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TIMELIKE SPATIALLY CLOSED TRAJECTORIES UNDER A POTENTIAL ON SPLITTING LORENTZIAN MANIFOLDS

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ABSTRACT: We study the periodic motions of a relativistic particle submitted to the action of an external potential V. On a wide class of Lorentzian manifolds, we find timelike solutions of a differential equation (depending on V) closed in the spatial component and satisfying a Dirichlet condition in the temporal one. We prove a multiplicity result for the critical points of the (strongly indefinite) functional associated to the problem, using a saddle type theorem based on the notion of relative category. The periodicity of the problem, the non–compactness of the manifold and the lack of some assumptions involving the relative category make necessary to use a suitable penalization scheme and a Galerkin approximation.

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1. INTRODUCTION

In this article we study in the ambit of Lorentzian Geometry (cf. the basic references Beem et al [2], O'Neill [20]) the classical Riemannian topic of the existence of periodic orbits of Lagrangian systems (cf. e.g. Benci [3]).

Indeed, by using variational methods, we investigate on a wide class of Lorentzian manifolds $(\mathcal{M}, g)$ about the existence and multiplicity of solutions z, in the sense specified below, of the differential equation

$$D_s \dot{z} + \nabla_L V(z) = 0\,, \tag{1.1}$$

where D_s denotes the covariant derivative with respect to g, $V \in C^1(\mathcal{M}, \mathbf{R})$ and ∇_L is the gradient with respect to g.

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We consider *orthogonal splitting Lorentzian manifolds*, i.e. manifolds $(\mathcal{M}, g)$ with

$$\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}\,,$$

where $\mathcal{M}_0$ is a smooth connected finite dimensional manifold endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$ and the metric g is given by

$$g(z)[\zeta,\zeta] = \langle\alpha(z)\xi,\xi\rangle - \beta(z)\tau^2 \tag{1.2}$$

for any $z = (x,t) \in \mathcal{M}$, $\zeta = (\xi,\tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbf{R}$, where $\alpha(z) : T_x\mathcal{M}_0 \to T_x\mathcal{M}_0$ is a smooth, symmetric, strictly positive linear operator and $\beta : \mathcal{M} \to \mathbf{R}$ is a smooth, strictly positive, scalar field.

Note that by a result of Geroch, it is usually accepted that every globally hyperbolic time–oriented Lorentzian manifold is isometric to an orthogonal splitting one (cf. Geroch [13]), hence many physically relevant spacetimes as Robertson–Walker, Schwarzschild and Reissner–Nördstrom are isometric to products as above.

In the remainder of the article, for simplicity of notation, g will be also denoted by $\langle\cdot,\cdot\rangle_L$.

A vector $\zeta \in T\mathcal{M}$ is said timelike (resp. lightlike; causal; spacelike) if $\langle\zeta,\zeta\rangle_L < 0$, (resp. $\langle\zeta,\zeta\rangle_L = 0, \zeta \neq 0; \langle\zeta,\zeta\rangle_L \leq 0; \langle\zeta,\zeta\rangle_L > 0$ or $\zeta = 0$). A curve z on $\mathcal{M}$ is said to be *timelike, lightlike, causal* or *spacelike* according to the causal character of $\dot z$.

As in the case of autonomous Lagrangian systems on Riemannian manifolds, it is easy to verify that for each solution $z : I \to \mathcal{M}, I \subset \mathbf{R}$ interval, of (1.1) a constant $E_z \in \mathbf{R}$ exists such that

$$E_z = \frac{1}{2}\langle\dot z(s),\dot z(s)\rangle_L + V(z(s)) \quad \forall s \in I. \tag{1.3}$$

If we assume that the coefficients of the metric α, β and the potential V are periodic in the variable t, that is $T > 0$ exists such that

$$\alpha(x,t+T) = \alpha(x,t) \quad \beta(x,t+T) = \beta(x,t) \quad V(x,t+T) = V(x,t) \tag{1.4}$$

for any $z = (x,t) \in \mathcal{M}_0 \times \mathbf{R}$, it makes sense to study existence (and multiplicity) of solutions of the following problem.

For a fixed $k \in \mathbf{N}$ (in this paper we denote by $\mathbf{N}$ the set of the natural integers $n, n \geq 1$), we look for *kT–periodic trajectories under* V, i.e. smooth curves $z = (x,t) : [0,1] \to \mathcal{M}$ such that

$$\begin{cases} D_s\dot z + \nabla_L V(z) = 0 \\ E_z < 0 \\ x(1) = x(0) \quad \dot x(1) = \dot x(0) \\ t(1) = t(0) + kT \quad \dot t(1) = \dot t(0). \end{cases} \tag{1.5}$$

We point out that our solutions are really periodic in the spatial component, while satisfy a Dirichlet condition in the temporal one.

Note that, when $V \equiv 0$, (1.1) is the geodesic equation and the solutions of (1.5) are the timelike kT–periodic trajectories introduced by Benci and Fortunato [4]. These trajectories make sense for stationary metrics (i.e. metrics endowed with a smooth timelike Killing vector field) and also for metrics periodically depending on the time variable. Periodic trajectories can be of any causal character and in the timelike and lightlike cases they are the relativistic version of periodic trajectories in Newtonian Mechanics.

In our case, from a physical point of view, each causal solution (we refer to Remark 5.7 where it is shown that we find timelike solutions of (1.5)) of equation (1.1) represents the world line of a relativistic particle moving under the action of a gravitational field (described by the metric) and of an external potential V.

For $V \equiv 0$, existence and multiplicity results for periodic trajectories on stationary manifolds have been obtained in Benci and Fortunato [4], Benci et al [5], Benci et al [6], Sánchez [23] (cf. also the references therein) by using variational methods, in Greco [15] – by means of a variant of the shortening procedure used in the study of closed geodesics on Riemannian manifolds and in Sánchez [24], where periodic trajectories are studied by using the tools of Causality Theory. Existence results on splitting Lorentzian manifolds can be found in Candela et al [8], Greco [16], Masiello [19], Sánchez [24] only for the cases $\mathcal{M}_0 = \mathbf{R}^N$ or $\mathcal{M}_0$ compact.

As far as we know, there are no results for a non–trivial potential, except for an unpublished paper of Romero and Sánchez [21] where in particular it is studied the completeness of the solutions of (1.1); moreover we deal with orthogonal splitting Lorentzian manifolds having a possibly non–compact base $\mathcal{M}_0$, which were not taken into account not even for $V \equiv 0$.

Next we will discuss the assumptions of our main result (see Theorem 1.2 below). At first, we recall that in Benci et al [7] (cf. also Chapter 8, Masiello [18]) it has been studied the geodesic connectedness of orthogonal splitting Lorentzian manifolds under the following assumptions on the metric (1.2):

(h_1) $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold;

(h_2) $\lambda > 0$ exists such that

$$\langle \alpha(z)\xi, \xi\rangle \geq \lambda\langle \xi, \xi\rangle \quad \forall z \in \mathcal{M},\ \forall \xi \in T_x\mathcal{M}_0;$$

(h_3) $\nu, N > 0$ exist such that for any $z \in \mathcal{M}$

$$\nu \leq \beta(z) \leq N.$$

We point out that there is not loss of generality in assuming the completeness of the Riemannian metric $\langle\cdot,\cdot\rangle$ (cf. Sánchez [22]). The previous three conditions imply the global hyperbolicity of $\mathcal{M}$ (cf. e.g. Sánchez [22]), nevertheless, in the quoted references also a control on the growth of the coefficients was needed in order to get geodesic connectedness:

(h_4) denoted by α_t, β_t the partial derivatives of α and β with respect to t, $\Lambda > 0$ exists such that
$$\sup\{\|\alpha_t(z)\|, |\beta_t(z)| \mid z \in \mathcal{M}\} = \Lambda;$$

(h_5) uniformly in $x \in \mathcal{M}_0$ and $\xi \in T_x\mathcal{M}_0$ unitary vector
$$\limsup_{t\to+\infty}\langle\alpha_t(x,t)\xi,\xi\rangle \le 0 \quad \liminf_{t\to-\infty}\langle\alpha_t(x,t)\xi,\xi\rangle \ge 0.$$

First of all, due to the periodicity of the problem, we have to impose some topological assumptions (cf. Bartolo et al [1] for more details); furthermore, taking into account also the unboundedness of $\mathcal{M}_0$ and the presence of the potential V, we restate the previous assumptions as follows (see also Remarks 1.1):

(H_1) $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a complete not contractible Riemannian manifold and its fundamental group $\pi_1(\mathcal{M}_0)$ is finite or it has infinitely many coniugacy classes;

(H_2) $\lambda_1, \lambda_2 > 0$ exist such that
$$\langle\alpha(z)\xi,\xi\rangle \ge \lambda_1\langle\xi,\xi\rangle \quad \forall z \in \mathcal{M},\ \forall\xi \in T_x\mathcal{M}_0 \tag{1.6}$$
$$\langle\alpha(x,0)\xi,\xi\rangle \le \lambda_2\langle\xi,\xi\rangle \quad \forall x \in \mathcal{M}_0,\ \forall\xi \in T_x\mathcal{M}_0; \tag{1.7}$$

(H_3) $\nu, \eta, N, M > 0$ exist such that, uniformly in t,
$$\lim_{d(x,x_0)\to+\infty}\beta(x,t) = N = \sup_{z\in\mathcal{M}}\beta(z) \quad \nu \le \beta(z) < N \quad \forall z \in \mathcal{M} \tag{1.8}$$
$$\lim_{d(x,x_0)\to+\infty}V(x,t) = M = \sup_{z\in\mathcal{M}}V(z) \quad \eta \le V(z) < M \quad \forall z \in \mathcal{M} \tag{1.9}$$
where d denotes the distance induced by the Riemannian metric of $\mathcal{M}_0$;

(H_4) denoting by α_t, β_t, V_t the partial derivatives of α, β, V with respect to t, $\Lambda > 0$ exists such that
$$\sup\{\|\alpha_t(z)\|, |\beta_t(z)|, |V_t(z)| \mid z \in \mathcal{M}\} = \Lambda; \tag{1.10}$$

(H_5) for any $z = (x,t) \in \mathcal{M}, \xi \in T_x\mathcal{M}_0$
$$\langle\alpha_t(z)\xi,\xi\rangle \ge 0 \tag{1.11}$$

and, uniformly in $x \in \mathcal{M}_0$ and $\xi \in T_x\mathcal{M}_0$,

$$\limsup_{t\to+\infty} \langle \alpha_t(x,t)\xi, \xi\rangle = 0 \tag{1.12}$$

$$\liminf_{t\to+\infty} V_t(x,t) \geq 0 \qquad \limsup_{t\to-\infty} V_t(x,t) \leq 0. \tag{1.13}$$

Moreover, again due to the possible lack of compactness of $\mathcal{M}_0$ and the periodicity of the problem, there could exist Palais–Smale sequences which are not bounded (cf. Section 2).

This problem firstly appears in the Riemannian case to ensure the existence of closed geodesics; a detailed discussion of hypothesis at infinity in this case is carried out in Bartolo et al [1]. To avoid further technical problems, we choose here the simplest hypothesis: namely we impose a global condition on $\mathcal{M}_0$ by assuming the existence of an additional function convex outside a compact subset (cf. also Benci et al [6]); thus we assume that

(H_6) for some $x_0 \in \mathcal{M}_0$, $U \in C^2(\mathcal{M}_0, \mathbf{R})$ and $R, \mu > 0$ exist such that for any $x \in \mathcal{M}_0$ with $d(x, x_0) \geq R$:

$$H^U(x)[\xi,\xi] \geq \mu\langle\xi,\xi\rangle \qquad \forall \xi \in T_x\mathcal{M}_0, \tag{1.14}$$

where $H^U(x)[\xi,\xi]$ denotes the Hessian of U with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$ at x in the direction of ξ.

Nevertheless, we need to ensure now the compatibility between the role of U (at infinity) and the other elements of our problem. This compatibility holds under the following technical assumptions (see also Remarks 1.1):

(H_7) denoted by ∇ the gradient with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$,

$$\lim_{d(x,x_0)\to+\infty} |\nabla\beta(x,t)||\nabla U(x)| = 0 \qquad \lim_{d(x,x_0)\to+\infty} |\nabla V(x,t)|\,|\nabla U(x)| = 0$$

$$\lim_{d(x,x_0)\to+\infty} |\nabla\alpha(x,t)||\nabla U(x)| = 0 \qquad \lim_{d(x,x_0)\to+\infty} |\alpha_t(x,t)||\nabla U(x)| = 0$$

uniformly in $t \in \mathbf{R}$.

Remarks 1.1. (1) By (1.6), for any $z \in \mathcal{M}$, $\alpha(z) - \lambda_1 I$ is a positive self–adjoint operator on $T_z\mathcal{M}$. In particular this implies that $\alpha(z)$ is invertible, and this will be essential in the proof of some a priori bounds (see Section 3). Remark 3.3 will clarify the role of assumption (1.7).

(2) Assumption (1.11) has been also used in Giannoni and Masiello [14] where splitting Lorentzian manifolds conformally equivalent to a manifold satisfying property (1.11) have been considered; as our trajectories are not independent, up to

parametrizations, on conformal changes of the metric, we have to require it explicitly. Clearly, as (1.11) holds, the first inequality in (h_5) becomes an equality and the second one is no more necessary (cf. (1.12)).

(3) Also in the stationary case, assumptions analogous to those in (H_3) and (H_7) were imposed (cf. Theorem 1.4, Benci et al [6]).

Before stating our main result, we remind that for geometrically distinct kT–periodic trajectories we mean curves having different support (cf. Proposition 2.3). Moreover in (1.5) we will assume $t(0) = 0$ otherwise one could consider as distinct the curves $(x(s), t(s))$ and $(x(s), t(s) + t(0))$ which is not interesting.

Theorem 1.2. *Let $(\mathcal{M}, g)$ be an orthogonal splitting Lorentzian manifold and $V \in C^1(\mathcal{M}, \mathbf{R})$. Assume that hypothesis (H_1)–(H_7) hold. Then*

(i) $k_0 \in \mathbf{N}$ exists such that problem (1.5) admits at least one (timelike) solution for any $k \in \mathbf{N}, k \geq k_0$;

(ii) denoted by $N(k)$ the number of geometrically distinct (timelike) solutions of (1.5), it is

$$\lim_{k \to +\infty} N(k) = +\infty.$$

Remark 1.3. A kT–periodic trajectory $z(s) = (x(s), t(s))$ is said *trivial* if $x = x(s)$ is a constant curve. Note that, in spite of what occurs in the stationary case (when trivial trajectories are the critical points of β, thus for instance they always exist when $\mathcal{M}_0$ is compact), generally trivial periodic trajectories do not appear when the coefficients of the metric depend on the time variable. If $V \neq 0$ their existence is even more rare (cf. equations (3.1)). Thus we do not take into account trivial periodic trajectories under V.

As in the Riemannian case, periodic trajectories under V are the critical points of an action functional on a suitable infinite dimensional manifold, but, due to the indefiniteness of the metric, this functional is now strongly indefinite (i.e. it is unbounded both from below and from above and the Morse index of its critical points is infinite), therefore the modern theory of critical points for unbounded functionals is needed.

In particular we will use a multiplicity result (see Theorem 4.4) proved in Candela et al [8] for critical points of saddle type which is based on the notion of relative category (cf. Fournier and Willem [11], Fournier et al [12], Szulkin [25] and Section 4 where for convenience of the reader we recall the main properties and results). Such abstract theorem can not be directly applied; indeed in our case the action functional does not verify the well known Palais–Smale compactness condition and moreover the assumption involving the relative category fails. In order to bypass these problems we will introduce two approximation schemes: a suitable penalization of the action functional (cf. Section 2) and a Galerkin approximation (cf. Section 5). Finally some a priori bounds obtained in Section 3 will allow us to get solutions of (1.5).

On the other hand, when one deals with stationary Lorentzian manifolds, under reasonable assumptions on the coefficients of the metric, an easier approach is also possible. Indeed, by a variational principle (proved in Benci et al [5] by Benci, Fortunato and Giannoni) the problem is equivalent to the search of the critical points of a purely Riemannian functional which is bounded from below. We point out that when the metric is as in (1.2) this kind of approach fails due to the dependence on the time variable.

Clearly if we work in more restrictive conditions on the manifold, the assumptions become milder. Note for instance that if $\mathcal{M}_0$ is compact we only have to ensure a non–trivial topology (compare with Theorem 1.10 in Candela et al [8]).

Corollary 1.4. *Let $(\mathcal{M}, g)$ be an orthogonal splitting Lorentzian manifold and $V \in C^1(\mathcal{M}, \mathbf{R})$ be such that $V > 0$. Assume that $\mathcal{M}_0$ is a compact manifold such that its fundamental group $\pi_1(\mathcal{M}_0)$ is finite or it has infinitely many coniugacy classes. Then the thesis of Theorem 1.2 holds.*

Finally, let us point out that in particular, with arguments simpler than those used in the proof of Theorem 1.2, we obtain a result for trajectories joining two events (cf. Benci et al [7] for the case of geodesical connectedness).

Theorem 1.5. *Let $(\mathcal{M}, g)$ be an orthogonal splitting Lorentzian manifold and $V \in C^1(\mathcal{M}, \mathbf{R})$. Assume that (h_1)–(h_5) hold and that*

(h_6) *some positive constants η, M, Λ_1 exist such that*

$$\eta \leq V(z) \leq M \qquad |V_t(z)| \leq \Lambda_1 \quad \forall z = (x, t) \in \mathcal{M};$$

(h_7) *uniformly in $x \in \mathcal{M}_0$*

$$\liminf_{t \to +\infty} V_t(x, t) \geq 0 \qquad \limsup_{t \to -\infty} V_t(x, t) \leq 0.$$

Then for all $z_0, z_1 \in \mathcal{M}$ there exists a solution $z : [0, 1] \to \mathcal{M}$ of (1.1) such that $z(0) = z_0, z(1) = z_1$.

2. THE VARIATIONAL SETTING AND THE PENALIZATION ARGUMENT

Let $\mathcal{M}$ be an orthogonal splitting Lorentzian manifold. It is well known that $\mathcal{M}_0$ can be regarded as a submanifold of $\mathbf{R}^N$ for N sufficiently large and the metric $\langle\cdot,\cdot\rangle$ as the restriction to $\mathcal{M}_0$ of the Euclidean metric of $\mathbf{R}^N$. Thus we can define

$$H^1([0,1],\mathcal{M}_0) = \left\{x \in H^1([0,1],\mathbf{R}^N) \mid x([0,1]) \subset \mathcal{M}_0\right\},$$

where $H^1([0,1],\mathbf{R}^N)$ is the usual Sobolev space. We consider the set of closed curves on $\mathcal{M}_0$

$$\Lambda^1(\mathcal{M}_0) = \left\{x \in H^1([0,1],\mathcal{M}_0) \mid x(0) = x(1)\right\}.$$

It is an infinite dimensional Hilbert manifold (cf. e.g. Klingenberg [17]) whose tangent space at $x \in \Lambda^1(\mathcal{M}_0)$ can be identified with

$$T_x\Lambda^1(\mathcal{M}_0) = \left\{\xi \in H^1([0,1],T\mathcal{M}_0) \mid \xi(0) = \xi(1)\right\}.$$

Fix $k \in \mathbf{N}$ and take $T > 0$. We define the set

$$W_k = \left\{t \in H^1([0,1],\mathbf{R}) \mid t(0) = 0 \;\; t(1) = kT\right\}.$$

Denoting by t_k^* the segment joining 0 and kT, i.e.

$$t_k^*(s) = kTs \quad \forall s \in [0,1], \tag{2.1}$$

there results

$$W_k = H_0^1([0,1],\mathbf{R}) + t_k^*$$

where

$$H_0^1([0,1],\mathbf{R}) = \left\{t \in H^1([0,1],\mathbf{R}) \mid t(0) = 0 = t(1)\right\}.$$

Thus W_k is a closed affine submanifold of $H_0^1([0,1],\mathbf{R})$ whose tangent space at any point is $H_0^1([0,1],\mathbf{R})$. Finally, we set

$$Z_k = \Lambda^1(\mathcal{M}_0) \times W_k$$

which is a manifold of curves whose tangent space at any $z = (x,t) \in Z_k$ is

$$T_zZ_k = T_x\Lambda^1(\mathcal{M}_0) \times H_0^1([0,1],\mathbf{R}).$$

Consider the functional $f_k : Z_k \to \mathbf{R}$ defined by

$$f_k(z) = \int_0^1 \left(\frac{1}{2}\langle\dot z,\dot z\rangle_L - V(z)\right) ds \quad \forall z \in Z_k, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_L$ is the metric defined in (1.2). Under our assumptions on the coefficients of the metric and V, f_k is smooth and verifies the following variational principle.

Proposition 2.1. *Let α, β, V satisfy (1.4) and let $z = (x, t) \in Z_k$ be a critical point of f_k such that $\dot{t}(0)\,\dot{t}(1) \geq 0$. Then z is a solution of (1.1) and*

$$\dot{x}(1) = \dot{x}(0) \quad \dot{t}(1) = \dot{t}(0).$$

Proof. By classical regularization methods each critical point $z \in Z_k$ of f_k is smooth and is a solution of (1.1). Moreover, integrating by parts, for any $\zeta = (\xi, \tau) \in T_z Z_k$

$$0 = f_k'(z)[\zeta] = \int_0^1 \Big(\langle \dot{z}, \dot{\zeta} \rangle_L - \langle \nabla_L V(z), \zeta \rangle_L \Big)\, ds$$

$$= \int_0^1 \langle -D_s \dot{z} - \nabla_L V(z), \zeta \rangle_L ds + \langle \dot{z}(1), \zeta(1) \rangle_L - \langle \dot{z}(0), \zeta(0) \rangle_L$$

$$= \langle \dot{z}(1), \zeta(1) \rangle_L - \langle \dot{z}(0), \zeta(0) \rangle_L.$$

Then, as $\tau(0) = 0 = \tau(1)$

$$\langle \alpha(z(1))\dot{x}(1), \xi(1) \rangle = \langle \alpha(z(0))\dot{x}(0), \xi(0) \rangle,$$

and, by the periodicity assumption on α, we deduce $\dot{x}(1) = \dot{x}(0)$. Moreover, by the expression of E_z and (1.4), we also get $\dot{t}(1)^2 = \dot{t}(0)^2$, hence, taking into account that $\dot{t}(0)\ \dot{t}(1) \geq 0$, the proof is complete. □

Corollary 2.2. *Let α, β, V satisfy (1.4) and $V > 0$. If $z = (x, t) \in Z_k$ is a critical point of f_k such that $E_z < 0$, then z is a solution of (1.5).*

Proof. Observe that, if for some $s \in [0, 1]$ $\dot{t}(s) = 0$, then (cf. (1.3))

$$E_z = \frac{1}{2}\langle \alpha(z(s))\dot{x}(s), \dot{x}(s) \rangle + V(z(s)) > 0$$

which is a contradiction. Then $\dot{t}$ is strictly positive or strictly negative on $[0, 1]$. As $t \in W_k$ and $T > 0$, it must be $\dot{t} > 0$, so we can apply Proposition 2.1. □

The next proposition shows that distinct solutions of (1.5) have different supports.

Proposition 2.3. *Let V be strictly positive and $z_i = (x_i, t_i)$, $i = 1, 2$ be two solutions of (1.5) such that $z_1 \neq z_2$ and $t_i(0) = 0$, $i = 1, 2$. Then z_1 and z_2 are geometrically distinct.*

Proof. Assume by contradiction that $\sigma \in C^2([0, 1], [0, 1])$ exists such that $z_2(s) = z_1(\sigma(s))$ for any $s \in [0, 1]$. As in Corollary 2.2, t_i is strictly increasing so it must be

$$\sigma([0, 1]) = [0, 1] \qquad \dot{\sigma} > 0. \tag{2.3}$$

Then $s_0 \in [0,1]$ exists such that $\dot\sigma(s_0) = 1$ which easily implies $E_{z_1} = E_{z_2}$ (cf. (1.3)). So σ satisfies $\dot\sigma^2(s) = 1$ for any $s \in [0,1]$. By (2.3), we have $\dot\sigma = 1$ and $\sigma(0) = 0$. Thus $\sigma(s) = s$ for any $s \in [0,1]$ which is a contradiction because $z_1 \neq z_2$. $\square$

Here on we will assume that all the hypothesis of Theorem 1.2 hold.

By Corollary 2.2, it is clear that our problem is reduced to prove the existence of critical points on Z_k of the functional f_k defined in (2.2). Unfortunately f_k does not satisfy the Palais–Smale condition. We recall that if (X, h) is a Hilbert manifold and $F : X \to \mathbf{R}$ is a C^1 functional, F is said to satisfy the *Palais–Smale condition* at a level $c \in \mathbf{R}$ if every sequence $(x_n)_{n\in\mathbf{N}} \subset X$ such that

$$\lim_{n\to+\infty} F(x_n) = c \qquad \lim_{n\to+\infty} \|F'(x_n)\|_* = 0 \tag{2.4}$$

has a converging subsequence. (The norm $\|\cdot\|_*$ is the norm induced by h on $T_{x_n}X$.)

This condition is essential in critical point theory, so in order to deal with functionals verifying it, we will use a penalization argument. The non–compactness of $\mathcal{M}_0$ and the fact that our curves are closed in the spatial component require the use of two penalization terms: one in order to obtain bounded Palais–Smale sequences (cf. e.g. Benci et al [7]) and the second one in order to prevent the existence of Palais–Smale sequences with constant spatial components going to infinity (cf. e.g. Benci et al [6]).

To this aim, for any $\epsilon > 0$ consider the function $\psi_\epsilon : [0, +\infty[\to \mathbf{R}$ defined by

$$\psi_\epsilon(s) = \begin{cases} 0 & 0 \leq s \leq 1/\epsilon \\ \displaystyle\sum_{n=3}^{+\infty} \frac{1}{n!}\left(s - \frac{1}{\epsilon}\right)^n & s > 1/\epsilon. \end{cases} \tag{2.5}$$

Note that ψ_ϵ verifies the following properties: for any $\epsilon > 0$ two positive constants a_ϵ, b_ϵ exist such that

$$\psi'_\epsilon(s) \geq \psi_\epsilon(s) \geq a_\epsilon s - b_\epsilon \quad s\psi'_\epsilon(s) \geq \psi_\epsilon(s) \quad \forall s \geq 0 \tag{2.6}$$

and, if $0 < \epsilon \leq \epsilon'$,

$$\psi_\epsilon(s) \leq \psi_{\epsilon'}(s) \quad \psi'_\epsilon(s) \leq \psi'_{\epsilon'}(s) \quad \forall s \geq 0. \tag{2.7}$$

Let us consider the following family of penalized functionals $f_{k,\epsilon} : Z_k \to \mathbf{R}$,

$$f_{k,\epsilon}(z) = f_k(z) - \psi_\epsilon(\|\dot t\|^2) + \int_0^1 U_\epsilon(x)ds$$

$$= \int_0^1 \left(\frac{1}{2}\langle \dot z, \dot z\rangle_L - V(z)\right) ds - \psi_\epsilon(\|\dot t\|^2) + \int_0^1 U_\epsilon(x)ds \tag{2.8}$$

for any $z = (x, t) \in Z_k$, where

$$\|\dot{t}\|^2 = \int_0^1 \dot{t}^2 ds$$

and, for U as in assumption (H_6),

$$U_\epsilon(x) = \psi_\epsilon(U(x)) \quad \forall x \in \mathcal{M}_0.$$

We end this section proving that each $f_{k,\epsilon}$ satisfies the Palais–Smale condition. To this aim, we will use the following lemma, whose proof can be found in Lemma 2.2, Benci et al [6].

Lemma 2.4. *Let U, x_0, μ be as in assumption (H_6). Then $C_1, C_2, C_3 > 0$ exist such that for any $x \in \mathcal{M}_0$*

$$\langle \nabla U(x), \nabla U(x)\rangle^{1/2} \geq \mu d(x, x_0) - C_1 \tag{2.9}$$

$$U(x) \geq \frac{\mu}{2} d^2(x, x_0) - C_2 d(x, x_0) - C_3. \tag{2.10}$$

Proposition 2.5. *For any $\epsilon > 0$, the functional $f_{k,\epsilon}$ defined in (2.8) satisfies the Palais–Smale condition at any $c \in \mathbf{R}$.*

Proof. Let $c \in \mathbf{R}$ and $(z_n)_{n\in\mathbf{N}} \subset Z_k$ be such that

$$\lim_{n\to+\infty} f_{k,\epsilon}(z_n) = c \qquad \lim_{n\to+\infty} \|f'_{k,\epsilon}(z_n)\|_* = 0. \tag{2.11}$$

If $z_n = (x_n, t_n)$, setting $\tau_n = t_n - t_k^* \in H_0^1([0,1], \mathbf{R})$, by (2.11) an infinitesimal sequence $(\delta_n)_{n\in\mathbf{N}}$ exists, such that, from (1.10) and (1.8), for any $n \in \mathbf{N}$:

$$\delta_n \|\dot{t}_n - \dot{t}_k^*\| = f'_{k,\epsilon}(z_n)[(0, \tau_n)]$$

$$= \int_0^1 \left(\frac{1}{2}\langle \alpha_t(z_n)\dot{x}_n, \dot{x}_n\rangle - \frac{1}{2}\beta_t(z_n)\dot{t}_n^2 - V_t(z_n)\right)(t_n - t_k^*)ds - \int_0^1 \beta(z_n)\dot{t}_n^2 ds$$

$$+kT \int_0^1 \beta(z_n)\dot{t}_n ds - 2\psi'_\epsilon(\|\dot{t}_n\|^2) \int_0^1 \dot{t}_n^2 ds + 2kT\psi'_\epsilon(\|\dot{t}_n\|^2) \int_0^1 \dot{t}_n ds$$

$$\leq \Lambda \left(\frac{1}{2}\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle ds + \frac{1}{2}\|\dot{t}_n\|^2 + 1\right) \|t_n - t_k^*\|_\infty - \nu\|\dot{t}_n\|^2$$

$$+kTN \int_0^1 |\dot{t}_n| ds - 2\psi'_\epsilon(\|\dot{t}_n\|^2)\|\dot{t}_n\|^2 + 2k^2T^2\psi'_\epsilon(\|\dot{t}_n\|^2). \tag{2.12}$$

Moreover, by (2.11) and (2.8), (1.6), (1.8), (1.9) imply

$$\frac{\Lambda}{2}\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle \leq K_1 + \frac{N}{2}\|\dot{t}_n\|^2 + M + \psi_\epsilon(\|\dot{t}_n\|^2) \tag{2.13}$$

for some $K_1 > 0$. Now observe that, as

$$\|t_n - t_k^*\|_\infty \leq \|\dot{t}_n - \dot{t}_k^*\| \tag{2.14}$$

by (2.12), (2.13), (2.14)

$$\delta_n \|\dot{t}_n - t_k^*\| + \nu \|\dot{t}_n\|^2 + 2\psi'_\epsilon(\|\dot{t}_n\|^2)\|\dot{t}_n\|^2$$

$$\leq \left(K_2 + K_3\|\dot{t}_n\|^2 + K_4\psi_\epsilon(\|\dot{t}_n\|^2)\right)\|\dot{t}_n - \dot{t}_k^*\| + kTN\|\dot{t}_n\| + 2k^2T^2\psi'_\epsilon(\|\dot{t}_n\|^2) \tag{2.15}$$

for some $K_2, K_3, K_4 > 0$. By (2.15) and the properties (2.6), (2.7), $K_5 > 0$ exists such that

$$\|\dot{t}_n\| \leq K_5 \quad \forall n \in \mathbf{N} \tag{2.16}$$

and, by (2.13), $K_6 > 0$ exists such that

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds \leq K_6 \quad \forall n \in \mathbf{N}. \tag{2.17}$$

Now, we claim that

$$\sup\{d(x_n(s), x_0) \mid s \in [0,1]\ n \in \mathbf{N}\} < +\infty. \tag{2.18}$$

Indeed, if by contradiction (2.18) is not true, by (2.17) also

$$\lim_{n\to+\infty} \inf_{s\in[0,1]} d(x_n(s), x_0) = +\infty,$$

then, from (2.10) and the definition of U_ϵ,

$$\lim_{n\to+\infty} \int_0^1 U_\epsilon(x_n) ds = +\infty. \tag{2.19}$$

As, by (1.8), (1.9) and (2.16)

$$f_{k,\epsilon}(z_n) \geq -\frac{N}{2}K_5^2 - M - \psi_\epsilon(K_5^2) + \int_0^1 U_\epsilon(x_n) ds$$

also

$$\lim_{n\to+\infty} f_{k,\epsilon}(z_n) = +\infty$$

which is in contradiction with (2.11). So (2.18) is proved.

By (2.16), the Poincaré inequality, (2.17) and (2.18), we deduce that $(z_n)_{n\in\mathbf{N}}$ is bounded in Z_k. Then it admits a subsequence weakly and uniformly convergent to a curve $z \in Z_k$. Standard arguments imply that the convergence is strong (see e.g. Benci et al [5]). □

3. SOME A PRIORI ESTIMATES

In this section we will prove some estimates about the critical points of the penalized functionals $f_{k,\epsilon}$ introduced in Section 2. First, we observe that, if $z = (x, t) \in Z_k$ is a critical point of $f_{k,\epsilon}$, z is smooth and satisfies the following system of differential equations:

$$\begin{cases} D_s(\alpha(z)\dot{x}) - \frac{1}{2}\langle \dot{x}, \dot{x}\rangle \nabla\alpha(z) + \frac{1}{2}\dot{t}^2\nabla\beta(z) + \nabla V(z) - \nabla U_\epsilon(x) = 0 \\ \frac{d}{ds}(\beta(z)\dot{t}) + \frac{1}{2}\langle \alpha_t(z)\dot{x}, \dot{x}\rangle - \frac{1}{2}\beta_t(z)\dot{t}^2 - V_t(z) + 2\psi'_\epsilon(\|\dot{t}\|^2)\ddot{t} = 0 \end{cases} \tag{3.1}$$

where here D_s denotes the covariant derivative induced by the Riemannian metric $\langle\cdot,\cdot\rangle$. Our main result in this section is Proposition 3.5, where we prove that, if ϵ is sufficiently small, each critical point of $f_{k,\epsilon}$ is a critical point of the unperturbed functional f_k. This will be achieved by various lemmas.

Lemma 3.1. *Let $k \in \mathbf{N}$. Then $D_1 > 0$ exists such that for any $\epsilon \in]0, 1]$ and $z = (x, t) \in Z_k$ critical point of $f_{k,\epsilon}$ it results*

$$\|t\|_\infty \le D_1.$$

Proof. By (1.12) and the first inequality in (1.13), for any $\delta > 0$, a constant $M_\delta > 0$ exists such that

$$\begin{aligned} &\frac{1}{2}\langle \alpha_t(x, t)\xi, \xi\rangle \le \delta \quad \forall t \ge M_\delta,\ \forall x \in \mathcal{M}_0,\ \forall \xi \in T_x\mathcal{M}_0 \\ &V_t(x, t) \ge -\delta \qquad\qquad \forall t \ge M_\delta,\ \forall x \in \mathcal{M}_0 \end{aligned} \tag{3.2}$$

and by the second inequality in (1.13)

$$V_t(x, t) \le \delta \qquad \forall t \le -M_\delta,\ \forall x \in \mathcal{M}_0. \tag{3.3}$$

We can choose

$$\delta < \min\left\{1, \left(\frac{\nu}{2 + \frac{\Lambda}{2}}\right)^2\right\} \tag{3.4}$$

and M_δ so that $M_\delta > kT$.

We prove that, for any $\epsilon \in]0, 1]$, each critical point $z = (x, t) \in Z_k$ of $f_{k,\epsilon}$ verifies

$$\|t\|_\infty \le M_\delta + 1. \tag{3.5}$$

Assume by contradiction that, for some $\epsilon \in]0, 1]$, a critical point $z = (x, t) \in Z_k$ of $f_{k,\epsilon}$ exists such that (3.5) is not true. If $\|t\|_\infty$ is achieved at a point $s_1 \in]0, 1[$, we suppose at first that

$$\|t\|_\infty = t(s_1) > M_\delta + 1. \tag{3.6}$$

Consider the set

$$A = \{s \in [0,1] \mid t(s) > M_\delta\}$$

which is an open subset of $[0,1]$. Note that s_1 is an interior point of A. Define the function

$$u(s) = \beta(z(s))\dot{t}(s) + 2\psi'_\epsilon(\|\dot{t}\|^2)\dot{t}(s) \quad \forall s \in [0,1]. \tag{3.7}$$

By the second equation of (3.1), u verifies

$$-\dot{u} = \frac{1}{2}\langle \alpha_t(z)\dot{x}, \dot{x}\rangle - \frac{1}{2}\beta_t(z)\dot{t}^2 - V_t(z). \tag{3.8}$$

By (1.10) and (3.2) for any $s \in A$

$$-\dot{u}(s) \leq 2\delta + \frac{\Lambda}{2}\dot{t}^2(s). \tag{3.9}$$

Let B be the maximal component of A such that $s_1 \in B$ and $\overline{s} = \inf B$ so that

$$\overline{s} < s_1, \quad t(\overline{s}) = M_\delta.$$

Applying the mean value theorem to t in $[\overline{s}, s_1]$, $r \in]\overline{s}, s_1[$ exists such that $\dot{t}(r) \geq 1$. Thus, we can consider

$$s_0 = \sup\left\{s \in [r, s_1] \mid \dot{t}(s) = \sqrt{\delta}\right\}.$$

We have

$$\dot{t}(s_0) = \sqrt{\delta}, \quad 0 \leq \dot{t}(s) \leq \sqrt{\delta} \quad \forall s \in [s_0, s_1].$$

Integrating (3.9) in $[s_0, s_1]$ and observing that $u(s_1) = 0$ (since $\dot{t}(s_1) = 0$) we obtain

$$u(s_0) = u(s_0) - u(s_1) = -\int_{s_0}^{s_1} \dot{u}\,ds \leq 2\delta + \frac{\Lambda}{2}\delta.$$

Then

$$\nu\sqrt{\delta} \leq \beta(z(s_0))\sqrt{\delta} + 2\psi'_\epsilon(\|\dot{t}\|^2)\sqrt{\delta} \leq \left(2 + \frac{\Lambda}{2}\right)\delta$$

which is in contradiction with (3.4). Hence the lemma is proved when (3.6) holds. If

$$\|t\|_\infty = -t(s_1) > M_\delta + 1$$

the proof is similar, using (1.11) and (3.3). □

Lemma 3.2. *Let $k \in \mathbf{N}$. Then $D_2 > 0$ exists such that for any $\epsilon \in]0,1]$ and $z = (x,t) \in Z_k$ critical point of $f_{k,\epsilon}$ it results*

$$\|\dot{t}\|_\infty \leq D_2.$$

Proof. For any $\epsilon \in]0,1]$ and $z = (x,t)$ critical point of $f_{k,\epsilon}$, consider the function u defined in (3.7). By (1.11), (1.10) and (1.8) it is

$$\begin{aligned} \dot{u} &= -\frac{1}{2}\langle \alpha_t(z)\dot{x}, \dot{x}\rangle + \frac{1}{2}\beta_t(z)\dot{t}^2 + V_t(z) \\ &\leq \frac{1}{2}\frac{\Lambda}{\nu}\beta(z)\dot{t}^2 + \Lambda \leq \frac{1}{2}\frac{\Lambda}{\nu}u\dot{t} + \Lambda. \end{aligned} \tag{3.10}$$

Consider the set

$$B = \left\{ s \in [0,1] \mid |\dot{t}(s)| > 1 \right\}$$

and $]s_0, s_1[$ be a maximal component of B. We can assume that $\dot{t}(s) > 0$ in $]s_0, s_1[$ (the other case is similar). Note that, by (1.8)

$$u(s) \geq \beta(z)\dot{t} \geq \nu \quad \forall s \in]s_0, s_1[$$

then, by (3.10)

$$\frac{\dot{u}}{u} \leq \frac{1}{2}\frac{\Lambda}{\nu}\dot{t} + \frac{\Lambda}{u} \leq \frac{1}{2}\frac{\Lambda}{\nu}\dot{t} + \frac{\Lambda}{\nu} \quad \forall s \in]s_0, s_1[. \tag{3.11}$$

Integrating (3.11) from s_0 to $s \in]s_0, s_1[$ we obtain

$$\log u(s) - \log u(s_0) \leq \frac{1}{2}\frac{\Lambda}{\nu}\left(t(s) - t(s_0)\right) + \frac{\Lambda}{\nu}(s - s_0) \leq \frac{\Lambda}{\nu}D_1 + \frac{\Lambda}{\nu} = C_1$$

where D_1 is as in Lemma 3.1. Then

$$u(s) \leq u(s_0)e^{C_1} \quad \forall s \in]s_0, s_1[. \tag{3.12}$$

As $\dot{t}(s_0) = 1$, by (3.12) and (1.8) we get

$$\begin{aligned} \nu\dot{t}(s) &\leq C_2\left[\beta(z(s_0)) + 2\psi'_\epsilon(\|\dot{t}\|^2)\right] - 2\psi'_\epsilon(\|\dot{t}\|^2)\dot{t}(s) \\ &\leq C_2 N + 2\psi'_\epsilon(\|\dot{t}\|^2)(C_2 - \dot{t}(s)), \end{aligned}$$

where $C_2 = e^{C_1}$. So if $\dot{t}(s) \geq C_2$

$$\dot{t}(s) \leq \frac{C_2 N}{\nu}.$$

Thus

$$\|\dot{t}\|_\infty \leq \max\left\{1, C_2, \frac{C_2 N}{\nu}\right\}$$

and the proof is complete. □

The following remark will be useful in the sequel.

Remark 3.3. Let $\alpha(z)$ be the linear operator defined at (1.2). Assumptions (1.7) in (H_2) and (1.10) in (H_4) imply that

$$\langle \alpha(z)\xi, \xi \rangle \le (\lambda_2 + \Lambda|t|)\langle \xi, \xi \rangle \tag{3.13}$$

for any $z = (x, t) \in \mathcal{M}$, $\xi \in T_x\mathcal{M}_0$. Then, from (3.13) we easily get

$$\langle \alpha^{-1}(z)\xi, \alpha^{-1}(z)\xi \rangle \ge \frac{1}{(\lambda_2 + \Lambda|t|)^2}\langle \xi, \xi \rangle \tag{3.14}$$

and finally by (1.6) and (3.14)

$$\langle \alpha^{-1}(z)\xi, \xi \rangle \ge \frac{\lambda_1}{(\lambda_2 + \Lambda|t|)^2}\langle \xi, \xi \rangle \quad \forall z = (x, t) \in \mathcal{M}, \xi \in T_x\mathcal{M}_0. \tag{3.15}$$

Set now, for any $k \in \mathbf{N}$

$$d_0(k) = -\frac{1}{2}Nk^2T^2 - M, \tag{3.16}$$

where N and M are as in (1.8), (1.9).

Lemma 3.4. *Let $k \in \mathbf{N}$, $M_0 \in \mathbf{R}$ and $\delta > 0$. Then $\epsilon_0, D_3 > 0$ exist such that for any $\epsilon \in]0, \epsilon_0]$ and for any $z = (x, t)$ critical point of $f_{k,\epsilon}$ such that*

$$\delta + d_0(k) \le f_{k,\epsilon}(z) \le M_0, \tag{3.17}$$

it results

$$d(x(s), x_0) \le D_3 \quad \forall s \in [0, 1].$$

Proof. By Lemma 3.2 and (2.5), if $\epsilon < 1/D_2^2$, then

$$\psi_\epsilon(\|\dot{t}\|^2) = 0. \tag{3.18}$$

For such ϵ, by (2.8), (3.18), (1.8) and (1.9), for any $z = (x, t) \in Z_k$ critical point of $f_{k,\epsilon}$ satisfying (3.17) it results

$$\frac{\lambda_1}{2}\int_0^1 \langle \dot{x}, \dot{x} \rangle ds \le M_0 + \frac{1}{2}N\|\dot{t}\|^2 + M \le M_0 + \frac{1}{2}ND_2^2 + M. \tag{3.19}$$

Assume by contradiction the existence of an infinitesimal sequence $(\epsilon_n)_{n\in\mathbf{N}}$ and a sequence $(z_n = (x_n, t_n))_{n\in\mathbf{N}}$ of critical points of f_{k,ϵ_n} satisfying (3.17), such that

$$\sup\{d(x_n(s), x_0) \mid n \in \mathbf{N}\ s \in [0, 1]\} = +\infty. \tag{3.20}$$

By (3.19), $F_1 > 0$ exists such that for n sufficiently large

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds \le F_1. \tag{3.21}$$

By (3.20) and (3.21) also

$$\lim_{n\to+\infty}\inf_{s\in[0,1]} d(x_n(s),x_0)=+\infty. \tag{3.22}$$

We set, for simplicity, $f_{k,\epsilon_n}=f_n$, $\psi_{\epsilon_n}=\psi_n$ and we define the function

$$u_n(s)=U(x_n(s)) \quad \forall s\in[0,1].$$

As z_n is a smooth critical point of f_n, we easily get $\dot{x}_n(0)=\dot{x}_n(1)$ (cf. Proposition 2.1), so

$$\dot{u}_n(0)=\dot{u}_n(1). \tag{3.23}$$

Moreover, by (3.22) and (1.14), for n large enough and for any $s\in[0,1]$ we have

$$\begin{aligned}\ddot{u}_n(s)&=H^U(x_n(s))[\dot{x}_n(s),\dot{x}_n(s)]+\langle D_s\dot{x}_n(s),\nabla U(x_n(s))\rangle\\ &\ge\mu\langle\dot{x}_n(s),\dot{x}_n(s)\rangle+\langle D_s\dot{x}_n(s),\nabla U(x_n(s))\rangle.\end{aligned} \tag{3.24}$$

Then, by (3.23), (3.24) and the first equation of (3.1), we obtain

$$\begin{aligned}0&=\int_0^1\ddot{u}_n(s)ds\ge\mu\int_0^1\langle\dot{x}_n,\dot{x}_n\rangle ds+\frac{1}{2}\int_0^1\langle\dot{x}_n,\dot{x}_n\rangle\langle\alpha^{-1}(z_n)\nabla\alpha(z_n),\nabla U(x_n)\rangle ds\\ &-\frac{1}{2}\int_0^1\dot{t}_n^2\langle\alpha^{-1}(z_n)\nabla\beta(z_n),\nabla U(x_n)\rangle ds-\int_0^1\langle\nabla\alpha(z_n),\dot{x}_n\rangle\langle\alpha^{-1}(z_n)\dot{x}_n,\nabla U(x_n)\rangle ds\\ &-\int_0^1\dot{t}_n\langle\alpha^{-1}(z_n)\alpha_t(z_n)\dot{x}_n,\nabla U(x_n)\rangle ds-\int_0^1\langle\alpha^{-1}(z_n)\nabla V(z_n),\nabla U(x_n)\rangle ds\\ &+\int_0^1\psi_n'(U(x_n))\langle\alpha^{-1}(z_n)\nabla U(x_n),\nabla U(x_n)\rangle ds.\end{aligned} \tag{3.25}$$

Note that, by the invertibility of α (see Remarks 1.1), it is easy to see that

$$\|\alpha^{-1}(z)\|\le\frac{1}{\lambda_1} \quad \forall z\in\mathcal{M}$$

so that, by (3.25), (3.22), (3.21), Lemma 3.2 and assumption (H_7) of Theorem 1.2, we have

$$0\ge\mu\int_0^1\langle\dot{x}_n,\dot{x}_n\rangle ds+a_n+\int_0^1\psi_n'(U(x_n))\langle\alpha^{-1}(z_n)\nabla U(x_n),\nabla U(x_n)\rangle ds, \tag{3.26}$$

where $(a_n)_{n\in\mathbf{N}}$ is an infinitesimal sequence.

Now, using the first inequality of (3.17), Lemma 3.1 and (3.13), we get

$$\begin{aligned}&\frac{1}{2}(\lambda_2+\Lambda D_1)\int_0^1\langle\dot{x}_n,\dot{x}_n\rangle ds\\ &\ge\delta+d_0(k)+\frac{1}{2}\int_0^1\beta(z_n)\dot{t}_n^2ds+\int_0^1 V(z_n)ds-\int_0^1\psi_n(U(x_n))ds.\end{aligned} \tag{3.27}$$

By (1.8) and (1.9), for a fixed $\gamma > 0$, for n sufficiently large

$$\beta(z_n) \geq N - \frac{\gamma}{k^2T^2} \qquad V(z_n) \geq M - \gamma, \tag{3.28}$$

so that

$$\int_0^1 \beta(z_n)\dot{t}_n^2 ds \geq \left(N - \frac{\gamma}{k^2T^2}\right)\left(\int_0^1 \dot{t}_n ds\right)^2 = Nk^2T^2 - \gamma. \tag{3.29}$$

By (3.16), (3.27), (3.28), (3.29) and choosing $\gamma < \delta/3$

$$\begin{aligned}\frac{1}{2}(\lambda_2 + \Lambda D_1)\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle ds &\geq \delta - \frac{3}{2}\gamma - \int_0^1 \psi_n(U(x_n))ds \\ &\geq \frac{1}{2}\delta - \int_0^1 \psi_n(U(x_n))ds. \end{aligned}\tag{3.30}$$

Observe also that, by (3.15), Lemma 3.1, (3.22) and (2.9) of Lemma 2.4, for n sufficiently large we have

$$\langle \alpha^{-1}(z_n)\nabla U(x_n), \nabla U(x_n)\rangle \geq \frac{\lambda_1}{(\lambda_2 + \Lambda D_1)^2}\langle \nabla U(x_n), \nabla U(x_n)\rangle \geq \frac{2\mu}{\lambda_2 + \Lambda D_1}. \tag{3.31}$$

By (3.26), (3.30), (3.31) and (2.5), we obtain, for n large enough

$$\begin{aligned} 0 &\geq \frac{\mu}{\lambda_2 + \Lambda D_1}\left(\delta - 2\int_0^1 \psi_n(U(x_n))ds\right) + a_n + \frac{2\mu}{\lambda_2 + \Lambda D_1}\int_0^1 \psi_n'(U(x_n))ds \\ &= \frac{\mu\delta}{\lambda_2 + \Lambda D_1} + a_n + \frac{2\mu}{\lambda_2 + \Lambda D_1}\int_0^1 \left(\psi_n'(U(x_n)) - \psi_n(U(x_n))\right) ds \geq \frac{\mu\delta}{\lambda_2} + a_n \end{aligned}\tag{3.32}$$

Finally, taking the limit as n goes to $+\infty$ in (3.32), we get a contradiction. □

By Lemmas 3.2, 3.4 and the expression of the penalization (see (2.5),(2.8)) we easily get the following proposition.

Proposition 3.5. *Let $M_0 \in \mathbf{R}$ and $\delta > 0$. Then for any $k \in \mathbf{N}$ $\epsilon_1 > 0$ exists such that, for any $\epsilon \in]0, \epsilon_1]$, each $z = (x,t) \in Z_k$ critical point of $f_{k,\epsilon}$ satisfying (3.17) is a critical point of f_k.*

4. THE RELATIVE CATEGORY

In order to prove Theorem 1.2, we will apply a result in Candela et al [8] about the existence and multiplicity of critical points for strongly indefinite functionals. It is based on the notion of relative category and its main properties which are recalled in the sequel. We refer to Fournier and Willem [11], Fournier et al [12], Szulkin [25], Candela et al [8] for more details.

Definition 4.1. Let Y, W be two closed subsets of a topological space Z. The category of W in Z relative to Y, denoted by $\text{cat}_{Z,Y}W$, is the minimum natural integer h (possibily $+\infty$) such that there exist $h+1$ closed and contractible subsets $W_0, \dots, W_h$, covering W and $h+1$ functions $f_i \in C([0,1] \times W_i, Z)$, $i = 0, \dots, h$, such that

$$\begin{aligned}
&f_i(0,w) = w && \forall w \in W_i, i = 0, \dots, h \\
&f_0(1,w) \in Y && \forall w \in W_0 \\
&f_0(s,y) \in Y && \forall y \in W_0 \cap Y, s \in [0,1] \\
&f_i(1,w) = w_i && \forall w \in W_i \text{ for some } w_i \in Z, i = 1, \dots, h.
\end{aligned}$$

We observe that, denoted by $\text{cat}_Z W$ the classical Ljusternik–Schnirelmann category (cf. e.g. Masiello [18]) of W in Z it results

$$\text{cat}_Z W = \text{cat}_{Z,\emptyset} W.$$

Proposition 4.2. *Let W_1, W_2, Y be two closed subsets of a topological space Z.*

(i) *If $W_1 \subset W_2$ then $\text{cat}_{Z,Y}W_1 \le \text{cat}_{Z,Y}W_2$;*
(ii) $\text{cat}_{Z,Y}(W_1 \cup W_2) \le \text{cat}_{Z,Y}W_1 + \text{cat}_Z W_2$;
(iii) $\text{cat}_{Z,Y}W_1 \le \text{cat}_Z W_1$.

Moreover, by Definition 4.1, the following proposition easily follows.

Proposition 4.3. *Let Z be a topological space and C, Λ be two subsets of Z such that C is a strong deformation retract of $Z \setminus \Lambda$. Then*

$$\text{cat}_{Z,C}\,(Z \setminus \Lambda) = 0.$$

We will use the following abstract result (for the proof see Theorem 1.5, Candela et al [8]).

Theorem 4.4. *Let Z be a C^2 complete Riemannian manifold modelled on a Hilbert space and $F \in C^1(Z, \mathbf{R})$. Assume that there exist two subsets Λ and C of Z such that C is a closed strong deformation retract of $Z \setminus \Lambda$ and*

$$\inf_{z \in \Lambda} F(z) > \sup_{z \in C} F(z) \qquad \text{cat}_{Z,C} Z > 0. \tag{4.1}$$

Assume also that F satisfies the Palais–Smale condition. Define, for any $l \in \mathbf{N}$, $1 \le l \le \text{cat}_{Z,C}Z$,

$$c_l = \inf_{B \in \Gamma_l} \sup_{z \in B} F(z)$$

where

$$\Gamma_l = \{B \subset Z \mid B \text{ is closed, } \text{cat}_{Z,C}B \ge l\}.$$

Then each c_l is a critical value of F such that

$$c_l \geq \inf_{z \in \Lambda} F(z).$$

In order to apply Theorem 4.4, we need information about the relative category of the spaces of curves involved in our proof. To this aim, we will use the following result of Fadell and Husseini (see Corollary 4.6, Fadell and Husseini [10], Corollary 1.5, Proposition 3.1 in Fadell and Husseini [9]).

Proposition 4.5. *Let $\mathcal{M}_0$ be Riemannian manifold such that assumption (H_1) holds. Let D^m denote the unit disk in $\mathbf{R}^m$ and S^m its boundary. Then*

$$\mathrm{cat}_{\Lambda^1(\mathcal{M}_0)\times D^m, \Lambda^1(\mathcal{M}_0)\times S^m}(\Lambda^1(\mathcal{M}_0) \times D^m) = +\infty$$

and $\Lambda^1(\mathcal{M}_0) \times D^m$ contains compact subsets whose relative category in $\Lambda^1(\mathcal{M}_0) \times D^m$ with respect to $\Lambda^1(\mathcal{M}_0) \times S^m$ is arbitrarily large.

5. PROOF OF THEOREM 1.2

Our aim is now to apply Theorem 4.4 to the functionals $f_{k,\epsilon}$ and then, thanks to the a priori estimates of Section 3, to obtain critical points of f_k. We will make a Galerkin approximation on Z_k.

Indeed for any $m \in \mathbf{N}$, define

$$H_m = \mathrm{span}\,\{\sin(j\pi s) \mid j = 1, \ldots, m\}$$

and

$$W_{k,m} = H_m + t_k^*,$$

where t_k^* is as in (2.1). We define also

$$Z_{k,m} = \Lambda^1(\mathcal{M}_0) \times W_{k,m} \quad \Lambda_k = \Lambda^1(\mathcal{M}_0) \times \{t_k^*\} \quad f_{k,\epsilon}^m = f_{k,\epsilon}|_{Z_{k,m}}$$

(see Section 2). From now on, for a fixed k sufficiently large (which will be better defined in the sequel) we will consider

$$\epsilon < \frac{1}{k^2 T^2} = \epsilon(k) \tag{5.1}$$

so that, if $z = (x, t_k^*) \in \Lambda_k$, by (2.8), (1.8), (1.9) we easily obtain (cf. also (3.16))

$$f_{k,\epsilon}^m(z) \geq -\frac{1}{2} N k^2 T^2 - M = d_0(k). \tag{5.2}$$

The following lemma will define the set which plays the role of C in Theorem 4.4.

Lemma 5.1. *Set*

$$d_1(k) = 2d_0(k).$$

A continuous map $\rho_k : \Lambda^1(\mathcal{M}_0) \to]0, +\infty[$ *exists such that, if* $z = (x,t) \in Z_{k,m}$

$$\|\dot{t} - \dot{t}_k^*\| = \rho_k(x) \Rightarrow f_{k,\epsilon}^m(z) \leq d_1(k).$$

Proof. For any $z = (x,t) \in Z_{k,m}$, we set $t_m = t - t_k^*$. Observe that

$$\|t\|_\infty \leq \|\dot{t}\| \leq \|\dot{t}_m\| + kT. \tag{5.3}$$

Then, by (3.13), (5.3), (1.8), (1.9) and as ψ_ϵ in (2.5) is increasing we obtain

$$f_{k,\epsilon}^m(z) \leq$$

$$\frac{1}{2}\int_0^1 (\lambda_2 + \Lambda|t|)\langle \dot{x}, \dot{x}\rangle ds - \frac{1}{2}\int_0^1 \beta(z)(\dot{t}_m + kT)^2 ds - \int_0^1 V(z) ds + \int_0^1 U_\epsilon(x) ds$$

$$\leq \frac{1}{2}(\lambda_2 + \Lambda kT)\|\dot{x}\|^2 - \frac{\nu}{2}k^2T^2 - \eta + \int_0^1 U_1(x) ds + \frac{\Lambda}{2}\|\dot{x}\|^2\|\dot{t}_m\| - \frac{\nu}{2}\|\dot{t}_m\|^2. \tag{5.4}$$

The right hand side of (5.4) is equal to $d_1(k)$ if and only if

$$\|\dot{t}_m\| = \frac{\Lambda}{2\nu}\|\dot{x}\|^2$$

$$+\left(\frac{\Lambda^2}{4\nu^2}\|\dot{x}\|^4 - \frac{2}{\nu}d_1(k) - \frac{2\eta}{\nu} - k^2T^2 + \frac{1}{\nu}(\lambda_2 + \Lambda kT)\|\dot{x}\|^2 + \frac{2}{\nu}\int_0^1 U_1(x) ds\right)^{\frac{1}{2}}$$

$$= \frac{\Lambda}{2\nu}\|\dot{x}\|^2$$

$$+\left(\frac{\Lambda^2}{4\nu^2}\|\dot{x}\|^4 + \frac{2N-\nu}{\nu}k^2T^2 + \frac{2\,(2M-\eta)}{\nu} + \frac{\lambda_2 + \Lambda kT}{\nu}\|\dot{x}\|^2 + \frac{2}{\nu}\int_0^1 U_1(x) ds\right)^{\frac{1}{2}}$$

$$= \rho_k(x), \tag{5.5}$$

so the proof is complete. □

Now for any $m, k \in \mathbf{N}$ we define the set

$$C_{k,m} = \left\{ z = (x,t) \in Z_{k,m} \mid \|\dot{t} - \dot{t}_k^*\| = \rho_k(x) \right\}.$$

We want to apply Theorem 4.4 to the functionals $f_{k,\epsilon}^m$ where Z, Λ, C are respectively replaced by $Z_{k,m}$, Λ_k, $C_{k,m}$. Note that, by (5.2), the definition of $C_{k,m}$ and Lemma 5.1 it is

$$\inf_{z\in\Lambda_k} f_{k,\epsilon}^m(z) \geq d_0(k) > d_1(k) \geq \sup_{z\in C_{k,m}} f_{k,\epsilon}^m(z), \tag{5.6}$$

thus the first property in (4.1) is verified. Moreover the following lemma holds. □

Lemma 5.2. *The set $C_{k,m}$ is a closed strong deformation retract of $Z_{k,m} \setminus \Lambda_k$.*

Proof. Defining the map

$$h_m(z) = \frac{\rho_k(x)}{\|\dot{t} - \dot{t}_k^*\|}(t - t_k^*) + t_k^*$$

and as $\rho_k(x)$ is strictly positive (see (5.5)), we can argue exactly as in Lemma 5.3 in Candela et al [8]. □

Now, to get the existence of critical values of $f_{k,\epsilon}^m$, it remains to prove that the relative category of $Z_{k,m}$ in itself relative to $C_{k,m}$ is not trivial. Here the Galerkin approximation is essential in order to apply Proposition 4.5.

Lemma 5.3. *For any $l \in \mathbf{N}$ a compact subset $K_{k,m}(l)$ of $Z_{k,m}$ exists such that*

$$\operatorname{cat}_{Z_{k,m},C_{k,m}} K_{k,m}(l) \geq l. \tag{5.7}$$

Proof. Using Proposition 4.5, the proof can be carried out as in Lemma 5.5 in Candela et al [8]. Here we give only the main ideas which will be useful in the sequel of this section.

Define, for any $k, m \in \mathbf{N}$

$$B_{k,m} = \left\{t \in W_{k,m} \mid \|\dot{t} - \dot{t}_k^*\| \leq 1\right\}, \qquad \tilde{B}_{k,m} = \Lambda^1(\mathcal{M}_0) \times B_{k,m},$$

$$S_{k,m} = \partial B_{k,m}, \qquad \tilde{S}_{k,m} = \Lambda^1(\mathcal{M}_0) \times S_{k,m}.$$

For a fixed $l \in \mathbf{N}$, by Proposition 4.5, a compact subset $\tilde{K}_{k,m}(l)$ of $\tilde{B}_{k,m}$ exists such that

$$\operatorname{cat}_{\tilde{B}_{k,m},\tilde{S}_{k,m}} \tilde{K}_{k,m}(l) \geq l. \tag{5.8}$$

Define the continuous map $\Phi : Z_{k,m} \to Z_{k,m}$

$$\Phi(x,t) = (x, \phi(x,t)) \qquad \forall z = (x,t) \in Z_{k,m}\,,$$

where

$$\phi(x,t) = \rho_k(x)(t - t_k^*) + t_k^*.$$

By the properties of the relative category,

$$K_{k,m}(l) = \Phi(\tilde{K}_{k,m}(l))$$

is a compact subset of $Z_{k,m}$ such that, by (5.8),

$$\operatorname{cat}_{Z_{k,m},C_{k,m}} K_{k,m}(l) \geq \operatorname{cat}_{\tilde{B}_{k,m},\tilde{S}_{k,m}} \tilde{K}_{k,m}(l) \geq l$$

so the proof is complete. □

Remark 5.4. For any $k, m, l \in \mathbf{N}$, let $\tilde{K}_{k,m}(l)$ be the compact subset of $\tilde{B}_{k,m}$ satisfying (5.8). The arguments used in the proof of Proposition 4.5 show that, for a fixed l, the subsets $\tilde{K}_{k,m}(l)$ can be chosen in way that they have the same projection on $\Lambda^1(\mathcal{M}_0)$ for any k and m. Hence for any $k, m \in \mathbf{N}$ we can write

$$\tilde{K}_{k,m}(l) = V(l) \times G_{k,m}(l)\,,$$

where $V(l)$ is compact in $\Lambda^1(\mathcal{M}_0)$ and $G_{k,m}(l)$ is compact in $B_{k,m}$. By the definition of Φ, it is clear that also

$$K_{k,m}(l) = \Phi(\tilde{K}_{k,m}(l)) = V(l) \times \phi(G_{k,m}(l))$$

has spatial part independent of k and m. Moreover, by the expression of $\rho_k(x)$ given in (5.5), some positive constants $a(l)$, $b(l)$, $c(l)$ exist such that

$$\max_{x\in V(l)} \rho_k(x) \le a(l) + b(l)kT + c(l)\sqrt{kT}. \tag{5.9}$$

Note now that, reasoning as in the proof of Proposition 2.5, also each $f^m_{k,\epsilon}$ verifies the Palais–Smale condition. Then, by (5.6), Lemmas 5.2 and 5.3, we can apply Theorem 4.4. For any $\epsilon \le \epsilon(k)$ (see (5.1)), $l \in \mathbf{N}$, we define

$$\Gamma^l_{k,m} = \left\{ B \subset Z_{k,m} \mid B \text{ is closed, } \operatorname{cat}_{Z_{k,m},C_{k,m}} B \ge l \right\} \tag{5.10}$$

and

$$c^m_{k,\epsilon}(l) = \inf_{B\in\Gamma^l_{k,m}} \sup_{z\in B} f^m_{k,\epsilon}(z). \tag{5.11}$$

Hence $c^m_{k,\epsilon}(l)$ is a critical value of $f^m_{k,\epsilon}$. We claim that, denoting by $z^m_{k,\epsilon}(l)$ the corresponding critical point, we have

$$f^m_{k,\epsilon}(z^m_{k,\epsilon}(l)) \le d(l) + f(l)kT + g(l)\sqrt{kT} - \frac{\nu}{2}k^2T^2\,, \tag{5.12}$$

where $d(l)$, $f(l)$, $g(l)$ are positive constants independent of k, m, ϵ. Indeed, by (5.10) and (5.11)

$$f^m_{k,\epsilon}(z^m_{k,\epsilon}(l)) \le \max_{z\in K_{m,k}(l)} f^m_{k,\epsilon}(z)\,, \tag{5.13}$$

where $K_{m,k}(l)$ is as in Lemma 5.3. By the expression of Φ (see again Lemma 5.3), for any $z = (x,t) \in K_{m,k}(l)$

$$\|t\|_\infty \le \rho_k(x) + \|t^*_k\|_\infty = \rho_k(x) + kT. \tag{5.14}$$

Then, by Remark 3.3 for any $z = (x,t) \in K_{m,k}(l)$

$$f^m_{k,\epsilon}(z) \le \frac{1}{2}\left(\lambda_2 + \Lambda(\rho_k(x) + kT)\right)\|\dot{x}\|^2 - \frac{\nu}{2}k^2T^2 - \eta + \int_0^1 U_1(x)ds$$

so that, by Remark 5.4, (5.9) and (5.13), (5.12) follows.

Next we will show that, for some l, $c^m_{k,\epsilon}(l)$ verifies the first inequality of (3.17). To this aim, for t^*_k fixed, consider the functional $J_{k,\epsilon} : \Lambda^1(\mathcal{M}_0) \to \mathbf{R}$ defined by

$$J_{k,\epsilon}(x) = \int_0^1 \left[\frac{1}{2}\langle \alpha(x,t^*_k)\dot{x},\dot{x}\rangle - \frac{1}{2}\beta(x,t^*_k)k^2T^2 - V(x,t^*_k) + U_\epsilon(x) \right] ds$$

and consider for any $c \in \mathbf{R}$ its sublevels

$$J^c_{k,\epsilon} = \left\{ x \in \Lambda^1(\mathcal{M}_0) \mid J_{k,\epsilon}(x) \le c \right\}.$$

The following fundamental lemma holds.

Lemma 5.5. *A positive constant δ exists such that for any $k \in \mathbf{N}$ large enough and $\epsilon < \epsilon(k)$ it is*

$$\mathrm{cat}_{\Lambda^1(\mathcal{M}_0)} J^{\delta+d_0(k)}_{k,\epsilon} \le \dim \mathcal{M}_0, \tag{5.15}$$

(see (5.1) and (3.16)).

Proof. We set

$$A = \{x \in \mathcal{M}_0 \mid U(x) < U_0\},$$

where (by Lemma 2.4) we can choose U_0 in way that

$$U(x) \ge U_0 \Rightarrow d(x,x_0) \ge R+1$$

and R has been introduced in assumption (H_6). At first, we prove the existence of $\delta > 0$ such that

$$J^{\delta+d_0(k)}_{k,\epsilon} \subset \Lambda^1(A^c) = \left\{ x \in \Lambda^1(\mathcal{M}_0) \mid \forall s \in [0,1]\ x(s) \notin A \right\}. \tag{5.16}$$

Note that, again from Lemma 2.4, it is

$$D = \mathrm{diam}\, A < +\infty.$$

Moreover, by (1.8) and (1.9), $\delta_1, \delta_2 > 0$ exist such that

$$D_\beta = \sup\{\beta(x,t) \mid d(x,x_0) \le D+1, t \in \mathbf{R}\} < N - \delta_1, \tag{5.17}$$

$$D_V = \sup\{V(x,t) \mid d(x,x_0) \le D+1, t \in \mathbf{R}\} < M - \delta_2. \tag{5.18}$$

Setting $\delta = \delta_1 + \delta_2$ we can prove that

$$\sup\{d(x(s),x_0) \mid s \in [0,1]\} \ge D+1 \quad \forall x \in J^{\delta+d_0(k)}_{k,\epsilon}. \tag{5.19}$$

Indeed, if for some $x \in J_{k,\epsilon}^{\delta+d_0(k)}$, (5.19) is not true, from the definition of D_β, D_V and $J_{k,\epsilon}$

$$-\frac{1}{2}k^2T^2D_\beta - D_V \le J_{k,\epsilon}(x) \le \delta + d_0(k)$$

which is a contradiction because, by (5.17), (5.18), for $k \in \mathbf{N}$ such that $\frac{k^2T^2}{2} > 1$,

$$\frac{1}{2}k^2T^2D_\beta + D_V < \frac{1}{2}k^2T^2N - \frac{\delta_1 k^2T^2}{2} + M - \delta_2 \le -\delta - d_0(k),$$

so (5.19) is proved. Now we can prove (5.16): fix $x \in J_{k,\epsilon}^{\delta+d_0(k)}$ and define

$$l(x) = \int_0^1 \langle \dot{x}, \dot{x}\rangle^{1/2} ds.$$

By the definition of $J_{k,\epsilon}$, (1.8) and (1.9), we easily get

$$l(x) \le \left(\int_0^1 \langle \dot{x}, \dot{x}\rangle ds\right)^{1/2} \le \sqrt{\frac{2\delta}{\lambda_1}}. \tag{5.20}$$

By (5.19) and (5.20), if we choose δ in such a way that $2\delta < \lambda_1^2$, we also get

$$\inf\{d(x(s), x_0) \mid s \in [0,1]\} \ge D + 1 - \sqrt{\frac{2\delta}{\lambda_1}} > D$$

which implies (5.16).

Now, reasoning as in Lemma 3.1 in Benci et al [6], using the gradient flow of U, it is possible to "project" each curve of $J_{k,\epsilon}^{\delta+d_0(k)}$ on ∂A. Indeed, setting

$$\Lambda^1(\partial A) = \left\{x \in \Lambda^1(\mathcal{M}_0) \mid x(s) \in \partial A \ \forall s \in [0,1]\right\},$$

a continuous map $\Pi : \Lambda^1(A^c) \to \Lambda^1(\partial A)$ exists such that

$$\begin{gathered}\Pi(x) = x \ \ \forall x \in \Lambda^1(\partial A) \\ \exists L > 0 \text{ s.t. } l(\Pi(x)) \le L l(x) \ \ \forall x \in \Lambda^1(A).\end{gathered} \tag{5.21}$$

By (5.16), (5.20) and (5.21) $\Pi(J_{k,\epsilon}^{\delta+d_0(k)}) \subset \Lambda^1(\partial A, \delta)$, where

$$\Lambda^1(\partial A, \delta) = \left\{ y \in \Lambda^1(\partial A) \mid l(y) \le L\sqrt{\frac{2\delta}{\lambda_1}} \right\}.$$

So, since ∂A is compact, applying classical properties of Ljusternik–Schnirelmann category, we finally obtain

$$\mathrm{cat}_{\Lambda^1(\mathcal{M}_0)} J_{k,\epsilon}^{\delta+d_0(k)} \le \mathrm{cat}_{\Lambda^1(\mathcal{M}_0)} \Lambda^1(\partial A, \delta) \le 1 + \dim \partial A = \dim \mathcal{M}_0$$

and the proof is complete. □

As a consequence of Lemma 5.5 the following property holds: for any $l \in \mathbf{N}$ with $l > \dim \mathcal{M}_0$ and for any $B \in \Gamma^l_{k,m}$

$$B \cap \left((J_{k,\epsilon})_{\delta+d_0(k)} \times \{t^*_k\}\right) \neq \emptyset, \tag{5.22}$$

where

$$(J_{k,\epsilon})_{\delta+d_0(k)} = \left\{x \in \Lambda^1(\mathcal{M}_0) \mid J_{k,\epsilon}(x) > \delta + d_0(k)\right\}.$$

Indeed, if $B \in \Gamma^l_{k,m}$ would exist with

$$B \cap \left((J_{k,\epsilon})_{\delta+d_0(k)} \times \{t^*_k\}\right) = \emptyset,$$

then

$$B \subset \left(J^{\delta+d_0(k)}_{k,\epsilon} \times \{t^*_k\}\right) \cup (Z_{k,m} \setminus \Lambda_k).$$

By Proposition 4.2, Lemma 5.2, Proposition 4.3 and Lemma 5.5 we get

$$l \leq \operatorname{cat}_{Z_{k,m},C_{k,m}} B \leq \operatorname{cat}_{Z_{k,m},C_{k,m}}(Z_{k,m} \setminus \Lambda_k) + \operatorname{cat}_{Z_{k,m}}(J^{\delta+d_0(k)}_{k,\epsilon} \times \{t^*_k\})$$

$$\leq \operatorname{cat}_{\Lambda^1(\mathcal{M}_0)} J^{\delta+d_0(k)}_{k,\epsilon} \leq \dim \mathcal{M}_0$$

which is a contradiction. So (5.22) is proved. Then, for any fixed $k \in \mathbf{N}$, for any $\epsilon < \epsilon(k)$, $l > \dim \mathcal{M}_0$, $B \in \Gamma^l_{k,m}$, choose $(\overline{x}, t^*_k) \in B \cap \left((J_{k,\epsilon})_{\delta+d_0(k)} \times \{t^*_k\}\right)$ so that

$$\sup_{z \in B} f^m_{k,\epsilon}(z) \geq f^m_{k,\epsilon}(\overline{x}, t^*_k) \geq J_{k,\epsilon}(\overline{x}) \geq \delta + d_0(k).$$

Then, for any $k \in \mathbf{N}$ we get

$$c^m_{k,\epsilon}(l) \geq \delta + d_0(k) \quad \forall m \in \mathbf{N}, l > \dim \mathcal{M}_0, \epsilon < \epsilon(k). \tag{5.23}$$

In order to prove Theorem 1.2, we will use the following lemma (for the proof see e.g. Benci et al [7], Candela et al [8]) which allows us to get over the Galerkin approximation.

Lemma 5.6. *Let $k \in \mathbf{N}$ and $\epsilon \in]0,1]$ be fixed. Assume that for all $m \in \mathbf{N}$, a critical point $z_m \in Z_{k,m}$ of $f^m_{k,\epsilon}$ exists. If for some $c_1, c_2 > 0$ it is*

$$c_1 \leq f^m_{k,\epsilon}(z_m) \leq c_2 \quad \forall m \in \mathbf{N},$$

then the sequence $(z_m)_{m\in\mathbf{N}}$ converges, up to subsequences, to a curve $z \in Z_k$ such that

$$f'_{k,\epsilon}(z) = 0 \qquad \lim_{m\to+\infty} f^m_{k,\epsilon}(z_m) = f_{k,\epsilon}(z) \quad c_1 \leq f_{k,\epsilon}(z) \leq c_2.$$

Proof of Theorem 1.2. Let us fix $l_0 = \dim \mathcal{M}_0 + 1$. As the right hand side of (5.12) negatively diverges as k goes to $+\infty$, we can take k_0 such that

$$f^m_{k,\epsilon}(z^m_{k,\epsilon}(l_0)) \leq -3M \qquad \forall k \in \mathbf{N}, k \geq k_0, m \in \mathbf{N}, \epsilon \leq \epsilon(k_0). \tag{5.24}$$

Moreover, from the choice of l_0, by (5.23)

$$f^m_{k,\epsilon}(z^m_{k,\epsilon}(l_0)) \geq \delta + d_0(k) \qquad \forall k \geq k_0, m \in \mathbf{N}, \epsilon \leq \epsilon(k_0). \tag{5.25}$$

Now, fix $k \in \mathbf{N}, k \geq k_0$, $\epsilon \leq \epsilon(k_0)$ and consider the sequence of critical points $(z^m_{k,\epsilon}(l_0))_{m\in\mathbf{N}}$ of $f^m_{k,\epsilon}$. By (5.24) and (5.25), we can apply Lemma 5.6 obtaining the existence of a critical point $z_{k,\epsilon} \in Z_k$ of $f_{k,\epsilon}$ such that

$$\delta + d_0(k) \leq f_{k,\epsilon}(z_{k,\epsilon}) \leq -3M \tag{5.26}$$

so, if we also choose $\epsilon < \epsilon_1$ (see Proposition 3.5) $z_{k,\epsilon} \equiv z_k$ is a critical point of f_k. Note that, by the second inequality in (5.26) and (1.9)

$$E_{z_k} = f_k(z_k) + 2\int_0^1 V(z_k)ds \leq -M \tag{5.27}$$

then, by Corollary 2.2, z_k is a solution of (1.5) and *(i)* is proved (see also Remark 5.7).

In order to prove *(ii)* consider the critical values $c^m_{k,\epsilon}(l)$ for any $l \geq l_0$. Reasoning as in the first part of the proof, for ϵ small enough, by Lemma 5.6 and Proposition 3.5, we get the existence of critical values $c_k(l)$ of f_k such that

$$\delta + d_0(k) \leq c_k(l) \leq d(l) + f(l)kT + g(l)\sqrt{kT} - \frac{\nu}{2}k^2T^2.$$

Then, fixed $l \in \mathbf{N}$, there exists $k_l \in \mathbf{N}$ such that for any $k \in \mathbf{N}, k > k_l$

$$c_k(l_0 + l) \leq -3M$$

so that $c_k(l_0) \leq c_k(l_0+1) \leq \ldots \leq c_k(l_0+l)$ are critical values of f_k whose critical points have negative energy. Moreover, reasoning as in Fournier et al [12], if for some $h \in \mathbf{N}$, $c_k(l) = c_k(l+1) = \ldots = c_k(l+h)$, at least h distinct critical points of f_k exist at the level $c_k(l)$. Thus, taking into account Proposition 2.3, if $k > k_l$ it is $N(k) \geq l$ and the proof is complete. $\square$

Remark 5.7. *Observe that all the trajectories z_k found in Theorem 1.2 are timelike curves. Indeed they satisfy (5.27), then, by (1.9)*

$$\frac{1}{2}\sup_{s\in I}\langle \dot{z}_k(s), \dot{z}_k(s)\rangle \leq -M - \eta < 0.$$

Moreover, by (5.12), $(f_k(z_k))_{n\in\mathbf{N}}$ is a negatively diverging sequence. Then, by (5.27), if $z_k = (x_k, t_k)$, also

$$\lim_{k\to+\infty} E_{z_k} = -\infty \qquad \lim_{k\to+\infty} \|\dot{t}_k\| = +\infty$$

as, by (1.9)

$$\frac{N}{2}\|\dot{t}_k\|^2 \geq -f_k(z_k) - \int_0^1 V(z_k)ds \geq -f_k(z_k) - M.$$

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