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THE STUDY OF A WAVE EQUATION WITH JUMPING NONLINEARITY BY DUALITY FORMULATION

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ABSTRACT: We investigate the existence of nontrivial solutions of a nonlinear wave equation under Dirichlet boundary condition with jumping nonlinearity. We use the fact that the solutions of the equation coincide with the critical points of its corresponding dual functional. Here we apply the Mountain Pass Theorem to find the nontrivial solutions.

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1. INTRODUCTION

In this paper we investigate the existence of solutions $u(x,t)$ for a perturbation $b[(u+1)^+ - 1]$ of the one-dimensional nonlinear wave equation

$$\begin{aligned} &u_{tt} - u_{xx} = b[(u+1)^+ - 1] \quad \text{in} \quad (c,d)\times\mathbb{R} \\ &u(c,t) = u(d,t) = 0, \qquad\qquad (1.1) \\ &u(x,t+T) = u(x,t), \end{aligned}$$

where $u^+ = \max\{u, 0\}$, b is a constant and the period T is given. This equation satisfies Dirichlet boundary condition on the interval $c < x < d$ and periodic condition on the variable t.

In Lazer and McKenna [6], it is pointed out that this kind of nonlinearity can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity has been extensively studied by many authors.

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They have studied the existence of solutions of the nonlinear equation with jumping nonlinearity for the fourth elliptic operator and other operators, see Lazer and McKenna [7], Micheletti and Pistoia [8], Micheletti and Pistoia [9], Tarantello [12]. However, there has been very little discussion on the wave operator with this kind of nonlinearity. The objective of this paper is to study the nonlinear wave equation with this jumping nonlinearity.

We assume that the period T is a rational multiple of the length $(d-c)$ of the $x-$interval where problem (1.1) is posed (As is well-known, serious difficulties of a number theoretical nature arise when that is not the case). For simplicity, only the case $T=\pi$ will be considered. By obvious changes of variables, problem (1.1) can be reduced to

$$\begin{aligned} &u_{tt}-u_{xx}=b[(u+1)^{+}-1] \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R} \\ &u(\pm\frac{\pi}{2},t)=0, \\ &u(x,t+\pi)=u(x,t). \end{aligned} \tag{1.2}$$

In this paper we investigate the existence of nontrivial solutions of Dirichlet boundary value problem (1.2). We show it by the Mountain Pass Theorem and duality principle. This technique has already been used in existence and multiplicity results for Hamiltonian system by Ambrosetti and Rabinowitz [1], Clark and Ekeland [2]. Choi and Mckenna used this method for finding numerical result for semilinear wave equation, see Choi et al [4].

In Section 2, we investigate some properties of the Banach space spanned by eigenfunctions of the wave operator. We show that the trivial solution exists for $-3<b<1$. In Section 3, we construct a duality formulation of this wave equation. We show the existence of nontrivial periodic solutions of the wave equation. The main result is the following: (1.2) has at least one nontrivial periodic solution for $-7<b<-3$.

2. THE BANACH SPACE SPANNED BY EIGENFUNCTIONS

In this section we investigate the properties of the Banach space spanned by the eigenfunctions of the wave operator. Let L be the wave operator in $\mathbb{R}^2$, i.e., $Lu=u_{tt}-u_{xx}$. When u is even in x and periodic in t with period π, the eigenvalue problem for $u(x,t)$

$$\begin{aligned} &Lu=\lambda u \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}, \\ &u(\pm\frac{\pi}{2},t)=0 \end{aligned} \tag{2.1}$$

has infinitely many eigenvalues $\lambda_{mn} = (2n+1)^2 - 4m^2 \quad (m, n = 0, 1, 2, ...)$ and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x && \text{for} \quad n \geq 0 \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x && \text{for} \quad m > 0, \quad n \geq 0 \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos(2n+1)x && \text{for} \quad m > 0, \quad n \geq 0. \end{aligned}$$

Let n be fixed and define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n + 1,$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n - 3.$$

Let $n \to \infty$, we obtain that

$$\lim_{n \to \infty} \lambda_n^+ = +\infty,$$

$$\lim_{n \to -\infty} \lambda_n^- = -\infty.$$

It is easy to check that the only eigenvalues in the interval (-15, 9) are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Ω be the square $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and H the Hilbert space defined by

$$H_0 = \{u \in L^2(\Omega) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

(i) $|||u||| \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u,

(ii)$\|u\| = 0$ if and only if $|||u||| = 0$,

(iii) $Lu \in H$ implies $u \in H$.

Define $L_\beta u = Lu + \beta u$. Then we have the following lemma.

Lemma 2.1. *Let $\beta \in \mathbb{R}$, $\beta \neq -\lambda_{mn}$ $(m, n \geq 0)$. Then we have:*

(i) L^{-1} *is a bounded linear operator from H_0 to H_0.*

(ii) L_β^{-1} *is a bounded linear operator from H_0 to H.*

Proof. We know $\lambda_{mn} = (2n+1)^2 - 4m^2$. Since $2n+1$ is odd and $2m$ is even, we have $\lambda_{mn} \neq 0$ and $|(2n+1) - 2m| \geq 1$.

So

$$|\lambda_{mn}| \geq |(2n+1) - 2m| \cdot |(2n+1) + 2m| \geq 2n+1.$$

Thus it deduces that L^{-1} is a bounded linear operator from H_0 to H_0.

Suppose that $\beta \neq -\lambda_{mn}$. When n is fixed, $\lambda_n^+ = 4n+1, \lambda_n^- = -4n-3$. We see that $\lim_{n\to\infty} \lambda_n^+ = +\infty$, $\lim_{n\to-\infty} \lambda_n^- = -\infty$. . Hence the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |\beta|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}).$$

Then

$$L_\beta^{-1} u = \sum (\frac{1}{\lambda_{mn} + \beta} h_{mn}\phi_{mn} + \frac{1}{\lambda_{mn} + \beta} k_{mn}\psi_{mn}).$$

Hence we have the inequality

$$|||L_\beta^{-1} u||| = \sum \frac{|\lambda_{mn}|}{|\lambda_{mn} + \beta|^2} (h_{mn}^2 + k_{mn}^2) \leq C \sum (h_{mn}^2 + k_{mn}^2)$$

for some C, which means that

$$|||L_\beta^{-1} u||| \leq C_1 \|u\|, C_1 = \sqrt{C}.$$

So L_β^{-1} is a bounded linear operator from H_0 to H. □

Theorem 1. *Let $-3 < b < 1$. Then the equation*

$$Lu = b[(u+1)^+ - 1]$$

has only the trivial solution in H_0.

Proof. Since $\lambda_{10} = -3$ and $\lambda_{00} = 1$, let $\beta = -\frac{1}{2}(\lambda_{00} + \lambda_{10}) = -\frac{1}{2}(-3+1) = 1$. The equation is equivalent to

$$u = (L+\beta)^{-1} [(b+\beta)(u+1)^+ - \beta(u+1)^- - (b+\beta)], \tag{2.2}$$

where we use the equality $u = u^+ - u^-$.

By Lemma 2.1, $(L+\beta)^{-1}$ is a compact linear map from H_0 into H_0. Therefore its L^2-norm is $\frac{1}{2}$. We note that

$$\begin{aligned}
& \|(b+\beta)[(u_1+1)^+ - (u_2+1)^+] - \beta[(u_1+1)^- - (u_2+1)^-]\| \\
\leq\ & \max\{|b+\beta|, |\beta|\}\|u_1 - u_2\| \\
<\ & \tfrac{1}{2}(\lambda_{00} - \lambda_{10})\|u_1 - u_2\| \\
=\ & 2\|u_1 - u_2\|,
\end{aligned}$$

where we used the inequality $|s_1^+ - s_2^+| + |s_1^- - s_2^-| \leq |s_1 + s_2|$.

So the right hand side of (2.2) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$. Since $u \equiv 0$ is a solution of equation (2.2), $u \equiv 0$ is the unique solution. □

The following proposition is necessary to next section.

Proposition. *If $-7 < b < 1$, then the equation $Lu - bu^+ = 0$ admits only the trivial solution $v = 0$ in H_0.*

Proof. $H_1 = span\{\cos x \cos 2mt, m \geq 0\}$ is invariant under L and under the map $u \mapsto bu^+$. So the spectrum σ_1 of L retracted to H_1 contains $\lambda_{10} = -3$ in $(-7, 1)$ where $\lambda_{21} = -7, \lambda_{00} = 1$. The spectrum σ_2 of L retracted to $H_2 = H_1^\perp$ contains $\lambda_{10} = -3$ in $(-7, 1)$. From the symmetry theorem in Lazer and McKenna [5], any solution $y(t)\cos x$ of this equation satisfies $y'' + y - by^+ = 0$. This nontrivial periodic solution is periodic with period $\pi + \dfrac{\pi}{\sqrt{-b+1}} \neq \pi$. This shows that there is no nontrivial solution of $Lu - bu^+ = 0$. □

3. THE DUALITY FORMULATION

In this section we study the duality formulation of nonlinear wave equation (1.2) subject to the boundary condition and periodic condition.

Recall that λ_{mn} and φ_{mn}, ψ_{mn} $(m, n \geq 0)$ are the eigenvalues and normalized eigenfunctions of the operator L in the symmetric subspaces H. Let β be a given constant satisfying the conditions:

$$0 < -\lambda_{10} = 3 < -b < \beta < -\lambda_{21} = 7\,. \tag{3.1}$$

Define

$$L_\beta u = Lu + \beta u,$$

$$g_\beta(u) = \beta u + b[(u+1)^+ - 1].$$

So equation (1.2) can be written as

$$L_\beta u - g_\beta(u) = 0. \tag{3.2}$$

By the condition (3.1) and Lemma 2.1, L_β has a compact inverse from H to H. By the condition (3.1), $g_\beta : \mathbb{R} \to \mathbb{R}$ is monotone increasing function. Thus $g_\beta(u)$ has a monotone increasing inverse given by

$$\begin{aligned} g_\beta^{-1}(u) &= \begin{cases} \dfrac{1}{\beta+b}u, & \text{if} \quad u > -(\beta+b) \\ \dfrac{1}{\beta}(u+b), & \text{if} \quad u \le -(\beta+b) \end{cases} \\ &= \frac{1}{\beta+b}[u+(\beta+b)]^+ - \frac{1}{\beta}[u+(\beta+b)]^- - 1. \end{aligned} \tag{3.3}$$

Let $v = L_\beta u$, i.e., $u = L_\beta^{-1} v$. Then equation (3.2) can be written as

$$-L_\beta^{-1} v + g_\beta^{-1}(v) = 0. \tag{3.4}$$

Note that $v = 0$ is a trivial solution of (3.4). Equation (3.4) is a duality formulation of equation (1.2).

Define the energy functional $I(v) : H \to \mathbb{R}$ by

$$I(v) = \int_\Omega [-\frac{1}{2} L_\beta^{-1} v \cdot v + G(v)] dx dt\,,$$

where

$$G(v) = \frac{1}{2(\beta+b)}([v+(\beta+b)]^+)^2 + \frac{1}{2\beta}([v+(\beta+b)]^-)^2 - v - \frac{\beta+b}{2}.$$

Lemma 3.1. *I(v) is continuous and has a Frechét derivative with*

$$I'(v)w = \int_\Omega [-L_\beta^{-1} v + g_\beta^{-1}(v)] w dx dt$$

for any $v, w \in H$. Consequently, the solutions of (3.4) in H correspond to critical points of $I(v)$ in H.

Proof. Let v, w be in H. To prove the continuity of $I(v)$ we consider

$$\begin{aligned}
& I(v+w)-I(v) \\
= & \int_\Omega [-\frac{1}{2}L_\beta^{-1}(w+v)\cdot(w+v)+G(w+v)]dxdt \\
& -\int_\Omega [-\frac{1}{2}L_\beta^{-1}v\cdot v+G(v)]dxdt \\
= & \int_\Omega [-\frac{1}{2}L_\beta^{-1}v\cdot w-\frac{1}{2}L_\beta^{-1}w\cdot v-\frac{1}{2}L_\beta^{-1}w\cdot wdxdt \\
& +\frac{1}{2(\beta+b)}\int_\Omega \{([v+w+(\beta+b)]^+)^2-([v+(\beta+b)]^+)^2\}dxdt \\
& +\frac{1}{2\beta}\int_\Omega \{([v+w+(\beta+b)]^-)^2-[v+(\beta+b)]^-)^2\}dxdt \\
& -\int_\Omega wdxdt.
\end{aligned}$$

Let $v=\sum(h_{mn}\phi_{mn}+k_{mn}\psi_{mn})$, $w=\sum(\tilde{h}_{mn}\phi_{mn}+\tilde{k}_{mn}\psi_{mn})$. Then we have

$$\begin{aligned}
|\int \frac{1}{2}L_\beta^{-1}v\cdot w| &= |\sum \frac{1}{2(\lambda_{mn}+\beta)}(h_{mn}\tilde{h}_{mn}+k_{mn}\tilde{k}_{mn})| \\
&\leq D_1|\sum \lambda_{mn}(h_{mn}\tilde{h}_{mn}+k_{mn}\tilde{k}_{mn})| \\
&\leq D_1|\|v\||\cdot|\|w\||,
\end{aligned}$$

with some constant $D_1>\dfrac{1}{2|\lambda_{mn}+\beta|}>0$ for all $m,n\geq 0$.

For the same reason we get the following:

$$|\int \frac{1}{2}L_\beta^{-1}v\cdot w|\leq D_2|\|v\||\cdot|\|w\||$$

for some constant $D_2>0$; and

$$|\int L_\beta^{-1}v\cdot w|\leq D_3|\|v\||^2$$

for some constant $D_3>0$.

On the other hand we have

$$\begin{aligned}
& |\int(|(v+w+\beta+b)^+|^2-|(v+\beta+b)^+|^2)dxdt| \\
\leq\ & 2\|(v+\beta+b)^+\|\|w\|+\|w\|^2 \\
\leq\ & 2\|(v+\beta+b)^+\|\cdot|\|w\||+|\|w\||^2\,,
\end{aligned}$$

where we used the inequality $||(s+t)^+|^2-|s^+|^2|\leq 2s^+|t|+|t|^2$. By using $||(s+t)^-|^2-|s^-|^2|\leq 2s^-|t|+|t|^2$ the same result will be get for the negative part. With the above results, we see that $I(v)$ is continuous at v.

To prove that $I(v)$ is Frechét derivative at $v \in H$, it is enough to compute the following:

$$\begin{aligned}
& |I(v+w) - I(v) - I'(v)w| \\
= & \ |\int_\Omega [-\frac{1}{2}L_\beta^{-1}v \cdot w - \frac{1}{2}L_\beta^{-1}w \cdot v - \frac{1}{2}L_\beta^{-1}w \cdot w dxdt \\
& + \int_\Omega [G(v+w) - G(v)]dxdt - \int_\Omega -L_\beta^{-1}v \cdot w + g_\beta^{-1}(v) w dxdt| \\
= & \ |\int_\Omega [\frac{1}{2}L_\beta^{-1}v \cdot w - \frac{1}{2}L_\beta^{-1}w \cdot v - \frac{1}{2}L_\beta^{-1}w \cdot w dxdt \\
& + \frac{1}{2(\beta+b)} \int_\Omega \{([v+w+(\beta+b)]^+)^2 - ([v+(\beta+b)]^+)^2\} dxdt \\
& + \frac{1}{2\beta} \int_\Omega \{([v+w+(\beta+b)]^-)^2 - [v+(\beta+b)]^-)^2\} dxdt \\
& - \frac{1}{(\beta+b)} \int_\Omega ([v+(\beta+b)]^+)^2 w dxdt \\
& - \frac{1}{\beta} \int_\Omega ([v+(\beta+b)]^-)^2 w dxdt| \\
\le & \ \frac{3}{2}D\|\|w\|\|^2 + |\frac{1}{2(\beta+b)}| \int w^2 dxdt + |\frac{1}{2\beta}| \int w^2 dxdt \\
\le & \ (\frac{3}{2}D + \frac{1}{2|\beta+b|} + \frac{1}{2|\beta|})\|\|w\|\|^2
\end{aligned}$$

for some constant $D = \max\{D_1, D_2, D_3\} > 0$ and here we use the inequality

$$0 \le |(s+t)^+|^2 - |s^+|^2 - 2s^+t \le |t|^2$$

$$0 \le |(s+t)^-|^2 - |s^-|^2 - 2s^-t \le |t|^2.$$

Thus we complete the proof. □

Lemma 3.2. *If (3.1) is satisfied, and $\beta < (-b+7)/2$, then $v = 0$ is a strict local minimum of $I(v)$ in H.*

Proof. It is clear that $I(0) = 0$ and $I'(0) = 0$. For $s > 0$ and $\psi \in H$ with $\|\|\psi\|\| = 1$,

$$\begin{aligned}
I'(s\psi)\psi = & - s\int_\Omega L_\beta^{-1}\psi \cdot \psi dxdt + \int_\Omega \frac{1}{\beta+b}[s\psi + (\beta+b)]^+\psi \\
& - \frac{1}{\beta}[s\psi + (\beta+b)]^-\psi - \psi dxdt \\
= & - s\int_\Omega L_\beta^{-1}\psi \cdot \psi dxdt + \frac{s}{\beta+b}\int_\Omega \psi^2 dxdt \\
& + (\frac{1}{\beta+b} - \frac{1}{\beta})\int_\Omega [s\psi + (\beta+b)]^-\psi dxdt,
\end{aligned}$$

where we use the equality $s = s^+ - s^-$.

Let $H_1 = \text{span}\{\phi_{mn}, \psi_{mn} | \frac{1}{|\beta+\lambda_{mn}|} > \frac{1}{2\beta}\} \cap H$, which is finite dimensional. Let H_2 be the orthogonal complement of H_1 in H. Then $H = H_1 \oplus H_2$. Let $\psi = \psi_1 + \psi_2$, where $\psi_1 \in H_1$ and $\psi_2 \in H_2$. For any $\psi_2 \in H_2$, $|\int_\Omega L_\beta^{-1}\psi_2 \cdot \psi_2 dxdt| \leq \frac{1}{2\beta}\int_\Omega \psi_2^2 dxdt$. Hence we have

$$\begin{aligned} & I'(s\psi)\psi \\ = \; & -s\int_\Omega [L_\beta^{-1}\psi_1 \cdot \psi_1 + L_\beta^{-1}\psi_2 \cdot \psi_2]dxdt + \frac{s}{\beta+b}\int_\Omega (\psi_1^2 + \psi_2^2)dxdt \\ & +(\frac{1}{\beta+b} - \frac{1}{\beta})\int_\Omega [s(\psi_1+\psi_2) + (\beta+b)]^-(\psi_1+\psi_2)dxdt \\ \geq \; & s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|})\int_\Omega \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta})\int_\Omega \psi_2^2 dxdt \\ & -(\frac{1}{\beta+b} - \frac{1}{\beta})\int_\Omega [s(\psi_1+\psi_2) + (\beta+b)]^-(|\psi_1| + |\psi_2|)dxdt\,, \end{aligned}$$

where $|\beta+\mu| = \min\{|\beta+3|, |\beta-7|\}$.

Since H_1 is finite dimensional, and $\|\psi_1\| \leq \|\psi\| \leq |||\psi||| = 1$, there is a sufficient small $s_0 > 0$, which is dependent of β, b such that for any $0 < s < s_0$,

$$s|\psi_1(x,t)| \leq \frac{1}{2}(\beta+b).$$

Let

$$\Omega_s = \{(x,t) \in \Omega | s\psi_2 \leq -\frac{1}{2}(\beta+b) \text{ i.e. } s\psi_2 < 0\}.$$

Then we have

$$\begin{aligned} I'(s\psi)\psi \geq & \; s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|})\int_\Omega \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta})\int_\Omega \psi_2^2 dxdt \\ & -(\frac{1}{\beta+b} - \frac{1}{\beta})\int_\Omega [s(\psi_1+\psi_2) + (\beta+b)]^-(|\psi_1| + |\psi_2|)dxdt \\ \geq & \; s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|})\int_\Omega \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta})\int_\Omega \psi_2^2 dxdt \\ & -(\frac{1}{\beta+b} - \frac{1}{\beta})\int_\Omega [s\psi_2 + \frac{1}{2}(\beta+b)]^-(|\psi_1| + |\psi_2|)dxdt \\ = & \; s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|})\int_\Omega \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta})\int_\Omega \psi_2^2 dxdt \\ & -(\frac{1}{\beta+b} - \frac{1}{\beta})\int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^-(|\psi_1| + |\psi_2|)dxdt \\ \geq & \; s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|})\int_\Omega \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta})\int_\Omega \psi_2^2 dxdt \end{aligned}$$

$$+ s(\frac{1}{\beta+b} - \frac{1}{2\beta}) \int_{\Omega_s} \psi_2^2 dxdt$$

$$- (\frac{1}{\beta+b} - \frac{1}{\beta}) \int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^-(|\psi_1|) dxdt$$

$$\geq s(\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|}) \int_{\Omega} \psi_1^2 dxdt + s(\frac{1}{\beta+b} - \frac{1}{2\beta}) \int_{\Omega} \psi_2^2 dxdt$$

$$- (\frac{1}{\beta+b} - \frac{1}{\beta}) \int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^-(|\psi_1|) dxdt$$

$$\geq C_1 s - C_2 \int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^- |\psi_1| dxdt,$$

where $C_1 = \min\{\frac{1}{\beta+b} - \frac{1}{|\beta+\mu|}, \frac{1}{2\beta}\} > 0$, $C_2 = \frac{1}{\beta+b} - \frac{1}{\beta} > 0$.

Since $\beta < (-b+7)/2$, $C_1 > 0$,

$$1 = \int_{\Omega} (\psi_1^2 + \psi_2^2) dxdt \geq \int_{\Omega_s} \psi_2^2 dxdt \geq \frac{[-\frac{1}{2}(\beta+b)]^2}{s^2} \text{mes}(\Omega_s).$$

Take δ such that $0 < \delta \leq [\frac{1}{2}(\beta+b)]$. Thus $\text{mes}(\Omega_s) \leq \frac{s^2}{\delta^2}$. It is easy to know $\|\psi_1\|_{L^\infty} \leq C_3$, for some contant $C_3 > 0$. Since $\|\psi_1\| \leq 1$,$\|\psi_2\| \leq 1$,

$$\begin{aligned} I'(s\psi)\psi &\geq C_1 s - C_2 \int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^- |\psi_1| dxdt \\ &\geq C_1 s - C_2 C_3 \int_{\Omega_s} [s\psi_2 + \frac{1}{2}(\beta+b)]^- dxdt \\ &\geq C_1 s - C_2 C_3 \int_{\Omega_s} s\psi_2^- dxdt \\ &\geq C_1 s - sC_2 C_3 \|\psi_2\| \cdot \sqrt{\text{mes}(\Omega_s)}, \end{aligned}$$

where we used the fact that $s\psi_2 < s\psi_2 + \frac{1}{2}(\beta+b) < 0$ in Ω_s. So

$$\begin{aligned} I'(s\psi)\psi &\geq C_1 s - s\frac{s}{\delta} C_2 C_3 \|\psi_2\| \\ &\geq C_1 s - C_2 C_3 \frac{s^2}{\delta} \geq (C_1 - C_2 C_3 \frac{s}{\delta}) s. \end{aligned}$$

We choose s_1 so small that $C_1 - C_2 C_3 \frac{s}{\delta} \geq \frac{C_1}{4}$ for any $s \leq s_1$. Then for all $0 < s < \min\{s_0, s_1\}$,

$$I(s\psi) = \int_0^s I'(\tau\psi)\psi d\tau \geq \int_0^s \frac{C_1}{4} \tau d\tau = \frac{C_1}{8} s^2.$$

Thus $v = 0$ is a strictly local minimum of $I(v)$ in H. □

Lemma 3.3. *For $-7 < b < -3$ and β taken as in condition (3.1), $I(v)$ satisfies the (PS) condition in H.*

Proof. Suppose that any sequence $\{v_n\}$ is in H such that $I(v_n)$ is bounded and $I'(v_n) \to 0$. By the defination of (PS) condition we want to claim that $\{v_n\}$ has a convergent subsequence.

Assume the (PS) condition does not hold, that is, $|||(v_n)||| \to \infty$ as $n \to \infty$. Since $I'(v_n) \to 0$, we have

$$-L_\beta^{-1} v_n + g_\beta^{-1}(v_n) \to 0.$$

Dividing by $|||v_n|||$ and taking $w_n = \frac{v_n}{|||v_n|||}$, we have

$$-L_\beta^{-1} w_n + \frac{1}{\beta + b}[w_n + \frac{\beta + b}{|||v_n|||}]^+ - \frac{1}{\beta}[w_n + \frac{\beta + b}{|||v_n|||}]^- - \frac{1}{|||v_n|||} \to 0.$$

Since $|||w_n||| = 1$, there exists a subsequence of $\{w_n\}$ renamed so that $w_n \to w$ weakly in H. From Lemma 2.1 we know that L_β^{-1} is compact from H_0 to H. Due to the compactness of the operator L_β^{-1} which maps weakly convergent sequences, we have

$$\lim_{n\to\infty} L_\beta^{-1} w_n = L_\beta^{-1} w.$$

Then we get the following:

$$\frac{1}{\beta + b}[w_n + \frac{\beta + b}{|||v_n|||}]^+ - \frac{1}{\beta}[w_n + \frac{\beta + b}{|||v_n|||}]^- \to L_\beta^{-1} w.$$

Standard calculation shows that

$$\lim_{n\to\infty} (w_n + \frac{\beta + b}{|||v_n|||}) = (\beta + b)[L_\beta^{-1} w]^+ - \beta[L_\beta^{-1} w]^-.$$

Since the weak limit is unique, we have $\lim_{n\to\infty} w_n = w$ in H, which is the same as

$$w = (\beta + b)[L_\beta^{-1} w]^+ - \beta[L_\beta^{-1} w]^-,$$

Let $u = L_\beta^{-1} w$. Then u satisfies

$$Lu - bu^+ = 0.$$

By proposition, we have a trivial solution $u = 0$. Therefore $w = 0$. Thus $\lim_{n\to\infty} w_n = 0$ in H. This is a contradiction, since $|||w_n||| = 1$. Hence we have $|||v_n|||$ is bounded. Therefore $I(v)$ satisfies the (PS) condition. □

Lemma 3.4. *If additionally, β satisfies also $\frac{2}{3-\beta} + \frac{1}{\beta + b} + \frac{1}{\beta} < 0$, then $\lim_{s\to\infty} I(s\psi_{10}) = -\infty$.*

Proof. By calculation we have

$$\int_\Omega (\psi_{10}^+)^2 dxdt = \int_\Omega (\psi_{10}^-)^2 dxdt = \frac{1}{2}\int_\Omega (\psi_{10})^2 dxdt\,.$$

For any $\theta > 0$ and s being sufficiently large,

$$\begin{aligned}
& I(s\psi_{10}) \\
= & \frac{1}{2}\int_\Omega \{-\frac{1}{\beta-3}(s\psi_{10})^2 + \frac{1}{\beta+b}[(s\psi_{10}+\frac{\beta+b}{s})^+]^2 \\
& +\frac{1}{\beta}[(s\psi_{10}+\frac{\beta+b}{s})^-]^2\}dxdt - s\int_\Omega \psi_{10} - \frac{\beta+b}{2}dxdt \\
= & \frac{s^2}{2}\int_\Omega \{\frac{1}{3-\beta}\psi_{10}^2 + \frac{1}{\beta+b}[(\psi_{10}+\frac{\beta+b}{s})^+]^2 \\
& +\frac{1}{\beta}[(\psi_{10}+\frac{\beta+b}{s})^-]^2\}dxdt - s\int_\Omega \psi_{10}dxdt - \frac{\beta+b}{2}\int_\Omega 1dxdt \\
\le & \frac{s^2}{2}\int_\Omega \{\frac{1}{3-\beta}\psi_{10}^2 + \frac{1}{\beta+b}[(\psi_{10}^+)^2+\theta] + \frac{1}{\beta}[(\psi_{10}^-)^2+\theta]\}dxdt \\
& -s\int_\Omega \psi_{10}dxdt - \frac{\beta+b}{2}\int_\Omega 1dxdt \\
= & \frac{s^2}{2}\{[\frac{2}{3-\beta}+\frac{1}{\beta+b}+\frac{1}{\beta}] + 8\theta[\frac{1}{\beta+b}+\frac{1}{\beta}]\}\int_\Omega \psi_{10}dxdt \\
& -s\int_\Omega \psi_{10}dxdt - \frac{\beta+b}{2}\int_\Omega 1dxdt.
\end{aligned}$$

If we choose $\theta > 0$ so small, then

$$[\frac{2}{3-\beta}+\frac{1}{\beta+b}+\frac{1}{\beta}] + 8\theta[\frac{1}{\beta+b}+\frac{1}{\beta}] < 0.$$

We conclude that $\lim_{s\to\infty} I(s\psi_{10}) = -\infty$. □

To employ the Mountain Pass Theorem, we prove the main result of this paper.

Theorem 2. *Let $-7 < b < -3$. Then there exists at least one nontrivial solution of the equation*

$$\begin{aligned}
u_{tt} - u_{xx} &= b[(u+1)^+ - 1] \quad in \quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}\,, \\
u(\pm\frac{\pi}{2},t) &= 0, \\
u(x,t+\pi) &= u(x,t).
\end{aligned}$$

Proof. Let $-7 < b < -3$. By Lemma 3.2, there exists a small open neighborhood B of 0 in H such that in B, $v = 0$ is a strict local point of minimum of I. So there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$. The statement (I_1) of the Mountain Pass Theorem is valid. Since $I \to -\infty$ as $|||v||| \to \infty$, by Lemma 3.4 there is an $e \in E\backslash \bar{B}_\rho$

such that $I(e) \leq 0$. So the statement (I_2) of the Mountain Pass Theorem is satisfied. And $I \in C^1(\Omega, \mathbb{R})$ satisfies the Palais-Smale condition by Lemma 3.3. Thus according to the Mountain Pass Theorem there exists at least one nontrivial solution. □

4. APPENDIX

The Mountain Pass Theorem concerns with proving the existence of critical points of functional $I \in C^1(E, \mathbb{R})$ which satisfy the Palais-Smale (PS) condition, which occurs repeatedly in critical point theory.

Definition. We say that I satisfies the Palais-Smale condition, if any sequence $\{u_m\} \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

We state now the Mountain Pass Theorem.

Theorem A. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS) condition. Suppose $I(0) = 0$ and*

(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and

(I_2) there is an $e \in E \backslash \bar{B}_\rho$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, g(1) = e\}.$$

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