u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\right)$ are strongly convergent in $l_{a_i}^2(H)$ as $\lambda \searrow 0$. To this purpose, we subtract (3.2) for λ and for μ, multiply by $a_i\left(u_i^{\lambda\omega} - u_i^{\mu\omega}\right)$ and sum for $i = 1$ to $i = \infty$. Thus we estimate

$$\sum_{i=1}^{\infty} a_i(u_{i+1}^{\lambda\omega} - u_{i+1}^{\mu\omega} - u_i^{\lambda\omega} + u_i^{\mu\omega}, u_i^{\lambda\omega} - u_i^{\mu\omega})$$

$$-\sum_{i=1}^{\infty} a_i\theta_i(u_i^{\lambda\omega} - u_i^{\mu\omega} - u_{i-1}^{\lambda\omega} + u_{i-1}^{\mu\omega}, u_i^{\lambda\omega} - u_i^{\mu\omega})$$

$$= \sum_{i=1}^{\infty} a_ic_i(A_\lambda u_i^{\lambda\omega} - A_\mu u_i^{\mu\omega}, u_i^{\lambda\omega} - u_i^{\mu\omega}) + \omega\sum_{i=1}^{\infty} a_i||u_i^{\lambda\omega} - u_i^{\mu\omega}||^2. \tag{3.9}$$

A simple computation and the equality

$$J_\lambda u_i^{\lambda\omega} + \lambda A_\lambda u_i^{\lambda\omega} = u_i^{\lambda\omega} \tag{3.10}$$

imply that

$$\sum_{i=1}^{\infty} a_{i-1}||u_i^{\lambda\omega} - u_i^{\mu\omega} - u_{i-1}^{\lambda\omega} + u_{i-1}^{\mu\omega}||^2 = \sum_{i=1}^{\infty} a_ic_i(A_\lambda u_i^{\lambda\omega} - A_\mu u_i^{\mu\omega}, J_\lambda u_i^{\lambda\omega} - J_\mu u_i^{\mu\omega})$$

$$+\sum_{i=1}^{\infty} a_i c_i (A_\lambda u_i^{\lambda\omega} - A_\mu u_i^{\mu\omega}, \lambda A_\lambda u_i^{\lambda\omega} - \mu A_\mu u_i^{\mu\omega}) + \omega \sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega} - u_i^{\mu\omega}||^2, \tag{3.11}$$

therefore

$$\omega \sum_{i=1}^{\infty} a_i ||u_i^{\lambda\omega} - u_i^{\mu\omega}||^2 + \sum_{i=1}^{\infty} a_{i-1} ||u_i^{\lambda\omega} - u_i^{\mu\omega} - u_{i-1}^{\lambda\omega} + u_{i-1}^{\mu\omega}||^2$$

$$\leq K_6 (\lambda + \mu), \tag{3.12}$$

with $K_6 > 0$ independent of $\lambda,\ \mu$ and ω. We conclude that $\left(u_i^{\lambda\omega}\right)_\lambda$ and $\left(u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\right)_\lambda$ are convergent sequences, both in $l_{a_i}^2(H)$ and in H, as $\lambda \searrow 0$. Say $u_i^{\lambda\omega} \to u_i^\omega$. Observe that (3.10) implies the convergence $J_\lambda u_i^{\lambda\omega} \to u_i^\omega$, as $\lambda \searrow 0$. Let $A_\lambda u_i^{\lambda\omega} \rightharpoonup v_i^\omega$ as $\lambda \searrow 0$ in $l_{a_i}^2(H)$. We have denoted by " $\to$ " and " $\rightharpoonup$ " the strong and the weak convergence in every Hilbert space we use. The inclusion $A_\lambda u_i^{\lambda\omega} \in A\left(J_\lambda u_i^{\lambda\omega}\right)$ and the maximal monotonicity of A imply that $u_i^\omega \in D(A)$ and $v_i^\omega \in Au_i^\omega$.

Now we may pass to the limit in (3.2) as $\lambda \searrow 0$ and obtain that u_i^ω verifies the problem

$$\begin{cases} u_{i+1}^\omega - (1+\theta_i)\, u_i^\omega + \theta_i u_{i-1}^\omega \in c_i A u_i^\omega + \omega u_i^\omega + f_i, \quad i \geq 1 \\ u_0^\omega = a, \qquad (u_i^\omega)_{i\geq 1} \in l_{a_i}^2(H). \end{cases} \tag{3.13}$$

In view of (3.8) we deduce

$$\sum_{i=1}^{\infty} a_i ||u_i^\omega||^2 \leq K_5. \tag{3.14}$$

One establishes that $\left(u_i^\omega - u_{i-1}^\omega\right)_\omega$ is strongly convergent in $l_{a_i}^2(H)$ with respect to ω. Indeed, subtracting (3.13) for ω and ν, multiplying by $a_i\left(u_i^\omega - u_i^\nu\right)$ and summing for $i = 1$ to $i = \infty$, we arrive as in (3.12) (with the aid of (3.14) and of the monotonicity of A) to

$$\sum_{i=1}^{\infty} a_{i-1} ||u_i^\omega - u_{i-1}^\omega - u_i^\nu + u_{i-1}^\nu||^2 \leq K_5 (\omega + \nu). \tag{3.15}$$

It follows that $\left(u_i^\omega - u_{i-1}^\omega\right)_\omega$ is strongly convergent in $l_{a_i}^2(H)$ and consequently in H. Since A is maximal monotone and u_i^ω is weakly convergent (denote $u_i^\omega \rightharpoonup u_i$), we may pass to the limit as $\omega \searrow 0$ in (3.13) written under the form

$$u_{i+1}^\omega - u_i^\omega - \theta_i \left(u_i^\omega - u_{i-1}^\omega\right) - \omega u_i^\omega - f_i \in c_i A u_i^\omega \tag{3.16}$$

and we conclude that u_i is a solution of (1.1), (1.2). The uniqueness can be easily proved. □

4. AN EXAMPLE

We now give some applications to partial differential equations. We work in the usual Sobolev space $H = H^{-1}(\Omega)$. Consider $\beta : \mathbb{R} \to \mathbb{R}$ a maximal monotone operator in $\mathbb{R}$, such that the range of β is $R(\beta) = \mathbb{R}$ or equivalently

$$\lim_{|r|\to\infty} j(r)/ \mid r \mid= \infty, \tag{4.1}$$

where j is such that $\beta = \partial j$ (Brezis [10]). We are concerned with the boundary value problem

$$\begin{cases} u_{i+1}(x) - (1+\theta_i)u_i(x) + \theta_i u_{i-1}(x)\,, \\ \in -\Delta\beta(u_i(x)) + f_i\,(x)\,,\ x \in \Omega\,, \\ \beta(u_i(x)) \ni 0,\ x \in \partial\Omega\,, \\ u_0(x) = a(x),\ x \in \Omega\,, \end{cases} \tag{4.2}$$

where c_i and θ_i satisfy the assumptions

$$0 < c \le c_i, \quad \theta_i > 0, \quad k = \sup_{i\ge 1} \theta_i \in (0,1)\,. \tag{4.3}$$

Let $A : H^{-1}(\Omega) \to H^{-1}(\Omega)$ be the nonlinear diffusion operator defined by

$$Au = -\Delta\beta(u), \tag{4.4}$$

$$D(A) = \{u \in H^{-1}(\Omega) \cap L^1(\Omega), (\exists) v \in H_0^1(\Omega),$$

$$v(x) \in \beta(u(x))\ a.e.\ x \in \Omega\}. \tag{4.5}$$

It is known that A is the subdifferential of the convex, lower semicontinuous, proper function $\varphi : H^{-1}(\Omega) \to (-\infty, +\infty]$,

$$\varphi(u) = \begin{cases} \int\limits_\Omega j(u)dx, & \text{if } \ u \in L^1(\Omega),\ j(u) \in L^1(\Omega) \\ +\infty, & \text{otherwise} \end{cases}$$

(see Barbu [9] or Brezis [10]). As a consequence of Theorem 3.1, we can state the following existence result.

Corollary 4.1. *If* $a \in H^{-1}(\Omega)\,,\ (c_i)_{i\ge1}\,,\ (\theta_i)_{i\ge1}$ *verify* (4.3), β *satisfies the above conditions and* $(f_i)_{i\ge1} \in l^2_{a_i}(H^{-1}(\Omega))$, *then problem* (4.2) *has a unique solution* $u_i \in l^2_{a_i}(H^{-1}(\Omega))\,,\ u_i \in H^{-1}(\Omega) \cap L^1(\Omega)\,,\ (\forall)\, i \ge 1.$

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