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EULER-POINCARÉ FORMALISM OF COUPLED KDV TYPE SYSTEMS AND DIFFEOMORPHISM GROUP ON S^1

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ABSTRACT: In this paper we show that almost all the coupled KdV equations follow from the geodesic flows of L^2 metric on the semidirect product space $\widehat{Diff^s(S^1) \bigodot C^\infty(S^1)}$, where $Diff^s(S^1)$ is the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. We also study the geodesic flow of a H^1 metric.

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1. INTRODUCTION

It is known that the periodic Korteweg-de Vries (KdV) equation can be interpreted as geodesic flow of the right invariant metric on the Bott-Virasoro group, which at the identity is given by the L^2-inner product, Ovsienko and Khesin [24], Segal [27], Witten [30].

Recently Misiolek [22] and others (Kupershmidt [18], Shkoller [26]) showed that an analogous correspondence can be established for the Camassa-Holm equation, Camassa and Holm [7]. It gives rise to a geodesic flow of a certain right invariant Sobolev metric H^1 on the Bott-Virasoro group. The Camassa-Holm equation is sometimes also called the Fokas-Fuchssteiner-Camassa-Holm equation, because it was obtained first by Fuchssteiner and Fokas [11]. In fact, the Hunter-Saxton (or Harry-Dym) system also follows from geodesic flow on diffeomorphism groups, Guha [12].

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Thus we see the KdV and the Camassa-Holm equations arise in a unified geometric construction, both are integrable systems which describe geodesic flows on the Bott-Virasoro group. Earlier it was known that both the KdV and the Camassa-Holm are obtained from different regularisations of the Euler equation for a one dimensional compressible fluid. The Euler equation, of course, describes geodesic motion on the group of orientation preserving diffeomorphisms of the circle $Diff(S^1)$ with respect to L^2 metric, Ebin and Marsden [9].

Following Ebin and Marsden [9], we enlarge $Diff(S^1)$ to a Hilbert manifold $Diff^s(S^1)$, the diffeomorphism of Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local charts, one can check whether the derivation of order $\leq s$ are square integrable.

In this paper we will consider the semi-direct product space $Diff^s(S^1) \bigodot C^\infty(S^1)$. The Lie algebra of $Diff^s(S^1) \bigodot C^\infty(S^1)$ is known in physical literature (Arbarello et al [4], Guha [13], Harnad and Kupershmidt [15], Marsden and Ratiu [21]). It has a three dimensional extension (explained in the next section)

$$Vect^s(S^1) \bigodot C^\infty(S^1) \oplus \mathbf{R}^3.$$

Then a typical element of this algebra would be

$$(f\frac{d}{dx}, u(x), \alpha)\,, \quad \text{where } f\frac{d}{dx} \in Vect(S^1), \quad u(x) \in C^\infty(S^1), \quad \alpha \in \mathbf{R}^3.$$

It was shown by Ovsienko and Rogers [25] that the cocycles define the universal central extension of the Lie algebra of $Vect^s(S^1) \bigodot C^\infty(S^1)$. This means

$$H^2(Vect^s(S^1) \bigodot C^\infty(S^1)) = \mathbf{R}^3.$$

The $Diff^s(\widehat{S^1) \bigodot} C^\infty(S^1)$ is the non-trivial extension of $Diff^s(S^1) \bigodot C^\infty(S^1)$.

In this paper we study a geodesic flows on the $Diff^s(\widehat{S^1) \bigodot} C^\infty(S^1)$, which at the identity is given by the L^2 inner product. These are all completely integrable coupled nonlinear third order partial differential equations.

1.1. BRIEF HISTORY AND FORMULATION OF COUPLED KDV TYPE SYSTEMS

Since 80's, the coupled KdV systems are considered to be important mathematical models. In 1981, Fuchssteiner made a detailed study of four coupled KdV equation and formulated the bihamiltonian structure of them. One of them turned out to be a complex version of the KdV. Immediately after that Hirota and Satsuma[1] introduced a coupled KdV equations

$$p_t + p_{xxx} + pp_x - qq_x = 0, \qquad q_t + q_{xxx} + pq_x + p_x q = 0.$$

The theoretists have studied the bihamiltonian and Lax pair of these class of systems. So it became necessary to have a Lie algebraic framework to study such systems. Given an isospectral flow of an appropriate eigenvalue problem, it is shown by Fokas and Anderson [10] that the strong symmetry constructed via the spectral gradient approach is a hereditary operator provided the spectral gradient functions are dense. A number of applications are given, among them several matrix systems, the Boussinesq equation as well as its modified form, and others. The bi-Hamiltonian structure of these equations is shown.

In the late eighties, Antonowicz and Fordy [3] investigated second order energy dependent spectral parameter and found their isospectral flows have multi-Hamiltonian structure. This approach gave a very simple and elegant construction of the associated Hamiltonian operators. This method can be applied to Lazutkin and Penkratova [19] nonstandard Lax operators and also to super Lax equations. Let us apply the Antonowicz-Fordy scheme to Kupershmidt's nonstandard Lax operators

$$\mathbf{L}\phi = (\epsilon\partial^2 - r\partial + q)\phi = 0, \tag{1}$$

where ϵ, r and q are now polynomials in λ and construct the associated Hamiltonian operators. This follows directly from the compatibility condition of (1) and

$$\phi_t = \mathbf{P}\phi \equiv \frac{1}{2}(P\partial + Q)\phi,$$

where P and Q are functions of u_i and the spectral parameter λ.

Taking some special values of ϵ_i, they derive a tri-Hamiltonian dispersive water waves hierarchy. The first nontrivial member of this hierarchy is

$$u_{0t} = \frac{1}{4}u_{1xxx} + \frac{1}{2}u_1 u_{0x} + u_0 u_{1x} \qquad u_{1t} = u_{0x} + \frac{3}{2}u_1 u_{1x}.$$

An invertible change of variables

$$q = u_0 + \frac{1}{4}{u_1}^2 - \frac{1}{2}u_{1x}, \qquad r = u_1.$$

transforms the above equation into a standard dispersive water waves equation.

$$q_t = \frac{1}{2}(q_x + 2qr)_x, \qquad r_t = \frac{1}{2}(r_x + 2q + r^2)_x.$$

Recently, Alber et al showed that in case of certain potentials, a limiting procedure can be applied to generic solution, which results in solutions with peaks (see Alber et al [2] and references therein).

[1]The history and detail structure of these equations can be found in Ablowitz and Clarkson [1] and Wang [29].

1.2. MOTIVATION AND RESULT

In this paper we will demonstrate that almost all the coupled KdV type systems derived from the Antonowicz-Fordy scheme can be obtained from the geodesic flows on the $Diff^s(\widehat{S^1)\bigodot} C^\infty(S^1)$.

We begin with a prototypical example, the dispersive water waves equation

$$w_{0t} = w_{1xxx} + 3(w_1 w_0)_x + w_0 w_{1x} \tag{2}$$

$$w_{1t} = w_{0x} + 4w_1 w_{1x}. \tag{3}$$

We show that this is a geodesic flow on the extension of the Bott-Virasoro group, and this flow

$$\begin{pmatrix} w_{0t} \\ w_{1t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}D^3 + Dw_0 + w_0 D & w_1 D \\ Dw_1 & D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{w_0} \\ \frac{\delta H}{w_1} \end{pmatrix} \tag{4}$$

is connected to a hyperplane in the coadjoint orbit of the Bott-Virasoro group.

It is possible to define a Miura map for this system

$$u = w_1, \qquad v = w_0 + \frac{3}{4}w_1^2. \tag{5}$$

The Miura map transformed this equation to

$$v_t = u_{xxx} + uv_x + 2(uv)_x - \frac{3}{2}u^2 u_x \tag{6}$$

$$u_t = v_x. \tag{7}$$

It is easy to see that the Hamiltonian structure also transformed to

$$\mathcal{O} = \begin{pmatrix} \frac{1}{2}D^3 + D(v-2u^2) + (v-2u^2)D + 2v_x & uD \\ Du & D \end{pmatrix}$$

with the Hamiltonian functionals satisfy

$$\frac{\delta H_1}{\delta v} = 2u, \qquad \frac{\delta H_1}{\delta u} = v - 2u^2. \tag{8}$$

There are several equations arose from their scheme can be manifested as the geodesic flows on $Diff^s(\widehat{S^1)\bigodot} C^\infty(S^1)$. Hence, we unify the geometry these systems. In our earlier paper Guha [13], we already showed that the Ito and various dispersive water waves equations follow from the geodesic flows. We also formulated Guha [14] the Kaup-Newell equation as a Euler-Poincaŕe flow on the space of first order differential operators on S^1.

This paper is organized as follows: In Section 2, we discuss the geodesic flows on $Diff^s(\widehat{S^1)\bigodot} C^\infty(S^1)$ with respect to L^2 metric. Then we study a geodesic flow of the right invariant inner metric on the $Diff^s(\widehat{S^1)\bigodot} C^\infty(S^1)$, which at the identity is given by the H^1 inner product.

2. COUPLED KDV TYPE EQUATIONS AND L^2 METRIC ON BOTT-VIRASORO GROUP

Let $Diff^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. It is known that the group $Diff^s(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{id}Diff^s(S^1) = Vect^s(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{D}iff^s(S^1)$ and the Virasoro algebra Vir respectively, Guha [14], Harnad and Kupershmidt [15].

The Lie algebra $Vect^s(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}. \tag{9}$$

One parameter family of $Vect^s(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$L^{(\mu)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) - \mu f'(x)a(x), \tag{10}$$

where

$$L^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} - \mu f'(x)$$

is the derivative with respect to the vector field $f(x)\frac{d}{dx}$.

The Lie algebra of $Diff^s(S^1) \bigodot C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathcal{G} = Vect^s(S^1) \bigodot C^\infty(S^1).$$

An element of $\mathcal{G}$ is a pair $(f(x)\frac{d}{dx}, a(x))$, where $f(x)\frac{d}{dx} \in Vect^s(S^1)$ and $a(x) \in C^\infty(S^1)$.

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\omega_1((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f'(x)g''(x)dx\,, \tag{11}$$

$$\omega_2((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f''(x)b(x) - g''a(x))dx\,, \tag{12}$$

$$\omega_3((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = 2\int_{S^1} a(x)b'(x)dx. \tag{13}$$

The first cocycle ω_1 is the well known Gelfand-Fuchs cocycle. The Virasoro algebra is the unique non-trivial central extension of $Vect(S^1)$ via this ω_1 cocycle. Hence we define the Virasoro algebra

$$Vir = Vect^s(S^1) \oplus \mathbf{R}.$$

The space $C^\infty(S^1) \oplus \mathbf{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part*, and the pairing between this space and the Virasoro algebra is given by:

$$< (u(x), a), (f(x)\frac{d}{dx}, \alpha) >= \int_{S^1} u(x)f(x)dx + a\alpha.$$

Similarly we consider an extension of $\mathcal{G}$. This extended algebra is given by

$$\hat{\mathcal{G}} = Vect^s(S^1) \bigodot C^\infty(S^1) \oplus \mathbf{R}^3. \tag{14}$$

The Lie algebra $\hat{\mathcal{G}}$ has been considered in various places (Arbarello et al [4], Guha [13], Harnad and Kupershmidt [15], Marcel et al [20]). It was shown in Ovsienko and Rogers [25] that the cocycles define the universal central extension the Lie algebra $Vect^s(S^1) \bigodot C^\infty(S^1)$. This means $H^2(Vect(S^1) \bigodot C^\infty(S^1)) = \mathbf{R}^3$.

Definition 1. The commutation relation in $\hat{\mathcal{G}}$ is given by

$$[(f\frac{d}{dx}, a, \alpha), (g\frac{d}{dx}, b, \beta)] := ((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega)\,, \tag{15}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 2. The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: Consider

$$\hat{\mathcal{G}}^*_{reg} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3.$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $< \cdot, \cdot >$: $\hat{\mathcal{G}}^*_{reg} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$< \hat{u}, \hat{f} >= \int_{S^1} f(x)u(x)dx + \int_{S^1} a(x)v(x)dx + \alpha \cdot \gamma, \tag{16}$$

where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \alpha)$.

Extend this to a right invariant metric on the semi-direct product space $Diff^s(\widehat{S^1)\bigodot C^\infty}(S^1)$ by setting

$$< \hat{u}, \hat{f} >_{\hat{\xi}} = < d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{u}, d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{f} >_{L^2} \tag{17}$$

for any $\hat{\xi} \in \hat{\mathcal{G}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\widehat{\mathcal{G}}$, where

$$R_{\hat{\xi}} \; : \; \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$$

is the right translation by $\hat{\xi}$.

We shall show that the Antonowicz-Fordy equation is precisely the Euler-Poincaŕe equation on the dual space of $\hat{\mathcal{G}}$ associated with the L^2 inner product.

Given any three elements

$$\hat{f} = (f\frac{d}{dx}, a, \alpha), \quad \hat{g} = (g\frac{d}{dx}, b, \beta) \;\; \hat{u} = (u\frac{d}{dx}, v, c)$$

in $\hat{\mathcal{G}}$.

Lemma 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1 f''' + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \\ 0 \end{pmatrix}.$$

Proof. This follows from

$$< ad^*_{\hat{f}}\hat{u}, \hat{g} >_{L^2} = < \hat{u}, [\hat{f}, \hat{g}] >_{L^2}$$

$$=< (u(x)\frac{d}{dx}, v(x), c), [(fg' - f'g)\frac{d}{dx}, fb' - ga', \omega) >_{L^2}$$

$$= -\int_{S^1} (fg' - f'g)u(x)dx - \int_{S^1} (fb' - ga')vdx - c_1 \int_{S^1} f'(x)g''(x)dx$$
$$-c_2 \int_{S^1} (f''(x)b(x) - g''(x)a(x))dx - 2c_3 \int_{S^1} a(x)b'(x)dx.$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$R.H.S. =< (2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) - c_1 f'''(x)$$
$$+c_2 a''(x), f'(x)v(x) + f(x)v'(x) - c_2 f'' b(x) + 2c_3 a'(x), 0) \qquad \square$$

Let us choose the *hyperplane* in the dual space. The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1, c_2 = c_3 = \frac{1}{2}$.

Corollary 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) + f''' \\ f'v(x) + f(x)v'(x) \\ 0 \end{pmatrix}.$$

The Euler-Poincaŕe equation is the Hamiltonian flow on the coadjoint orbit in $\hat{\mathcal{G}}^*$, generated by the Hamiltonian

$$H(\hat{u}) \equiv H(u,v) =< (u(x),v(x)),(u(x),v(x)) >, \tag{18}$$

given by

$$\frac{d\hat{u}}{dt} = -ad^*_{\hat{u}(t)}u(t). \tag{19}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^*, Arnold [5], Cendra et al [8], Marsden and Ratiu [21].

Proposition 1. *Let $G \bigodot V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $< \cdot,\cdot >$ which is right translation. A curve $t \to c(t)$ in $G \bigodot V$ is a geodesic of this metric if and only if $u(t) = d_{c(t)}R_{c(t)^{-1}}\dot{c}(t)$ satisfies the Euler-Poicaŕe equation.*

If we assume

$$\frac{\delta H}{\delta w_0} = 2w_1, \qquad \frac{\delta H}{\delta w_1} = w_0,$$

then we prove the following theorem:

Theorem 1. *Let $t \mapsto \hat{c}$ be a curve in the $Diff^s(\widehat{S^1) \bigodot} C^\infty(S^1)$. Let $\hat{c} = (e,e,0)$ be the initial point, directing to the vector $\hat{c}(0) = (u(x)\frac{d}{dx}, v(x), \gamma_0)$, where $\gamma_0 \in \mathbf{R}^3$. Then $\hat{c}(t)$ is a geodesic of the (A) L^2 metric if and only if $(u(x,t)\frac{d}{dx}, v(x,t), \gamma)$ satisfies the Ito equation*

$$w_{0t} = w_{1xxx} + 3(w_1w_0)_x + w_0w_{1x}$$

$$w_{1t} = w_{0x} + 4w_1w_{1x}$$

$$\gamma_t = 0.$$

2.1. HAMILTONIAN STRUCTURES OF OTHER COUPLED SYSTEMS

In this section we will discuss the Euler-Poincaŕe framework of other coupled systems.

The Hamiltonian structure of the well known Ito system

$$u_t + u_{xxx} + 6uu_x + 2vv_x = 0, \qquad v_t + 2(uv)_x = 0$$

is given by

$$\mathcal{O}_{Ito} = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{pmatrix},$$

and it is connected to a hyperplane $c_1 = -1, c_2 = c_3 = 0$.

When we restrict to a hyperplane $c_1 = 0, c_2 = 1, c_3 = 0$, we obtain the modified dispersive water wave equation

$$u_t + 6uu_x + 2vv_x + v_{xx} = 0, \qquad v_t + 2(vu)_x - u_{xx} = 0,$$

and the Hamiltonian structure is

$$\mathcal{O}_2 = \begin{pmatrix} 4uD + 2u_x & 2vD \\ 2v_x + 2vD + D^2 & 0 \end{pmatrix}.$$

2.1.1. THE EULER-POINCAŔE FRAMEWORK OF HIROTA-SATSUMA EQUATION

By Lazutkin and Penkratova [19], this dual space can be identified with the space of Hill's operator or the space of projective connections

$$\Delta = \frac{d^2}{dx^2} + u(x), \tag{20}$$

where u is a periodic potential: $u(x+2\pi) = u(x) \in C^\infty(\mathbb{R})$. The Hill's operator maps

$$\Delta : \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{-\frac{3}{2}}. \tag{21}$$

The action of $Vect(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta] := \mathcal{L}^{-\frac{3}{2}}_{f(x)\frac{d}{dx}} \circ \Delta - \Delta \circ \mathcal{L}^{\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{22}$$

Certainly, the above equation yields the coadjoint action of $Vect(S^1)$. Hence we can extract the the operator $ad^*_{u(t)}$ from this information. The Euler-Poincaŕe equation is the Hamiltonian flow on the coadjoint orbit on the space of Hill's operator, generated by the Hamiltonian given by

$$\frac{du}{dt} = -ad^*_{u(t)}u(t). \tag{23}$$

The operator $(\frac{1}{2}\partial_x^3+2u\partial_x+u_x)$ is called the implectic operator of the KdV equation.

In the Hirota-Satsuma case the unknown variable is

$$u(x,t) = p(x,t) + iq(x,t),$$

where $p(x,t)$ is the real part of $u(x,t)$, $q(x,t)$ is the imaginary part of $u(x,t)$. Then the Hirota-Satsuma type equation follows from this scheme.

2.1.2. EULER-POINCAŔE FRAMEWORK OF NUTKU-OGUZ SYSTEM

There are several systems closely related to this system. A few years ago Nutku and Oguz [23] proposed a new class of coupled KdV type system

$$q_t = q_{xxx} + 2aqq_x + pp_x + (qp)_x, \qquad p_t = p_{xxx} + 2bpp_x + qq_x + (qp)_x,$$

where $a + b = 1$. If we change the variables to $u = q + p$ and $v = q + p$, then the above set is boiled down to

$$u_t = u_{xxx} + uu_x + vv_x, \qquad v_t = v_{xxx} + \lambda vv_x + (uv)_x,$$

where λ is a parameter which is assumed as real. If $\lambda = 0$, the system is a completely coupled KdV system discussed by Fuchssteiner. This system can be easily incorporated in our programme.

The symmetrically coupled system

$$u_t = u_{xxx} + v_{xxx} + 6uu_x + 4uv_x + 2u_xv, \qquad v_t = u_{xxx} + v_{xxx} + 6vv_x + 4vu_x + 2v_xu$$

is also a geodesic flow on the space of the Bott-Virasoro group. This can be easily checked if one replaces $\lambda = u + v$ and $H = \frac{1}{2}(u+v)^2 = \frac{1}{2}\lambda^2$.

2.1.3. EULER-POINCAŔE PROGRAMME FOR HÉNON-HEILES SYSTEM

The general Hénon-Heiles system can be related to the stationary flow of

$$u_t = (\partial^3 + 8au\partial + 4au_x)\frac{\delta H}{\delta u},$$

where $H = (-\frac{1}{3}bu^3 - \frac{1}{2}u_x^2)$. For this stationary flow, the gradient of the above Hamiltonian is in the kernel of the third order "KdV Hamiltonian structure". This can be transformed (Baker et al [6]) into more recognised form

$$y_{xx} + 2auy = 2ky^{-3}.$$

2.1.4. THE EULER-POINCAŔE FORMALISM OF KAUP-BOUSSINESQ SYSTEM

The Kaup-Boussinesq equation

$$u_t + uu_x + h_x = 0, \qquad h_t + (hu)_x - \frac{1}{4}u_{xxx} = 0$$

is another system apart from KdV which is often model for the shallow water undular bose. The KB system has a natural two wave structure, which enables one to capture the effects of interaction of unmodular bores or rarefaction waves araising in the decay of an jump discontinuity. This equation is also related to a hyperplane $c_1 = \frac{1}{4}$ and $c_3 = \frac{1}{2}$ in the coadjoint orbit of the extension of the Bott-Virasoro group. Its Hamiltonian structure is

$$\mathcal{O}_2 = \begin{pmatrix} 2hD + h_x + \frac{1}{4}D^3 & uD \\ Du & D \end{pmatrix},$$

with $\frac{\delta H}{\delta h} = u$ and $\frac{\delta H}{\delta u} = h$.

2.1.5. EULER-POINCAŔE FORMALISM OF BROER-KAUP SYSTEM

The Broer-Kaup system

$$u_t = -u_{xx} + 2(uv)_x + uu_x, \qquad v_t = v_{xx} + 2vv_x - 2u_x$$

is a geodesic flow associated to the hyperplane $c_2 = -1$ and $c_3 = -1$. Hence the Hamiltonian structure is

$$\mathcal{O}_{BK} = \begin{pmatrix} uD_x + D_x u & -D_x^2 + vD_x \\ D_x^2 + D_x v & -2D_x \end{pmatrix}, \qquad \text{with } H = \int_{S^1} uv.$$

2.1.6. EULER-POINCAŔE FLOW AND WADATI-KONNO-ICHIKAWA SYSTEM

In late seventies, Wadati et al [28] proposed two highly nonlinear equations

$$u_t = D_x^2(\frac{u}{\sqrt{1+uv}}), \qquad v_t = -D_x^2(\frac{v}{\sqrt{1+uv}}).$$

The Hamiltonian structure of this pair is associated to the hyperplane $c_2 = \kappa$, where κ is very large. Then the Hamiltonian structure becomes

$$\begin{pmatrix} uD_x + D_x u & -\kappa D_x^2 + vD_x \\ \kappa D_x^2 + D_x v & 0 \end{pmatrix}, \longrightarrow_{\kappa \to 0} \sim \begin{pmatrix} 0 & -D_x^2 \\ D_x^2 & 0 \end{pmatrix}.$$

If we use $H = 2\sqrt{1+uv}$, then we obtain WKI system.

3. H^1 METRIC AND INTEGRABLE EQUATION

Let us introduce H^1 norm on the algebra $\hat{\mathcal{G}}$

$$
\begin{aligned}
&< \hat{f}, \hat{g} >_{H^1} \\
&\quad = \int_{S^1} f(x)g(x)dx + \int_{S^1} a(x)b(x)dx \int_{S^1} \partial_x f(x)\partial_x g(x)dx \\
&\quad + \int_{S^1} \partial_x a(x)\partial_x b(x)dx + \alpha\cdot\beta,
\end{aligned} \tag{24}
$$

where $\hat{g}$ and $\hat{f}$ are as above.

Now we compute :

Lemma 2. *The coadjoint operator for H^1 norm is given by*

$$
ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) - c_1 f''' + c_2 a'' \\ f'(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) - c_2 f'' b(x) + 2c_3 a'(x) \\ 0 \end{pmatrix}.
$$

Proof. From the definition it follows that

$$
\begin{aligned}
&< ad^*_{\hat{f}}\hat{u}, \hat{g} >_{H^1} \\
&= -\int_{S^1}(fg' - f'g)u(x)dx - \int_{S^1}(fb' - ga')vdx - c_1\int_{S^1} f'(x)g''(x)dx \\
&\quad -c_2\int_{S^1}(f''(x)b(x) - g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx \\
&\quad -\int_{S^1}\partial_x(fg' - f'g)u(x)dx - \int_{S^1}\partial_x(fb' - ga')vdx.
\end{aligned}
$$

In the preceding section we have already computed the first five terms. After computing the last two terms by integrating by parts and using the fact that the functions $f(x), g(x), u(x)$ and $a(x), b(x), v(x)$ are periodic, this expression can be expressed as above.

Let us compute now the left hand side:

$$
\begin{aligned}
L.H.S. &= \int_{S^1}(ad^*_{\xi}\hat{q})\eta dx + \int_{S^1}(ad^*_{\xi}\hat{q})'\eta' dx \\
&= \int_{S^1}[(1-\partial^2)ad^*_{\xi}\hat{q}]\eta dx.
\end{aligned}
$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula. □

Corollary 2.

$$ad^*_{\hat{f}}\hat{u}$$
$$= \begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) + f''' \\ f'(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) \\ 0 \end{pmatrix}.$$

Hence the Hamiltonian operator is

$$\begin{pmatrix} \frac{1}{2}D^3 + D\tilde{w}_0 + \tilde{w}_0 D & \tilde{w}_1 D \\ D\tilde{w}_1 & D \end{pmatrix}, \tag{25}$$

where $\tilde{w}_i = (1-\partial_x^2)w_i$.

Thus we prove:

Theorem 2. *Let* $t \mapsto \hat{c}$ *be a curve in the* $Diff^s(\widehat{S^1)\bigodot C^\infty}(S^1)$. *Let* $\hat{c} = (e, e, 0)$ *be the initial point, directing to the vector* $\hat{c}(0) = (u(x)\frac{d}{dx}, v(x), \gamma_0)$*, where* $\gamma_0 \in \mathbf{R}^3$*. Then* $\hat{c}(t)$ *is a geodesic of the* H^1 *metric if and only if* $(u(x,t)\frac{d}{dx}, v(x,t), \gamma)$ *satisfies*

$$w_{0t} - w_{0xxt} = w_{1xxx} + 3(w_0 w_1)_x + w_0 w_{1x} - 2(w_{0xx}w_1)_x - 2w_{0xx}w_{1x} - w_{1xx}w_{0x}$$

$$w_{1t} - w_{1xxt} = w_{0x} + 4w_1 w_{1x} - 2(w_1 w_{1xx})_x.$$

Thus one can obtain the H^1 flows of other integrable systems following this prescription.

4. CONCLUSION AND OUTLOOK

In this paper we have shown that almost all the KdV systems can be derived as the geodesic flows with respect to the L^2 metrics on the Bott-Virasoro groups. We have also presented the method to obtain the H^1 analogue of these flows. It should be noticed that the superanalogue of all these coupled systems can be obtained from the L^2 geodesic flows on the superconformal group.

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