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COMPLEXIFIED FUETER OPERATORS IN CLASSICAL AND QUANTUM ELECTRODYNAMICS

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ABSTRACT: In this paper, we analyse some structural similarities between Maxwell and Dirac equations in the framework of a complexified Fueter operator formalism.

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1. INTRODUCTION

A first attempt for a generalization of Complex Analysis from two to four dimensions was presented by Fueter [1] in the later thirties. Since then other approaches have been made in order to accommodate the well known two-dimensional complex variable theory results in an extended analytical theory (e.g. Balabaev [2], Kassandrov [3], Vekua [4]). Recently we have developed a new approach such that Cauchy-Riemann like relations, hypercomplex differentiability (Machado and Borges [5]), and its relations with the Fueter theory of left and right quaternionic regular functions (Machado and Borges [6]), and hypercomplex conformal mappings (Machado and Borges [7]) are well established for a class of functions over bi-complex numbers and quaternions. Among all the main possible applications of our approach, we have quoted in op. cit. Machado and Borges [6] the appearance of a Klein-Gordon-Dirac quaternionic operator and the solution set for the quadriharmonic Laplace equations, whose relation with the Fueter theory is pointed out in the following theorem, whose proposition is:

Theorem. *Let f be an arbitrary quaternionic function with real coordinate functions, and let h be another arbitrary quaternionic function such that $\bar{\Gamma} f = h$. If h is*

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a left regular function of the Fueter theory, then f is a solution of the quadriharmonic equation $\nabla^2 f = 0$.

Proof. (op. cit. Machado and Borges [6]) It is immediate, since by the proposition $\bar{\Gamma} f = h$, and applying the operator Γ to both sides of this equation, one gets $\Gamma\bar{\Gamma} f = \nabla^2 f = \Gamma h$. Since h is left regular by hypothesis, $\Gamma h = 0$ and the conclusion holds.□

In the framework of applicability of our approach, based on Klein-Gordon-Dirac operator, we present in this paper structural analogies between Maxell and Dirac equations when analysed in the context of Fueter's left or right regularity.

2. COMPLEXIFIED FUETER OPERATORS AND PAULI MATRICES

Let us consider functions $\mathbf{f} : \mathcal{H} \to \mathcal{H}$ in the division algebra of quaternions

$$\begin{aligned} \mathbf{f}(t,x,y,z) &= f_0(t,x,y,z) + \mathbf{i} f_1(t,x,y,z) \\ &+ \mathbf{j} f_2(t,x,y,z) + \mathbf{k} f_3(t,x,y,z), \end{aligned}$$

having no restrictions over its coordinate functions $f_i : R^4 \to R$, except that they must be a desired number k of times partially differentiable in their independent variables. One complexifies the classical Fueter operator of the theory of regular functions, by introducing the operator Γ_c which is given by

$$\Gamma_c = i\frac{\partial}{\partial t} + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \tag{1}$$

where i is the complex unit ($i^2 = -1$) and the following multiplication rules stand for the quaternion units:

$$\begin{aligned} \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i^2} = -\mathbf{1}, \\ \mathbf{j^2} = -\mathbf{1}, \mathbf{k^2} = -\mathbf{1}, \\ \mathbf{1i} = \mathbf{i1} = \mathbf{i}, \mathbf{1j} = \mathbf{j1} = \mathbf{j}, \mathbf{1k} = \mathbf{k1} = \mathbf{k}, \end{aligned} \tag{2}$$

Hence the action of Γ_c over the quaternionic function $\mathbf{f}$ is written as

$$\begin{aligned} \Gamma_c \mathbf{f} = &\left(i\frac{\partial f_0}{\partial t} - \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z} \right) + \mathbf{i}\left(i\frac{\partial f_1}{\partial t} + \frac{\partial f_0}{\partial x} + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \\ &+\mathbf{j}\left(i\frac{\partial f_2}{\partial t} + \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \mathbf{k}\left(i\frac{\partial f_3}{\partial t} + \frac{\partial f_0}{\partial z} + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right). \end{aligned} \tag{3}$$

Equation (3) may be reinterpreted with help of the identifications

$$\Gamma_c = i\frac{\partial}{\partial t} + \nabla, \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \tag{4}$$

$$\mathbf{f} = f_0 + \vec{f}, \qquad \vec{f} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3, \tag{5}$$

in such a way that

$$\Gamma_c\mathbf{f} = i\frac{\partial f_0}{\partial t} + \nabla f_0 + i\frac{\partial \vec{f}}{\partial t} - \nabla\cdot\vec{f} + \nabla\times\vec{f}, \tag{6}$$

being ∇f_0, $\nabla\cdot\vec{f}$ and $\nabla\times\vec{f}$ respectively the gradient, divergent and curl operators as usual in Vector Analysis. By representing the quadri-vector $\mathbf{f}$ in column vector notation, (3) is also expressed as

$$\Gamma_c\mathbf{f} = i\frac{\partial}{\partial t}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$+ \frac{\partial}{\partial y}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{7}$$

This equality can be related to the Pauli matrices of Quantum Electrodynamics. In fact, replacing the matrix entries by complex units, one gets the equivalent form

$$\Gamma_c\mathbf{f} = i\frac{\partial}{\partial t}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} + i\frac{\partial}{\partial x}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$+ i\frac{\partial}{\partial y}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} + i\frac{\partial}{\partial z}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} =$$

$$= \left(i\mathbf{I}\frac{\partial}{\partial t} + i\gamma_1\frac{\partial}{\partial x} + i\gamma_2\frac{\partial}{\partial y} + i\gamma_3\frac{\partial}{\partial z} \right)\mathbf{f}, \tag{8}$$

being $\mathbf{I}$ the 4x4 identity matrix, and

$$\gamma_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_2 & \mathbf{0} \\ \mathbf{0} & -\sigma_2 \end{pmatrix}, \tag{9}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & i\sigma_3 \\ -i\sigma_3 & \mathbf{0} \end{pmatrix}, \tag{10}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & i\sigma_1 \\ -i\sigma_1 & \mathbf{0} \end{pmatrix}, \tag{11}$$

where σ_1, σ_2 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

As expected, the matrices

$$\bar{\gamma_1} = i\gamma_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \bar{\gamma_2} = i\gamma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{13}$$

and

$$\bar{\gamma_3} = i\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

together with the identity matrix $\mathbf{I}$, form a representation for the group of quaternions. This representation is not unique, but is more convenient for our purposes. For instance, a classical homomorphism for the quaternion group which appears very often in literature (Negi et al [8], Deavours [9]) is given by the identity $\mathbf{I}$ and somewhat different τ matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} i\sigma_2 & \mathbf{0} \\ \mathbf{0} & -i\sigma_2 \end{pmatrix}, \tag{15}$$

$$\tau_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix}, \tag{16}$$

$$\tau_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & i\sigma_2 \\ i\sigma_2 & \mathbf{0} \end{pmatrix}. \tag{17}$$

According to this representation, $\mathbf{f} = f_0\mathbf{I} + f_1\tau_1 + f_2\tau_2 + f_3\tau_3$, and equation (6) is preserved again, but now through a row vector formalism, namely

$$\left(i\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ -f_1 & f_0 & -f_3 & f_2 \\ -f_2 & f_3 & f_0 & -f_1 \\ -f_3 & -f_2 & f_1 & f_0 \end{pmatrix}$$

$$= \left(i\frac{\partial f_0}{\partial t} - \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z},\ i\frac{\partial f_1}{\partial t} + \frac{\partial f_0}{\partial x} + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z},\right.$$
$$\left. i\frac{\partial f_2}{\partial t} + \frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x},\ i\frac{\partial f_3}{\partial t} + \frac{\partial f_0}{\partial z} + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right), \tag{18}$$

a result which is not so close to the usual Dirac equation form as the representation (8). Another interesting feature is that (15,16 17) do not include the entire set of Pauli matrices.

By direct calculations it may be shown that the matrices γ_i obey the following anticommutation relations:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0, \quad i,j = 1,2,3; \quad i \neq j. \tag{19}$$

Furthermore they are all self-adjoint (Hermitian) and unitary,

$$\gamma_i = \left(\gamma_i^T\right)^*,$$
$$\gamma_i^{-1} = \left(\gamma_i^T\right)^*, \tag{20}$$

satisfying the following multiplication laws,

$$\gamma_1\gamma_2 = -\gamma_2\gamma_1 = -i\gamma_3,$$
$$\gamma_2\gamma_3 = -\gamma_3\gamma_2 = -i\gamma_1,$$
$$\gamma_3\gamma_1 = -\gamma_1\gamma_3 = -i\gamma_2,$$
$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = I. \tag{21}$$

They also have the same characteristic polynomial,

$$\lambda^4 - 2\lambda^2 + 1, \tag{22}$$

and hence two + 1 and two -1 eigenvalues. Notice that the above properties are in perfect correspondence with those required for the α_i matrix operators in the Dirac Hamiltonian, as we will see in next sections. From the combination of (6) and (8), a new representation for the action of Γ_c over the quadrivector $\mathbf{f}$ arises as

$$\Gamma_c\mathbf{f} = \left(i\mathbf{I}\frac{\partial}{\partial t} + i\gamma_1\frac{\partial}{\partial x} + i\gamma_2\frac{\partial}{\partial y} + i\gamma_3\frac{\partial}{\partial z}\right)\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$
$$= \begin{pmatrix} i\frac{\partial f_0}{\partial t} - \nabla\cdot\vec{f} \\ \left(\nabla f_0 + i\frac{\partial \vec{f}}{\partial t} + \nabla\times\vec{f}\right)_{\mathbf{i}} \\ \left(\nabla f_0 + i\frac{\partial \vec{f}}{\partial t} + \nabla\times\vec{f}\right)_{\mathbf{j}} \\ \left(\nabla f_0 + i\frac{\partial \vec{f}}{\partial t} + \nabla\times\vec{f}\right)_{\mathbf{k}} \end{pmatrix}. \tag{23}$$

Equation (23) plays a central role in the subsequent discussions about the formal, common mathematical structure from both Maxwell and Dirac equations.

3. CLASSICAL EQUATIONS AND COMPLEXIFIED FUETER OPERATORS

3.1. ON THE COVARIANCE OF WAVE EQUATIONS

The set of Maxwell's equations is given in general form by

$$\begin{cases} \nabla \cdot \vec{E} = \rho, \\ \nabla \cdot \vec{H} = 0, \\ \nabla \times \vec{H} = \frac{1}{c}\left(\vec{J} + \frac{\partial \vec{E}}{\partial t}\right), \\ \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \end{cases} \tag{24}$$

when $\vec{E}$ and $\vec{H}$ are respectively electric and magnetic fields, c is a constant, $\vec{J}$ is the density of current and ρ the density of charge. In (24), $\nabla \cdot \vec{H} = 0$ implies the existence of a vector potential $\vec{A}$ satisfying $\vec{H} = \nabla \times \vec{A}$, and $\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$ implies the existence of a scalar potential ϕ satisfying $\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\phi$. Maxwell's equations are usually expressed in the notation of 3-vectors and 3-vector differential operators in a Cartesian coordinate system. In this form their covariance under rotations and translations is obvious. That they are not covariant under the Galilean transformations is most easily seen by noting that the wave equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0, \tag{25}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0, \tag{26}$$

are a consequence of them. The quantities ϕ and $\vec{A}$ are the scalar and vector potentials, and c is an universal constant with the dimension of a velocity. Thus Maxwell's theory predicts the possibility of electromagnetism propagating with velocity c. That this constant is equal to the velocity of light indicated to Maxwell the well-known fact that light is an electromagnetic phenomenon. Nevertheless, the concept of a velocity as an universal constant is not consistent with the principle of relativity as formulated in terms of Galilean transformations, since velocities do not remain invariant under these transformations. Looking for a linear transformation on the four parameters x, y, z and t that will leave the electromagnetic wave equations (25,26) invariant, write x^1, x^2, x^3, x^4 for x, y, z, ct. Then

$$x'^{\mu} = x^{\nu}\left(a^{-1}\right)^{\mu}_{\nu}, \quad (\mu, \nu = 1, 2, 3, 4). \tag{27}$$

The operator in (25,26) is

$$\Box = \frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}, \tag{28}$$

where $\eta^{\mu\nu}$ is the 4x4 matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

The transformation law of $\frac{\partial}{\partial x^\mu}$ is

$$\frac{\partial}{\partial x'^\mu} = a^\nu_\mu \frac{\partial}{\partial x^\nu}. \tag{30}$$

Then, for (28) to be unchanged by the transformation, $\eta^{\mu\nu}$ must transform like a tensor (in the four dimensional space) and must retain the same form in order that $\Box$ shall retain the same form. The inverse matrix $\eta_{\mu\nu}$ has the same components as $\eta^{\mu\nu}$ and transform like

$$a^\alpha_\mu \eta_{\alpha\beta} a^\beta_\nu = \eta_{\mu\nu}. \tag{31}$$

The transformations (27) with a^ν_μ satisfying (31) are called Lorentz transformations. They include rotations in (x, y, z) space as well as the required generalization of Galilean transformations. Lorentz transformations leave the electromagnetic wave equations invariant. Another remarkable wave equation that is also Lorentz covariant, is the Dirac equation (Section 4), that is the appropriate equation for the description of particles with spin 1/2 and rest mass m. When the rest mass is zero, it is usually assumed to be the equation for free neutrinos. As far as an unifying approach for locally classical laws is desirable, physical laws must be covariant under Lorentz and not Galilean transformations. Moreover, it is expected that some of these classical equations (e.g. Dirac and Maxwell equations) would have an identical foundation. In the following sections we show that both Maxwell and Dirac equations have structural similarities when analysed in the context of Fueter formalism extended to our most recent results (op. cit. Machado and Borges [5], Machado and Borges [6], Machado and Borges [7]).

3.2. MAXWELL EQUATIONS IN A QUATERNIONIC FORM

Let us first consider the complex-valued quaternion function defined by

$$\mathcal{F} = \mathcal{E} - i\mathcal{H}, \tag{32}$$

and following (5),

$$\mathcal{E} = E_0 + \vec{E}, \quad \mathcal{H} = H_0 + \vec{H}, \tag{33}$$

where $\vec{E}$ and $\vec{H}$ are respectively electric and magnetic fields satisfying the Maxwell equations, and E_0 and H_0 will be choosen latter according to some physical requirements. By applying (6) and equating to zero, one gets

$$\Gamma_c \mathcal{F} = i\frac{\partial E_0}{\partial t} + \nabla E_0 + i\frac{\partial \vec{E}}{\partial t} - \nabla \cdot \vec{E} + \nabla \times \vec{E}$$

$$+\frac{\partial H_0}{\partial t} - i\nabla H_0 + \frac{\partial \vec{H}}{\partial t} + i\nabla \cdot \vec{H} - i\nabla \times \vec{H} = 0. \tag{34}$$

Grouping together the scalar and vector parts in the expression above, one reaches the equation

$$\Gamma_c \mathcal{F} = \left(\frac{\partial H_0}{\partial t} - \nabla \cdot \vec{E}\right) + i\left(\frac{\partial E_0}{\partial t} + \nabla \cdot \vec{H}\right)$$

$$+\left(\nabla \times \vec{E} + \frac{\partial \vec{H}}{\partial t} + \nabla E_0\right) + i\left(\frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} - \nabla H_0\right) = 0, \tag{35}$$

which gives the system

$$\frac{\partial H_0}{\partial t} - \nabla \cdot \vec{E} = 0,$$

$$\frac{\partial E_0}{\partial t} + \nabla \cdot \vec{H} = 0,$$

$$\nabla \times \vec{E} + \frac{\partial \vec{H}}{\partial t} + \nabla E_0 = 0,$$

$$\frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} - \nabla H_0 = 0. \tag{36}$$

Clearly (36) is equivalent to the set of Maxwell equations, since one makes the choices

$$E_0 = 0, \quad \nabla H_0 = -\vec{J}, \quad \frac{\partial H_0}{\partial t} = \rho. \tag{37}$$

By substitution of these relations for H_0 in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0, \tag{38}$$

one also reaches the familiar d'Alambertian form

$$\frac{\partial^2 H_0}{\partial t^2} - \frac{\partial^2 H_0}{\partial x^2} - \frac{\partial^2 H_0}{\partial y^2} - \frac{\partial^2 H_0}{\partial z^2} = 0. \tag{39}$$

Hence the following conclusion holds: *given a known distribution of current and charge densities, and their related solutions of Maxwell equations, a complexified quaternion function $\mathcal{F}$ may be found, such that $\mathcal{F}$ is left regular for the operator Γ_c:*

$$\Gamma_c \mathcal{F} = \Gamma_c (\mathcal{E} - i\mathcal{H}) = 0. \tag{40}$$

In other words, (40) may be considered as a compact representation for the Maxwell equations. Besides, by taking the quaternion conjugation of Γ_c, defined as

$$\bar{\Gamma}_c = i\frac{\partial}{\partial t} - \mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} - \mathbf{k}\frac{\partial}{\partial z}, \tag{41}$$

one gets

$$\bar{\Gamma}_c \Gamma_c = \Gamma_c \bar{\Gamma}_c = -\frac{\partial^2}{\partial t^2} + \nabla^2. \tag{42}$$

Therefore, from (40) it results

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)(\mathcal{E} - i\mathcal{H}) = 0, \tag{43}$$

which is equivalent to the wave equations for $\vec{E}$, $\vec{H}$ and the continuity equation in the form of (39). Finally and according to the results of previous section, from (23) and (37) one reaches a representation for the Maxwell equations based on the unitary matrices γ_i:

$$\Gamma_c \mathcal{F} = \left(i\mathbf{I}\frac{\partial}{\partial t} + i\gamma_1\frac{\partial}{\partial x} + i\gamma_2\frac{\partial}{\partial y} + i\gamma_3\frac{\partial}{\partial z}\right)\begin{pmatrix} -iH_0 \\ E_1 - iH_1 \\ E_2 - iH_2 \\ E_3 - iH_3 \end{pmatrix} = 0. \tag{44}$$

It should be also mentioned that (40) requires only a slight modification in order to include the existence of magnetic monopoles: conditions (37) are written for this case as

$$\frac{\partial E_0}{\partial t} = -\rho_m, \quad \nabla E_0 = \vec{J}_m,$$

$$\nabla H_0 = -\vec{J}, \quad \frac{\partial H_0}{\partial t} = \rho, \tag{45}$$

and magnetic densities will satisfy the same form (39) of continuity equation as the electric densities.

Several attempts to achieve a "Dirac-like" form for the Maxwell equations have been presented in literature, with the structure outlined in (44). Perhaps the most popular is the one due to Moses (Moses [10]), and reproduced frequently in texts about Relativistic Quantum Mechanics (Greiner [11]). This analogous form is given as

$$-\frac{1}{i}\sum_{j=0}^{3}\bar{\alpha^j}\frac{\partial}{\partial x^j}\psi=-\frac{4\pi}{c}\Phi, \tag{46}$$

where

$$\bar{\alpha^0}=\mathbf{I},\quad \bar{\alpha^1}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},$$

$$\bar{\alpha^2}=\begin{pmatrix}0&0&-1&0\\0&0&0&i\\-1&0&0&0\\0&-i&0&0\end{pmatrix},\quad \bar{\alpha_3}=\begin{pmatrix}0&0&0&-1\\0&0&-i&0\\0&i&0&0\\-1&0&0&0\end{pmatrix}, \tag{47}$$

being ψ and Φ column vectors such that

$$\psi_0=0,\ \ \psi_1=H_1-iE_1,\ \ \psi_2=H_2-iE_2,\ \ \psi_3=H_3-iE_3,$$

$$\phi_0=c\rho,\ \ \phi_1=j_1,\ \ \phi_2=j_2,\ \ \phi_3=j_3. \tag{48}$$

The operators $\bar{\alpha^i}$ for i = 1,2,3 are Hermitian and satisfy the commutation relations

$$\bar{\alpha^i}\bar{\alpha^j}+\bar{\alpha^j}\bar{\alpha^i}=2\delta_{ij}\mathbf{I}, \tag{49}$$

but (46) does not reproduce the action of a complexified Fueter operator exactly as in (6). It makes more difficult to establish the connection with a theory of complexified regular functions. Besides, it is not possible to relate easily the operators $\bar{\alpha^i}$ with the entire set of Pauli matrices.

4. DIRAC EQUATIONS IN A QUATERNIONIC FORM

Now we are in condition to apply the previous results in the framework of Quantum Electrodynamics. We start by considering the Dirac equation for vanishing mass, with $\hbar=c=1$:

$$i\mathbf{I}\frac{\partial\Psi}{\partial t}+i\alpha_1\frac{\partial\Psi}{\partial x}+i\alpha_2\frac{\partial\Psi}{\partial y}+i\alpha_3\frac{\partial\Psi}{\partial z}=0. \tag{50}$$

The following requirements must be satisfied by both the four-component wavefunction Ψ and the operators α_i:

a) The wavefunction Ψ is also solution for a Klein-Gordon type equation;

b) The operators α_i, i = 1,2,3 must be Hermitian and obey the commutation relations

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta_{ij}\mathbf{I}; \tag{51}$$

what leads to the property

$$Tr\,(\alpha_i) = 0, \tag{52}$$

and thus the eigenvalues of α_i are +1 and -1, with two as multiplicity;

c) The Dirac equation in the form (50) must be covariant under Lorentz transformations.

The operators γ_i as defined in (15), (16) and (17) fulfil the requirements of b), according to the properties (19) and (21), hence we may try to represent (50) as

$$i\mathbf{I}\frac{\partial\Psi}{\partial t} + i\gamma_1\frac{\partial\Psi}{\partial x} + i\gamma_2\frac{\partial\Psi}{\partial y} + i\gamma_3\frac{\partial\Psi}{\partial z} = 0. \tag{53}$$

Since (53) is equivalent to the application of Γ_c on Ψ, the requirement a) results immediately from (42). Therefore, in the column vector formalism, one gets

$$\Gamma_c\Psi = \left(i\mathbf{I}\frac{\partial}{\partial t} + i\gamma_1\frac{\partial}{\partial x} + i\gamma_2\frac{\partial}{\partial y} + i\gamma_3\frac{\partial}{\partial z}\right)\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = 0. \tag{54}$$

As the next step, following the interpretation (5) and the equality (23), (54) is written as

$$\begin{pmatrix} i\frac{\partial\psi_0}{\partial t} - \nabla\cdot\vec{\psi} \\ \left(\nabla\psi_0 + i\frac{\partial\vec{\psi}}{\partial t} + \nabla\times\vec{\psi}\right)_{\mathbf{i}} \\ \left(\nabla\psi_0 + i\frac{\partial\vec{\psi}}{\partial t} + \nabla\times\vec{\psi}\right)_{\mathbf{j}} \\ \left(\nabla\psi_0 + i\frac{\partial\vec{\psi}}{\partial t} + \nabla\times\vec{\psi}\right)_{\mathbf{k}} \end{pmatrix} = 0, \tag{55}$$

with $\vec{\psi} = (\psi_1, \psi_2, \psi_3)^T$.

Notice that the usual interpretation of probability densities and probability currents associated to the Dirac equation is not changed by the form (53), only $\alpha_i's$ are replaced by $\gamma_i's$ in the definition of $\vec{j}$, the probability current. As a final remark, we may speculate that the Lorentz covariance, as stated in c), probably results from the Maxwell-like form clearly shown by (55).

5. CONCLUDING REMARKS

Motivated by the Lorentz-covariance and the desire to set up an identical foundation for Physics laws under a local point of view, we have shown some structural analogies between quaternionic Maxwell and Dirac equations when analysed in the context of an extended Fueter theory. Maxwell and Dirac equations are among the two most relevant wave equations described in Mathematical Physics, the former relating light as an electromagnetic wave phenomenon and the later dealing with the special relativistic equation for the electron - it is, indeed, under Schrödinger's version a generally covariant wave equation (see for instance, Lord [12]), showing how to describe an electron in a gravitational field. Both equations appear also in geometrical constructions in the scenario of Quantum Field Theory (see also Witten [13] and Penrose [14]). Modern supersymmetric theories of Yang-Mills type constructed on the group-manifold are such that Bianchi identities are satisfied as both Dirac and Maxwell homogeneous equations hold. These results are intrinsically related to the closure of the algebra of supersymmetric transformations (Borges [15], Borges [16]). We have established throughout this paper that in a quaternionic form - with vanishing mass - Dirac and Maxwell equations have an analogue formulation. We hope to continue an investigation in order to build up, in the nearest future, a treatment for the Dirac equation with non-vanishing rest mass.

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