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STABILITY OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS AND APPLICATIONS

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ABSTRACT: A theorem on the error estimate of approximate solutions for difference functional equations of the Volterra type with an unknown function of one variable is presented. The error is estimated by a solution of an initial problem for nonlinear difference equation. This general result is applied to the investigation of the stability of the generalized Lax method for first order partial functional differential equations. Classical solutions of nonlinear problems are approximated in the paper by solutions of suitable quasilinear systems of difference equations. This new approach to the numerical solving of nonlinear equations is generated by a quasilinearization technique. Numerical examples are presented.

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1. INTRODUCTION

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let E be the Haar pyramid

$$E = \{ (t,x) \in \mathbb{R}^{1+n} : \ t \in [0,a], \ -b + Mt \le x \le b - Mt \},$$

where $x = (x_1, \ldots, x_n)$, $a > 0$, $M = (M_1, \ldots, M_n) \in \mathbb{R}^n_+$, $\mathbb{R}_+ = [0, +\infty)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $b > Ma$. Write $E_0 = [-b_0, 0] \times [-b, b] \subset \mathbb{R}^{1+n}$, where $b_0 \in \mathbb{R}_+$ and $\Omega = E \times \mathbb{R} \times \mathbb{R}^n$. Given the functions

$$f : \Omega \to \mathbb{R}, \ \ \varphi : E_0 \to \mathbb{R}, \ \ V : C(E_0 \cup E, \mathbb{R}) \to C(E, \mathbb{R}),$$

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we consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,V[z](t,x),\partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \text{ for } (t,x) \in E_0\,, \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$. A function $v : E_0 \cup E \to \mathbb{R}$ is called a classical solution of the above problem if:

(i) $v \in C(E_0 \cup E, \mathbb{R})$ and v is of class C^1 on E,

(ii) v satisfies equation (1) on E and condition (2) holds.

We are interested in the construction of a method for the numerical approximation of solutions to (1), (2) with solutions of associated difference equations and in the estimation of the difference between these solutions. Classical difference methods for (1), (2) consist in replacing partial derivatives $\partial_t z$ and $(\partial_{x_1} z, \dots, \partial_{x_n} z)$ by difference expressions $\delta_0 z$ and $(\delta_1 z, \dots, \delta_n z)$. Since equation (1) contains the functional variable $V[z] \in C(E, \mathbb{R})$ and approximate solutions of (1), (2) are functions defined on the mesh, we need also some interpolating operators T_h. This leads to a difference functional problem of the Volterra type which is stable under natural assumptions on given functions and on the mesh. The monograph of Kamont [6] contains an exposition of the theory of difference methods for hyperbolic functional differential problems.

In the paper we present another approach to the numerical solving of (1), (2). We transform nonlinear equation (1) into a quasilinear system of difference equations. Our main ideas are based on a quasilinearization of (1) with respect to the last variable and on the theory of bicharacteristics for nonlinear functional differential equations. We give sufficient conditions for the convergence of the method. The stability of quasilinear difference schemes is investigated by using a comparison technique with nonlinear estimates for given functions. Additional motivations for our research are given in Section 3.

In recent years, a number of papers concerning numerical methods for functional partial differential equations have been published (Jaruszewska-Walczak and Kamont [5], Kamont [7] - Leszczyński [9], Malec [12] - Prządka [16]). It is easy to construct an explicit or implicit difference method for nonlinear functional differential equation which satisfies the consistency conditions on all sufficiently regular solutions of a considered problem. The main task in these investigations is to find a finite difference approximation which is stable. The method of difference inequalities and simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference functional problems. The proof of the convergence are also based on a general theorem on an error estimate of approximate solutions to functional difference equations of the Volterra type with initial or initial boundary conditions and with an unknown function of several variables. All these considerations as a rule involved a lot of technical calculations for to reach the convergence result. The main property of the corresponding operators was not easy to be seen. In the paper we present a new approach to the investigation of the stability of difference problems generated by hyperbolic functional differential problems. We prove a general theorem on an error estimate of approximate solutions to abstract functional difference equations with an unknown function of one variable. The error of an approximate solution is estimated by a solution of an initial problem for a nonlinear difference equation.

We use the above general ideas to the investigations of the stability of difference functional problems generated by (1), (2).

The paper is organized as follows. In Section 2 we prove a general theorem on the error estimate for difference functional equations. In the next section we formulate a quasilinear system of difference equations for problem (1), (2). A convergence result and an error estimate of approximate solutions are presented in Section 4. Numerical examples are given in the last section.

In the paper, we use some general ideas for finite difference problems which were introduced in Brunner [1], Godlewski and Raviart [4], Magomedov and Kholodov [11], Redheffer and Walter [17], Reinhardt [18]. Note that existence results for Cauchy problem (1), (2) can be found in Kamont [6], Chapter 2.

2. APPROXIMATE SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS

For any two sets U and W we denote by $\mathbf{F}(U,W)$ the class of all functions defined on U and taking values in W. If $f \in \mathbf{F}(U,W)$ and $A \subset U$ then $f\big|_A$ is the restriction of f to the set A. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. Write $I = [-b_0, 0]$, $J = [0, a]$. We define a mesh on the set $I \cup J$ in the following way. Let $h_0 > 0$ denote the step of the mesh. Nodal points are given by $t^{(r)} = rh_0$, $r \in \mathbb{Z}$. We assume that there is $N_0 \in \mathbb{N}$ such that $N_0 h_0 = b_0$. Let $K \in \mathbb{N}$ be defined by $Kh_0 \le a < (K+1)h_0$. Write

$$I_h = \{t^{(r)} : \ -N_0 \le r \le 0\}, \ J_h = \{t^{(r)} : \ 0 \le r \le K\},$$

$$J'_h = \{ t^{(r)} : \ 0 \le r \le K-1 \}.$$

Let X be a linear space with the norm $\| \cdot \|_X$. For a function $y : I_h \cup J_h \to X$ and for a point $t^{(r)} \in I_h \cup J_h$ we write $y^{(r)} = y(t^{(r)})$. In the same way we define the numbers

$\beta^{(i)}$, $0 \le i \le K$, where $\beta : J_h \to \mathbb{R}$. For a function $y : I_h \cup J_h \to X$ and for a point $t^{(r)} \in J_h$ we put

$$\|y\|_{X.r} = \max\{\, \|y^{(i)}\|_X : \ \ -N_0 \le i \le r \,\}.$$

Let $\mathbf{F}_0(I_h \cup J_h, X) \subset \mathbf{F}(I_h \cup J_h, X)$ be a linear subspace. Suppose that

$$F : J'_h \times \mathbf{F}_0(I_h \cup J_h, X) \to X$$

is a given operator. For $(t^{(r)}, y) \in J'_h \times \mathbf{F}(I_h \cup J_h, X)$ we write $F[t, y] = F(t^{(r)}, y)$. We will say that F satisfies the Volterra condition if for each $t^{(r)} \in J'_h$, y, $\tilde{y} \in \mathbf{F}_0(I_h \cup J_h, X)$ such that $y^{(i)} = \tilde{y}^{(i)}$ for $-N_0 \le i \le r$ we have $F[r, y] = F[r, \tilde{y}]$. Note that the Volterra condition means that the value of F at the point $(t^{(r)}, y)$ depends on $t^{(r)}$ and on the restriction of y to the set $\{\, t^{(-N_0)}, t^{(-N_0+1)}, \dots, t^{(r)} \,\}$ only.

Given $\eta : I_h \to X$, we consider the functional difference equation

$$y^{(r+1)} = F[r, y], \ \ 0 \le r \le K - 1, \tag{3}$$

with the initial condition

$$y^{(i)} = \eta^{(i)} \ \text{ for } \ t^{(i)} \in I_h. \tag{4}$$

We assume that F satisfies the Volterra condition. Then there exists exactly one solution $\tilde{y} : I_h \cup J_h \to X$ of problem (3), (4).

Let $Y \subset \mathbf{F}_0(I_h \cup J_h, X)$ be a fixed subset. Suppose that $v \in Y$, $\gamma : J'_h \to \mathbb{R}$ and $\alpha_0 \in \mathbb{R}_+$ are such that

$$\|v^{(r+1)} - F[r, v]\|_X \le \gamma^{(r)} \ \text{ on } J'_h \tag{5}$$

and

$$\|v - \eta\|_{X.0} \le \alpha_0. \tag{6}$$

The function v satisfying the above conditions will be considered as an approximate solution of (3), (4). We give a theorem on the estimate of the difference between the solution and the approximate solution of (3), (4). We look for approximate solutions of (3), (4) in the space $Y \subset \mathbf{F}_0(I_h \cup J_h, X)$.

Theorem 1. *Suppose that*

1) *the functions $F : J'_h \times \mathbf{F}_0(I_h \cup J_h, X) \to X$ and $\sigma : J'_h \times \mathbb{R}_+ \to \mathbb{R}_+$ are such that*

   (i) *σ is nondecreasing with respect to the second variable,*

   (ii) *the estimate*

$$\|F[i, y] - F[i, \tilde{y}]\|_X \le \sigma(t^{(i)}, \|y - \tilde{y}\|_{X.i}) \tag{7}$$

   is satisfied for $y \in \mathbf{F}_0(I_h \cup J_h, X)$, $\tilde{y} \in Y$ and $0 \le i \le K - 1$, *2)* $\eta : I_h \to X$ *and* $u : I_h \cup J_h \to X$ *is a solution of (3), (4),*

3) the functions $v : I_h \cup J_h \to X$, $\gamma : J'_h \to \mathbb{R}_+$ *and the constant* $\alpha_0 \in \mathbb{R}_+$ *are such that* $v \in Y$ *and conditions (5), (6) are satisfied,*

4) the function $\beta : J_h \to \mathbb{R}_+$ *is nondecreasing and*

$$\beta^{(r+1)} \geq \sigma(t^{(r)}, \beta^{(r)}) \text{ for } 0 \leq r \leq K-1 \text{ and } \beta^{(0)} \geq \alpha_0.$$

Under these assumptions we have

$$\|v - u\|_{X.r} \leq \beta^{(r)} \text{ for } 0 \leq r \leq K. \tag{8}$$

Proof. We prove (8) by induction. According to (6) we have estimate (8) for $r = 0$. Suppose now that $\|v - u\|_{X.r} \leq \beta^{(r)}$ where $0 \leq r < K$. It follows from (5), (7) and from the monotonicity of $\sigma(t^{(r)}, \cdot)$ that for $0 \leq i \leq r$ we have

$$\begin{aligned} &\|u^{(i+1)} - v^{(i+1)}\|_X \\ &\leq \|F[i, u] - F[i, v]\|_X + \|F[i, v] - v^{(i+1)}\|_X \\ &\leq \sigma(t^{(i)}, \|u - v\|_{X.i}) + \gamma^{(i)} \leq \beta^{(i+1)} \leq \beta^{(r+1)}. \end{aligned}$$

This implies that

$$\|u - v\|_{X.r+1} \leq \beta^{(r+1)}.$$

This is the desired conclusion. □

Remark 2. It follows from (7) that F satisfies the Volterra condition.

Remark 3. It is important in our considerations that we have assumed estimate (7) for $y \in \mathbf{F}_0(I_h \cup J_h, X)$ and for $\tilde{y} \in Y$ only. We will show that the operator F generated by quasilinear functional differential system satisfies condition (ii) of Theorem 1 and it does not satisfy (7) for $y, \tilde{y} \in \mathbf{F}_0(I_h \cup J_h, X)$.

3. DISCRETIZATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS

We will denote by $M_{k \times n}$ the space of all $k \times n$ matrices with real elements. For $x, \bar{x} \in \mathbb{R}^n$, $X \in M_{k \times n}$ where

$$x = (x_1, \dots, x_n), \ \bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \ \text{ and } \ X = \left[\, x_{ij} \,\right]_{i=1,\dots,k,\, j=1,\dots,n},$$

we put

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad x \diamond \bar{x} = (x_1\bar{x}_1, \dots, x_n\bar{x}_n)$$

and

$$\|X\| = \max\Big\{\sum_{j=1}^{n} |x_{ij} :\ 1 \le i \le k\Big\}.$$

The product of two matrices is denoted by " $\star$ ". If $X \in M_{k\times n}$, then X^T is the transpose matrix. We use the symbol " $\circ$ " to denote the scalar product in $\mathbb{R}^n$. For functions $z \in C(E_0 \cup E, \mathbb{R})$, $u \in C(E_0 \cup E, \mathbb{R}^n)$ and for a point $t \in [0, a]$ we write

$$\|z\|_t = \max\{\, |z(\tau, x)| :\ (\tau, x) \in E_0 \cup E,\ \tau \le t\,\},$$

$$\|u\|_t = \max\{\, \|u(\tau, x)\| :\ (\tau, x) \in E_0 \cup E,\ \tau \le t\,\}.$$

We define a mesh on the set $E_0 \cup E$ in the following way. Suppose that (h_0, h'), $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. Let $t^{(r)}$, $-N_0 \le r \le K$, be the nodal points defined in Section 2. Write $x^{(m)} = m \diamond h'$ where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and

$$x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}).$$

Denote by H the set of all $h = (h_0, h')$ such that there is $(N_1, \dots, N_n) \in \mathbb{N}^n$ with the properties: $N \diamond h' = b$ and $h' = Mh_0$. Write

$$\mathbb{R}^{1+n}_h = \{\, (t^{(r)}, x^{(m)}) :\ (r, m) \in \mathbb{Z}^{1+n}\,\}$$

and

$$E_{h.0} = E_0 \cap \mathbb{R}^{1+n}_h, \quad E_h = E \cap \mathbb{R}^{1+n}_h,$$

$$E_{h.r} = \big(E_{h.0} \cup E_h\big) \cap \big([-b_0, t^{(r)}] \times \mathbb{R}^n\big), \quad 0 \le r \le K.$$

For functions $z : E_{h.0} \cup E_h \to \mathbb{R}$, $u : E_{h.0} \cup E_h \to \mathbb{R}^n$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$, $u^{(r,m)} = u(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h.i} = \max\{\, |z^{(r,m)}| :\ (t^{(r)}, x^{(m)}) \in E_{h.i}\,\},$$

$$\|u\|_{h.i} = \max\{\, \|u^{(r,m)}\| :\ (t^{(r)}, x^{(m)}) \in E_{h.i}\,\},$$

where $0 \le i \le K$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, 1 standing on the $i-$th place. Write

$$E'_h = \{\, (t^{(r)}, x^{(m)}) \in E_h :\ \ (t^{(r+1)}, x^{(m)}) \in E_h\,\}.$$

It is easily seen that

$$E'_h = \{\, (t^{(r)}, x^{(m)}) :\ -b + Mt^{(r+1)} \le x^{(m)} \le b - Mt^{(r+1)},\ 0 \le r \le K-1\,\}.$$

We need the following assumptions on f and V.

Assumption H $[f, V]$**.** Suppose that

1) $f \in C(\Omega, \mathbb{R})$ and the partial derivatives

$$\partial_t f, \ \left(\partial_{x_1} f, \dots, \partial_{x_n} f)\right) = \partial_x f, \ \ \partial_p f, \ \ \left(\partial_{q_1} f, \dots, \partial_{q_n} f)\right) = \partial_q f$$

exist on Ω and $\partial_t f$, $\partial_p f \in C(\Omega, \mathbb{R})$, $\partial_x f$, $\partial_q f \in C(\Omega, \mathbb{R}^n)$,

2) there is $A \in \mathbb{R}_+$ such that $|\partial_p f(P)| \leq A$ and $\|\partial_q f(P)\| \leq A$ for $P = (t, x, p, q) \in \Omega$,

3) the operator $V : C(E_0 \cup E, \mathbb{R}) \to C(E, \mathbb{R})$ satisfies the Volterra condition,

4) if $z : E_0 \cup E \to \mathbb{R}$ is of class C^1, then there exist the partial derivatives

$$\partial_t V[z], \ \left(\partial_{x_1} V[z], \dots, \partial_{x_n} V[z]\right) = \partial_x V[z]$$

and $\partial_t V[z] \in C(E, \mathbb{R})$, $\partial_x V[z] \in C(E, \mathbb{R}^n)$.

We apply a method of quasilinearization to problem (1), (2). Suppose that Assumption $H[f, V]$ is satisfied and that $\varphi : E_0 \to \mathbb{R}$ is of class C^1. We introduce first additional unknown functions $u_0 = \partial_t z$, $u = \partial_x z$, $u = (u_1, \dots, u_n)$, in (1). Then we consider the linearization of (1) with respect to the last variable

$$\partial_t z(t, x) = f(U[z, u; t, x]) + \partial_q f(U[z, u; t, x]) \circ \left(\partial_x z(t, x) - u(t, x)\right), \tag{9}$$

where $U[z, u; t, x] = \left(t, x, V[z](t, x), u(t, x)\right)$. By virtue of (1) we get the following equations for u_0 and u :

$$\partial_t u_0(t, x) = \partial_t f(U[z, u; t, x]) + \partial_p f(U[z, u; t, x])\, \partial_t V[z](t, x) \tag{10}$$

$$+\partial_q f(U[z, u; t, x]) \circ \partial_x u_0(t, x)$$

and

$$\partial_t u(t, x) = \partial_x f(U[z, u; t, x]) + \partial_p f(U[z, u; t, x])\, \partial_x V[z](t, x) \tag{11}$$

$$+\partial_q f(U[z, u; t, x]) \star \left[\, \partial_x u(t, x) \,\right]^T.$$

We consider the following initial condition for system (9)-(11)

$$z(t, x) = \varphi(t, x), \ \ u_0(t, x) = \partial_t \varphi(t, x), \ \ u(t, x) = \partial_x \varphi(t, x), \ \ (t, x) \in E_0. \tag{12}$$

Under natural assumptions on given functions, the following properties of (1), (2) and (9)-(12) may be proved (Kamont [6], Chapter IV):

(A) if $\tilde{v} : E_0 \cup E \to \mathbb{R}$ is a classical solution of (1), (2) and $\tilde{v}$ is of class C^2 on E then the functions $(\tilde{v}, \partial_t \tilde{v}, \partial_x \tilde{v})$ satisfy (9)-(12),

(B) if $(\tilde{v}, \tilde{u}_0, \tilde{u}) : E_0 \cup E \to \mathbb{R}^{2+n}$, $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$, is a classical solution of (9)-(12) then $\partial_t \tilde{v} = \tilde{u}_0$, $\partial_x \tilde{v} = \tilde{u}$ and $\tilde{v}$ satisfies (1), (2).

The method of quasilinearization and the theory of bicharacteristics for nonlinear first order partial differential equations and systems were first treated by Cinquini Cibrario [2], [3]. The method was extended in Leszczyński [9], Kamont [6] on nonlinear functional differential problems. We use these ideas to solve numerically (1), (2). We will approximate classical solutions of (1), (2) with solutions of difference problems generated by (1)-(2).

Quasilinear system (9)-(11) contains the functional variables $V[z]$, $\partial_t V[z]$, $\partial_x V[z]$. On the other hand, approximate solutions of (1), (2) are functions defined on the mesh $E_{h.0} \cup E_h$. Therefore we need the interpolating operators

$$T_h : \mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}) \to C(E, \mathbb{R})$$

and

$$\bar{L}_h : \mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}^{2+n}) \to C(E, \mathbb{R}^{1+n}),$$

where $\bar{L}_h = (L_{h.0}, L_h)$, $L_h = (L_{h.1}, \dots, L_{h.n})$.

Assumption H$[V, T_h, \bar{L}_h]$**.** Suppose that conditions 3), 4) of Assumption H $[f, V]$ are satisfied and

1) there is an operator $T_h : \mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}) \to C(E, \mathbb{R})$ such that

   (i) the Lipschitz condition

$$\|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \le L_0 \|z - \bar{z}\|_{h.r}, \quad 0 \le r \le K, \tag{13}$$

   is satisfied on $\mathbf{F}(E_{h.0} \cup E_h, \mathbb{R})$,

   (ii) there is $\mu > 0$ such that for each function $v : E_0 \cup E \to \mathbb{R}$ which is of class C^2, there is $\kappa_0 \in \mathbb{R}_+$ such that

$$\|V[v] - T_h[v_h]\|_t \le \kappa_0 h_0^{\mu}, \quad t \in [0, a],$$

   where v_h is the restriction of v to the set $E_{h.0} \cup E_h$,

2) there is an operator $\bar{L}_h : \mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}^{2+n}) \to C(E, \mathbb{R}^{1+n})$ where $\bar{L}_h = (L_{h.0}, L_h)$, $L_h = (L_{h.1}, \dots, L_{h.n})$, with the properties :

   (i) the Lipschitz condition

$$\|L_{h.0}[z, \bar{u}] - L_{h.0}[\tilde{z}, \bar{v}]\|_{t^{(r)}} + \|L_h[z, \bar{u}] - L_h[\tilde{z}, \bar{v}]\|_{t^{(r)}}$$
$$\le L_1 \big[\, \|z - \tilde{z}\|_{h.r} + \|u_0 - v_0\|_{h.r} + \|u - v\|_{h.r} \,\big]$$

   is satisfied on $\mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}^{2+n})$ where $\bar{u} = (u_0, u)$, $u = (u_1, \dots, u_n)$ and $\bar{v} = (v_0, v)$, $v = (v_1, \dots, v_n)$,

(ii) there is $\nu > 0$ such that for each function $\tilde{v} : E_0 \cup E \to \mathbb{R}$ which is of class C^2 there is $\kappa_1 \in \mathbb{R}_+$ such that

$$\|L_{h.0}[\tilde{v}_h, \partial_t \tilde{v}_h, \partial_x \tilde{v}_h] - \partial_t V[\tilde{v}]\|_t$$

$$+\|L_h[\tilde{v}_h, \partial_t \tilde{v}_h, \partial_x \tilde{v}_h] - \partial_x V[\tilde{v}]\|_t \leq \kappa_1 h_0^{\nu}, \quad t \in [0, a],$$

where $\tilde{v}_h$, $\partial_t \tilde{v}_h$, $\partial_x \tilde{v}_h$ are the restrictions of $\tilde{v}$, $\partial_t \tilde{v}$, $\partial_x \tilde{v}$ to the set $E_{h.0} \cup E_h$.

Remark 4. It follows from Assumption H $[V, T_h, \bar{L}_h]$ that T_h and $\bar{L}_h$ satisfy the following Volterra condition: if

$$z\big|_{E_{h.r}} = \tilde{z}\big|_{E_{h.r}}, \quad \bar{u}\big|_{E_{h.r}} = \bar{v}\big|_{E_{h.r}},$$

then

$$T_h[z](t^{(r)}, x) = T_h[\tilde{z}](t^{(r)}, x)$$

and

$$\bar{L}_h[z, \bar{u}](t^{(r)}, x) = \bar{L}[\tilde{z}, \bar{v}](t^{(r)}, x)$$

for $(t^{(r)}, x) \in E$.

The difference operators δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ are defined in the following way. For a function $\omega : E_{h.0} \cup E \to \mathbb{R}$ and for a point $(t^{(r)}, x^{(m)}) \in E'_h$ we put

$$\delta_0 \omega^{(r,m)} = \frac{1}{h_0}\big[\omega^{(r+1,m)} - \Delta\omega^{(r,m)}\big], \quad \Delta\omega^{(r,m)} = \frac{1}{2n}\sum_{i=1}^{n}\big[\omega^{(r,m+e_i)} + \omega^{(r,m-e_i)}\big]$$

and

$$\delta_i \omega^{(r,m)} = \frac{1}{2h_i}\big[\omega^{(r,m+e_i)} - \omega^{(r,m-e_i)}\big], \quad 1 \leq i \leq n.$$

Now we formulate a system of difference equations corresponding to (1), (2). Let us denote by $(z, \bar{u})$, $\bar{u} = (u_0, u)$, $u = (u_1, \dots, u_n)$, the unknown functions of the variables $(t^{(r)}, x^{(m)})$. Write

$$\delta z^{(r,m)} = \big(\delta_1 z^{(r,m)}, \dots, \delta_n z^{(r,m)}\big), \quad \delta_0 u^{(r,m)} = \big(\delta_0 u_1^{(r,m)}, \dots, \delta_0 u_n^{(r,m)}\big),$$

$$\delta u^{(r,m)} = \big[\delta_j u_i u^{(r,m)}\big]_{i,j=1,\dots,n}, \quad \Delta u^{(r,m)} = \big(\Delta u_1^{(r,m)}, \dots, \Delta u_n^{(r,m)}\big)$$

and

$$P^{(r,m)}[z, u] = \big(t^{(r)}, x^{(m)}, T_h[z]^{(r,m)}, u^{(r,m)}\big).$$

Let us denote by F_h and $\bar{G}_h = (G_{h.0}, G_h)$, $G_h = (G_{h.1}, \dots, G_{h.n})$, the operators defined on $\mathbf{F}(E_{h.0} \cup E_h, \mathbb{R}^{2+n})$ in the following way: if $(z, \bar{u}) = (z, u_0, u)$, $u = (u_1, \dots, u_n)$, then

$$F_h[z, \bar{u}]^{(r,m)} = \Delta z^{(r,m)} + h_0 f(P^{(r,m)}[z, u]) \tag{14}$$

$$+h_0\partial_q f(P^{(r,m)}[z,u])\circ\big[\delta z^{(r,m)}-u^{(r,m)}\big],$$

and

$$G_{h.0}[z,\bar u]^{(r,m)}=\Delta u_0^{(r,m)}+h_0\partial_t f(P^{(r,m)}[z,u]) \tag{15}$$

$$+h_0\partial_p f(P^{(r,m)}[z,u])\,L_{h.0}[z,\bar u]^{(r,m)}+h_0\partial_q f(P^{(r,m)}[z,u])\circ\delta u_0^{(r,m)},$$

and

$$G_h[z,\bar u]^{(r,m)}=\Delta u^{(r,m)}+h_0\partial_x f(P^{(r,m)}[z,u]) \tag{16}$$

$$+h_0\partial_p f(P^{(r,m)}[z,u])\,L_h[z,\bar u]^{(r,m)}+h_0\partial_q f(P^{(r,m)}[z,u])\star\big[\,\delta u^{(r,m)}\,\big]^T.$$

We consider the system of difference equations

$$z^{(r+1,m)}=F_h[z,\bar u]^{(r,m)} \tag{17}$$

$$\bar u^{(r+1,m)}=\bar G_h[z,\bar u]^{(r,m)} \tag{18}$$

with the initial condition

$$z^{(r,m)}=\varphi_h^{(r,m)},\ \ \bar u^{(r,m)}=\bar\psi_h^{(r,m)}\ \text{ on }\ E_{h.0}, \tag{19}$$

where

$$\varphi_h:E_{0.h}\to\mathbb{R},\ \bar\psi_h:E_{h.0}\to\mathbb{R}^{1+n},\ \bar\psi_h=(\psi_{h.0},\psi_h),\ \psi_h=(\psi_{h.1},\dots,\psi_{h.n}),$$

are given functions. Note that problem (17)-(19) is a discretization of (9)-(12).

The difference problem consisting of system (17), (18) and the initial condition (19) is called a generalized Lax method for problem (1), (2). It is easily seen that initial problem (17)-(19) with $h\in H$ has exactly one solution $(z_h,\bar u_h):E_{0.h}\cup E_h\to\mathbb{R}^{2+n}$ where $\bar u_h=(\bar u_{h.0},\bar u_h)$ and $\bar u_h=(\bar u_{h.1},\dots,\bar u_{h.n})$.

There are the following differences between the classical difference method and our approach.

(A) If we apply the classical Lax scheme then by using interpolating operators we approximate classical solution of (1), (2) in the norm of the space $C(E_0\cup E,\mathbb{R})$. Let us denote by $C^1(E_0\cup E,\mathbb{R})$ the set of all function $z:E_0\cup E\to\mathbb{R}$ which are of class C^1. For $z\in C^1(E_0\cup E,\mathbb{R})$ we consider the norm

$$\|z\|_1=\|z\|_0+\|\partial_t z\|_0+\|\partial_x z\|_0,$$

where $\|z\|_0$, $\|\partial_t z\|_0$, $\|\partial_x z\|_0$ are adequate maximum norms. In our method we approximate classical solutions of (1), (2) with solutions of difference problems in the norm of the space $C^1(E_0 \cup E, \mathbb{R})$. Note that $C^1(E_0 \cup E, \mathbb{R})$ is a natural function space in the existence and uniqueness theory for (1), (2), see Kamont [6], Chapter 2.

(B) Suppose that we have solved numerically problem (1), (2) by using the classical Lax scheme. Then the approximate solution z_h is known on the mesh $E_{h.0} \cup E_h$ and we can construct an interpolating polynomial defined on $E_0 \cup E$ with using nodal points $t^{(r)}, x^{(m)})$ and the values $z_h^{(r,m)}$. All the nodal points have the multiplicity one.

In our approach, the values $z_h^{(r,m)}$ and $u_{0.h}^{(r,m)}$, $u_h^{(r,m)}$ are known and $u_{0.h}$, u_h approximate the first order partial derivatives of a solution of (1), (2). Then we can construct an interpolating polynomial defined on $E_0 \cup E$ and all the nodal points have the multiplicity two.

(C) In the classical case we approximate solutions of the nonlinear functional differential problem with solutions of difference problems which are nonlinear. In our approach we use quasilinear difference problems in order to approximate solutions of nonlinear equations.

4. CONVERGENCE OF THE GENERALIZED LAX METHOD

We first prove that problem (17)-(19) is a particular case of (3), (4). Write

$$U_h = \{ x^{(m)} : \; -N \le m \le N \},$$

$$A_h^{(r)} = \{ x^{(m)} : \; -b + Mt^{(r)} \le x^{(m)} \le b - Mt^{(r)} \}, \; 0 \le r \le K,$$

and $X_h = \mathbf{F}(U_h, \mathbb{R}^{2+n})$. For $\alpha = (\beta, \bar{\gamma}) \in X_h$, $\bar{\gamma} = (\gamma_0, \gamma)$, $\gamma = (\gamma_1, \dots, \gamma_n)$, we define the norm

$$\|\alpha\|_{X_h} = \max \{ |\beta(x^{(m)})| : \; x^{(m)} \in U_h \}$$
$$+ \max \{ |\gamma_0(x^{(m)})| : \; x^{(m)} \in U_h \} + \max \{ \|\gamma(x^{(m)})\| : \; x^{(m)} \in U_h \}.$$

For $y : I_h \cup J_h \to X_h$, $y = (\xi, \bar{\zeta})$, $\bar{\zeta} = (\zeta_0, \zeta)$, $\zeta = (\zeta_1, \dots, \zeta_n)$, we write

$$y^{(r)} = y(t^{(r)}), \;\; y^{(r,m)} = y^{(r)}(x^{(m)}), \; x^{(m)} \in U_h,$$

and

$$\|y\|_{X_h.r} = \max \{ \|y^{(i)}\|_{X_h} : \;\; -N_0 \le i \le r \}, \;\; 0 \le r \le K.$$

Let $\mathbf{F}_0(I_h \cup J_h, X_h)$ be defined by

$$\mathbf{F}_0(I_h \cup J_h, X_h) = \{ y \in \mathbf{F}(I_h \cup J_h, X_h) : \; y^{(r,m)} = \theta_{2+n} \text{ for } x^{(m)} \in U_h \setminus A_h^{(r)}, \, t^{(r)} \in J_h \},$$

where $\theta_{2+n} = (0, \dots, 0) \in \mathbb{R}^{2+n}$. It follows that for $y \in \mathbf{F}_0(I_h \cup J_h, X_h)$, $y = (\xi, \bar{\zeta})$, $\bar{\zeta} = (\zeta_0, \zeta)$, $\zeta = (\zeta_1, \dots, \zeta_n)$, we have

$$\|y\|_{X_h.r} = \|\xi\|_{h.r} + \|\zeta_0\|_{h.r} + \|\zeta\|_{h.r}, \; 0 \le r \le K.$$

Let us denote by $\Xi_h : J'_h \times \mathbf{F}_0(I_h \cup J_h, X_h) \to X_h$ the operator given by

$$\Xi_h = \big(F_h, \bar{G}_h \big),\ \bar{G}_h = \big(G_{h.0}, G_h \big),\ G_h = \big(G_{h.1}, \dots , G_{h.n} \big),$$

and for $(z, \bar{u}) \in \mathbf{F}_0(I_h \cup J_h, X_h)$, $\bar{u} = (u_0, u)$, $u = (u_1, \dots , u_n)$, we put

(i) $F_h[z, \bar{u}]^{(r,m)}$, $G_{h.0}[z, \bar{u}]^{(r,m)}$, $G_h[z, \bar{u}]^{(r,m)}$ are defined by (14)-(16) for $(t^{(r)}, x^{(m)}) \in E'_h$,

(ii) if $t^{(r)} \in J'_h$, $x^{(m)} \in U_h$ and $(t^{(r)}, x^{(m)}) \notin E'_h$ then

$$F_h[z, \bar{u}]^{(r,m)} = 0,\ \ G_{h.0}[z, \bar{u}]^{(r,m)} = 0,\ \ G_h[z, \bar{u}]^{(r,m)} = \theta\,,$$

where $\theta = (0, \dots , 0) \in \mathbb{R}^n$.

Let $\eta_h = (\tilde{\eta}_h, \bar{\eta}_h) : I_h \to X_h$, $\bar{\eta}_h = (\eta_{h.0}, \eta_h)$, $\eta_h = (\eta_{h.1}, \dots , \eta_{h.n})$, be defined by

$$\tilde{\eta}^{(r)}_h(x^{(m)}) = \varphi^{(r,m)}_h,\ \bar{\eta}^{(r)}_h(x^{(m)}) = \bar{\psi}^{(r,m)}_h,\ x^{(m)} \in U_h.$$

Consider the difference functional equation

$$y^{(r+1)} = \Xi_h[\,r, y\,],\ \ 0 \le r \le K - 1, \tag{20}$$

with the initial condition

$$y^{(i)} = \eta^{(i)}_h \ \text{ for }\ t^{(i)} \in I_h. \tag{21}$$

It is easily seen that the following conditions are equivalent:

I. The functions $(z_h, u_{h.0}, u_h) : E_{0.h} \cup E_h \to \mathbb{R}^{2+n}$, $u_h = (u_{h.1}, \dots , u_{h.n})$, satisfy (17)-19).

II. The function $y_h : I_h \cup J_h \to X_h$, $y_h = (\xi_h, \bar{\zeta}_h)$, $\bar{\zeta}_h = (\zeta_{h.0}, \zeta_h)$, defined by

$$\xi^{(r)}_h(x^{(m)}) = z^{(r,m)}_h,\ \ \zeta^{(r)}_{h.0}(x^{(m)}) = u^{(r,m)}_{h.0},\ \ \zeta^{(r)}_h(x^{(m)}) = u^{(r,m)}_h,$$

where $(t^{(r)}, x^{(m)}) \in E_{h.0} \cup E_h$ and

$$\xi^{(r)}_h(x^{(m)}) = 0,\ \ \zeta^{(r)}_{h.0}(x^{(m)}) = 0,\ \ \zeta^{(r)}_h(x^{(m)}) = \theta\ \text{ for }\ x^{(m)} \in U_h \setminus A^{(r)}_h,\ 0 \le r \le K,$$

is a solution of (20, (21).

We formulate next assumptions on given functions.

Assumption H $[f, \sigma]$**.** Suppose that conditions 1), 2) of Assumption H $[f, V]$ are satisfied and there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

1) σ is continuous and it is nondecreasing with respect to both variables,

2) for each $c \geq 1$ the function $\bar{\omega}(t) = 0$, $t \in [0, a]$, is the unique solution of the problem

$$\omega'(t) = c\,\omega(t) + c\,\sigma(\,t, c\,\omega(t)\,), \;\; \omega(0) = 0,$$

3) the expressions

$$|\,\partial_t f(t,x,p,q) - \partial_t f(t,x,\tilde{p},\tilde{q})\,|, \;\; \|\,\partial_x f(t,x,p,q) - \partial_x f(t,x,\tilde{p},\tilde{q})\,\|,$$

$$|\,\partial_p f(t,x,p,q) - \partial_p f(t,x,\tilde{p},\tilde{q})\,|, \;\; \|\,\partial_q f(t,x,p,q) - \partial_q f(t,x,\tilde{p},\tilde{q})\,\|$$

are bounded from above by $\sigma(t, |p-\tilde{p}| + \|q-\tilde{q}\|)$.

Theorem 5. *Suppose that Assumptions $H\,[V, T_h, \bar{L}_h]$ and $H\,[f,\sigma]$ are satisfied and*

1) $h \in H$ and

$$\frac{1}{n} - \frac{h_0}{h_i}\,\big|\,\partial_{q_i} f(t,x,p,q)\,\big| \geq 0 \;\; on \;\; \Omega \;\; for \;\; 1 \leq i \leq n, \tag{22}$$

2) $w : E_0 \cup E \to \mathbb{R}$ is a classical solution of (1), (2) and w is of class C^2,

3) the functions $(z_h, u_{h.0}, u_h) : E_{h.0} \cup E_h \to \mathbb{R}^{2+n}$, $u_h = (u_{h.1}, \ldots, u_{h.n})$, satisfy (17)-(19) and there exists $\alpha_0 : H \to \mathbb{R}_+$ such that

$$|\varphi_h^{(r,m)} - \varphi^{(r,m)}| + |\psi_{h.0}^{(r,m)} - \partial_t \varphi^{(r,m)}| + \|\psi_h^{(r,m)} - \partial_x \varphi^{(r,m)}\| \leq \alpha_0(h)$$

on $E_{h.0}$ and $\lim_{h\to 0} \alpha_0(h) = 0$.

Then there is $\alpha : H \to \mathbb{R}_+$ and $\varepsilon > 0$ such that for $\|h\| < \varepsilon$ we have

$$|w^{(r,m)} - z_h^{(r,m)}| + |\partial_t w^{(r,m)} - u_{h.0}^{(r,m)}| + \|\partial_x w^{(r,m)} - u_h^{(r,m)}\| \leq \alpha(h) \; on \; E_h \tag{23}$$

and $\lim_{h\to 0} \alpha(h) = 0$.

Proof. We apply Theorem 1 to prove (23). Let us denote by $y_h : I_h \cup J_h \to X_h$, $y_h = (\xi_h, \bar{\zeta}_h)$, $\bar{\zeta}_h = (\zeta_{h.0}, \zeta_h)$ the function defined by

$$\xi_h^{(r)}(x^{(m)}) = z_h^{(r,m)}, \;\; \zeta_{h.0}^{(r)}(x^{(m)}) = u_{h.0}^{(r,m)}, \;\; \zeta_h^{(r)}(x^{(m)}) = u_h^{(r,m)},$$

where $(t^{(r)}, x^{(m)}) \in E_{h.0} \cup E_h$ and

$$\xi_h^{(r)}(x^{(m)}) = 0, \;\; \zeta_{h.0}^{(r)}(x^{(m)}) = 0, \;\; \zeta_h^{(r)}(x^{(m)}) = \theta \;\text{ for }\; x^{(m)} \in U_h \setminus A_h^{(r)}, \; 0 \leq r \leq K.$$

Then u_h satisfies (20), (21). Let $\bar{c} \in \mathbb{R}_+$ be such a constant that

$$|\partial_t w(t,x)| \leq \bar{c}, \;\; \|\partial_x w(t,x)\| \leq \bar{c} \;\text{ for }\; (t,x) \in E_0 \cup E.$$

It follows from Assumption H $[V, T_h, \bar{L}_h]$ that there is $\bar{a} \in \mathbb{R}_+$ such that

$$\|L_{h.0}[w_h, \bar{v}_h]\|_t + \|L_h[w_h, \bar{v}_h]\|_t \leq \bar{a} \tag{24}$$

for $t \in [0, a]$. Write

$$Y_h = \{\, y \in \mathbf{F}_0(I_h \cup J_h, X_h) : \quad y = (\xi, \bar{\zeta}), \quad \bar{\zeta} = (\zeta_0, \zeta), \quad \zeta = (\zeta_1, \dots, \zeta_n) \tag{25}$$
$$\text{and } \ |\delta \xi^{(r,m)}| \leq \bar{c}, \quad |\zeta_0^{(r,m)}| \leq \bar{c}, \quad \|\zeta^{(r,m)}\| \leq \bar{c} \ \text{ where } \ (t^{(r)} x^{(m)}) \in E'_h \,\}.$$

Suppose that

$$y = (z, \bar{u}) \in \mathbf{F}_0(I_h \cup J_h, X_h) \ \text{ and } \ y_h = (w_h, \bar{v}_h) \in Y_h$$

where $\bar{u} = (u_0, u)$, $u = (u_1, \dots, u_n)$ and $\bar{v}_h = (v_{h.0}, v_h)$, $v_h = (v_{h.1}, \dots, v_{h.n})$. We prove that there are $c_0 \in \mathbb{R}_+$, c_1, $\kappa \geq 1$ such that

$$\Big\| \Xi_h[\, r, y\,] - \Xi_h[\, r, y_h\,] \Big\|_{X_h} \leq \|y - \bar{y}\|_{X_h.r} \tag{26}$$
$$+ h_0 c_0 \|y - \bar{y}\|_{X_h.r} + h_0 c_1\, \sigma(t^{(r)}, \kappa \|y - \bar{y}\|_{X_h.r}), \quad 0 \leq r \leq K - 1.$$

The proof will be divided into three steps.

I. We first prove that

$$\Big|\, F_h[z, \bar{u}]^{(r,m)} - F_h[w_h, \bar{v}_h]^{(r,m)} \,\Big| \leq \|z - w_h\|_{h.r} + A h_0 \|u - v_h\|_{h.r} \tag{27}$$
$$+ A \kappa h_0 \big[\|z - w_h\|_{h.r} + \|u - v_h\|_{h.r} \big] + 2 h_0 \bar{c}\, \sigma(t^{(r)}, \kappa[\|z - w_h\|_{h.r} + \|u - v_h\|_{h.r}])$$

on E'_h, where

$$\kappa = \max\{1, \ L_0\,\}. \tag{28}$$

Let us denote by $\Gamma_{h.0}$, $\Lambda_{h.0} : E'_h \to \mathbb{R}$ the functions defined by

$$\Gamma_{h.0}^{(r,m)} = \Delta(z - w_h)^{(r,m)} + h_0 \partial_q f(P^{(r,m)}[z, u]) \circ \delta(z - w_h)^{(r,m)},$$

and

$$\Lambda_{h.0}^{(r,m)} = h_0 \big[\, f(P^{(r,m)}[z, u]) - f(P^{(r,m)}[w_h, v_h])\,\big]$$
$$+ h_0 \big[\, \partial_q f(P^{(r,m)}[z, u]) - \partial_q f(P^{(r,m)}[w_h, v_h])\,\big] \circ \delta w_h^{(r,m)}$$
$$+ h_0 \big[\, \partial_q f(P^{(r,m)}[w_h, v_h]) - \partial_q f(P^{(r,m)}[z, u])\,\big] \circ v_h^{(r,m)}$$
$$+ h_0 \partial_q f(P^{(r,m)}[z, u]) \circ \big(\, v_h - u\,\big)^{(r,m)}.$$

Then we have

$$F_h[z, \bar{u}]^{(r,m)} - F_h[w_h, \bar{v}_h]^{(r,m)} = \Gamma_{h.0}^{(r,m)} + \Lambda_{h.0}^{(r,m)} \ \text{ on } \ E'_h. \tag{29}$$

We see at once that

$$\Gamma_{h.0}^{(r,m)} = \frac{1}{2}\sum_{i=1}^{n}(z-w_h)^{(r,m+e_i)}\Big[\,\frac{1}{n}+\frac{h_0}{h_i}\partial_{q_i}f(P^{(r,m)}[z,u])\,\Big]$$

$$+\frac{1}{2}\sum_{i=1}^{n}(z-w_h)^{(r,m-e_i)}\Big[\,\frac{1}{n}-\frac{h_0}{h_i}\partial_{q_i}f(P^{(r,m)}[z,u])\,\Big].$$

According to assumption (22) we get

$$\big|\,\Gamma_{h.0}^{(r,m)]}\,\big| \le \|z-w_h\|_{h.r}, \;\; (t^{(r)},x^{(m)})\in E'_h. \tag{30}$$

It follows from Assumption H$[f,\sigma]$ that

$$\big|\,f(P^{(r,m)}[z,u])-f(P^{(r,m)}[w_h,v_h])\,\big| \le A\kappa\big[\,\|z-w_h\|_{h.r}+\|u-v_h\|_{h.r}\big]$$

and

$$\|\,\partial_q f(P^{(r,m)}[z,u])-\partial_q f(P^{(r,m)}[w_h,v_h])\,\| \tag{31}$$

$$\le \sigma(t^{(r)},\kappa[\|z-w_h\|_{h.r}+\|u-v_h\|_{h.r}]).$$

The above estimates and (25) imply

$$\big|\,\Lambda_{h.0}^{(r,m)}\,\big| \le \kappa A h_0\,\big[\|z-w_h\|_{h.r}+\|u-v_h\|_{h.r}\big] \tag{32}$$

$$+Ah_0\|u-v_h\|_{h.r}+2\bar{c}\,\sigma(t^{(r)},\kappa[\|z-w_h\|_{h.r}+\|u-v_h\|_{h.r}]).$$

Relations (29), (30), (32) imply (27).

II. Now we prove that

$$\|\,G_h[z,\bar{u}]^{(r,m)}-G_h[w_h,\bar{v}_h]^{(r,m)}\| \le \|u-v_h\|_{h.r} \tag{33}$$

$$+AL_1h_0\big[\|z-w_h\|_{h.0}+\|u_0-v_{h.0}\|_{h.0}+\|u-v_h\|_{h.r}\big]$$

$$+h_0(1+\bar{a}+\bar{c})\sigma(t^{(r)},\kappa[\|z-w_h\|_{h.r}+\|u-v_h\|_{h.r}])$$

on E'_h. Let the functions Γ_h, $\Lambda_h : E'_h \to \mathbb{R}^n$ be defined by

$$\Gamma_h^{(r,m)} = \Delta(u-v_h)^{(r,m)}+h_0\,\partial_q f(P^{(r,m)}[z,u])\star\big[\,\delta(u-v_h)^{(r,m)}\,\big]^T$$

and

$$\Lambda_h^{(r,m)} = h_0\big[\,\partial_q f(P^{(r,m)}[z,u])-\partial_q f(P^{(r,m)}[w_h,v_h])\,\big]\star\big[\,\delta v_h^{(r,m)}\,\big]^T$$

$$+h_0\,\big[\,\partial_x f(P^{(r,m)}[z,u])-\partial_x f(P^{(r,m)}[w_h,v_h])\,\big]$$

$$+h_0\,\partial_p f(P^{(r,m)}[z,u])\,L_h[z,\bar{u}]^{(r,m)}-h_0\,\partial_p f(P^{(r,m)}[w_h,v_h])\,L_h[w_h,\bar{v}_h]^{(r,m)}.$$

Then we have

$$G_h[z,\bar{u}]^{(r,m)}-G_h[w_h,\bar{v}_h]^{(r,m)} = \Gamma_h^{(r,m)}+\Lambda_h^{(r,m)} \;\text{ on }\; E'_h. \tag{34}$$

It follows easily that

$$\|\Gamma_h^{(r,m)}\| \le \frac{1}{2}\sum_{i=1}^{n} \|(u-v_h)^{(r,m+e_i)}\| \Big[\,\frac{1}{n} + \frac{h_0}{h_i}\partial_{q_i} f(P^{(r,m)}[z,u])\,\Big]$$

$$+\frac{1}{2}\sum_{i=1}^{n} \|(u-v_h)^{(r,m-e_i)}\| \Big[\,\frac{1}{n} - \frac{h_0}{h_i}\partial_{q_i} f(P^{(r,m)}[z,u])\,\Big]$$

and consequently

$$\|\,\Gamma_h^{(r,m)} \quad \le \|u-v_h\|_{h.r} \text{ on } E'_h. \tag{35}$$

It is easily seen that estimates analogous to (31) are satisfied for the derivatives $\partial_x f$ and $\partial_p f$. Then we have

$$\|\,\Lambda_h^{(r,m)}\,\| \le AL_1 h_0\big[\|z-w_h\|_{h.0} + \|u_0 - v_{h.0}\|_{h.0} + \|u-v_h\|_{h.r}\big]$$

$$+h_0(1+\bar a+\bar c)\sigma(t^{(r)},\kappa[\|z-w_h\|_{h.r} + \|u-v_h\|_{h.r}]) \text{ on } E'_h.$$

The above estimate and (35) imply (33).

III. In a similar way we prove that

$$\|\,G_{h.0}[z,\bar u]^{(r,m)} - G_{h.0}[w_h,\bar v_h]^{(r,m)}\| \le \|u_0 - v_{h.0}\|_{h.r} \tag{36}$$

$$+AL_1 h_0\big[\|z-w_h\|_{h.0} + \|u_0 - v_{h.0}\|_{h.0} + \|u-v_h\|_{h.r}\big]$$

$$+h_0(1+\bar a+\bar c)\sigma(t^{(r)},\kappa[\|z-w_h\|_{h.r} + \|u-v_h\|_{h.r}])$$

on E'_h.

Since

$$F_h[z,\bar u]^{(r,m)} = 0,\quad G_{h.0}[z,\bar u]^{(r,m)} = 0,\quad G_{h.0}[z,\bar u]^{(r,m)} = \theta$$

for $t^{(r)} \in J'_h$, $x^{(m)} \in U_h$ and $(t^{(r)},x^{(m)}) \notin E'_h$, then estimates (27), (33), (36) imply (26) with

$$c_0 = A(\kappa+1+2L_1),\quad c_1 = 2(1+\bar a+2\bar c)$$

and κ, $\bar a$ given by (28), (24).

It follows that there is $\gamma : H \to \mathbb{R}_+$ such that

$$\big|w_h^{(r+1,m)} - F_h[w_h,\bar v_h]^{(r,m)}\big| + \big|v_{h.0}^{(r+1,m)} - G_{h.0}[w_h,\bar v_h]^{(r,m)}\big|$$

$$+\big\|v_h^{(r+1,m)} - F_h[w_h,\bar v_h]^{(r,m)}\big\| \le h_0\gamma(h) \text{ on } E'_h$$

and $\lim_{h\to 0}\gamma(h) = 0$. Let $\beta_h : J_h \to \mathbb{R}_+$ be a solution of the difference equation

$$\beta^{(r+1)} = \beta^{(r)} + c_0 h_0 \beta^{(r)} + c_1 h_0\,\sigma(t^{(r)},\kappa\beta^{(r)}) + h_0\gamma(h),\quad 0 \le r \le K-1, \tag{37}$$

with the initial condition $\beta^{(0)} = \alpha_0(h)$. It follows from Theorem 1 that

$$|w^{(r,m)} - z_h^{(r,m)}| + |\partial_t w^{(r,m)} - u_{h.0}^{(r,m)}| + \|\partial_x w^{(r,m)} - u_h^{(r,m)}\| \le \beta_h^{(r)} \text{ on } E_h. \tag{38}$$

Consider the Cauchy problem

$$\omega'(t) = c_0\omega(t) + c_1\sigma(t, \kappa\, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h). \tag{39}$$

It follows from conditions 1), 2) of Assumption H $[f, \sigma]$ that there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ the maximal solution ω_h of the above problem is defined on $[0, a]$ and

$$\lim_{h \to 0} \omega_h(t) = 0 \text{ uniformly on } [0, a].$$

Moreover, we have the difference inequality

$$\omega_h^{(r+1)} \ge \omega_h^{(r)} + c_0 h_0 \omega_h^{(r)} + c_1 h_0\, \sigma(t^{(r)}, \kappa\omega_h^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K - 1. \tag{40}$$

Since $\beta^{(0)} = \omega_h^{(0)}$, relations (37), (40) show that $\beta^{(r)} \le \omega_h^{(r)}$ for $0 \le r \le K$. Then we obtain estimate (23) for $\alpha(h) = \omega_h(a)$. This proves the theorem. □

Remark 6. Suppose that all the assumptions of Theorem 5 are satisfied with

$$\sigma(t, \tau) = L\tau, \quad (t, \tau) \in [0, a] \times \mathbb{R}_+,$$

where $L \in \mathbb{R}_+$ and the solution $v : E_0 \cup E \to \mathbb{R}$ of (1), (2) is of class C^3. Then there are $C_1, C_2 \in \mathbb{R}_+$ such that

$$|w^{(r,m)} - z_h^{(r,m)}| + |\partial_t w^{(r,m)} - u_{h.0}^{(r,m)}| + \|\partial_x w^{(r,m)} - u_h^{(r,m)}\|$$

$$\le C_0\alpha_0(h) + C_1 h_0 \;\; \textit{on } E_h.$$

The above estimate may be proved by using the definition of $\alpha(h)$ and by solving problem (39).

In the result on error estimates we need estimates for the derivatives of the solution v of problem (1), (2). One may obtain them by the method of differential inequalities, see Kamont [6], Chapter 1.

5. NUMERICAL EXPERIMENTS

For $n = 1$ we put

$$E = \{\, (t, x) : \;\; t \in [0,\, 0.5], \;\; -4 + t \le x \le 4 - t \,\}.$$

Consider the equation with deviated variables

$$\partial_t z(t,x) = \cos(\,\partial_x z(t,x)\,) - \cos(\,8tx\,z(t,x)\,) \tag{41}$$

$$+4(2t-x^2)z(t,x) + z(0.5t, tx) - \exp\big[\,t^2(1-2tx^2)\,\big]$$

and the initial condition

$$z(0,x) = 1 \;\text{ for }\; x \in [-4,4]. \tag{42}$$

The solution of the above problem is given by $w(t,x) = \exp\big[\,4t(t-x^2)\,\big]$. For $h_0 = h_1 = h$ we put

$$t^{(r)} = rh,\;\; x^{(m)} = mh,\;\; r = 0,1,\ldots,K,\;\; m = -N, -N+1, \ldots, N-1, N,$$

where K, $N \in \mathbb{N}$ and $Kh = 0.5$, $Nh = 4$. Write

$$E_h = \{\,(t^{(r)}, x^{(m)}) : \;\; 0 \le r \le K,\;\; -N + r \le m \le N - r\,\}.$$

Let us denote by $(z_h, u_{h.0}, u_h) : E_h \to \mathbb{R}^3$ the solution of the generalized Lax method for (41), (42).

Write

$$\varepsilon_h^{(r)} = \frac{1}{2N-2r+1} \sum_{m=-N+r}^{N-r} \big|w^{(r,m)} - z_h^{(r,m)}\big|, \tag{43}$$

and

$$\eta_h^{(r)} = \frac{1}{2N-2r+1} \sum_{m=-N+r}^{N-r} \big|\partial_z w^{(r,m)} - u_h^{(r,m)}\big|, \tag{44}$$

where $0 \le r \le K$. The numbers $\varepsilon_h^{(r)}$ and $\eta_h^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$.

In the table below we give experimental values of the functions ε_h and η_h.

Table of errors

	$(\varepsilon_h, \eta_h),\; h = 0.01$	$(\varepsilon_h, \eta_h),\; h = 0.001$
$t^{(r)} = 0.30$	$0.0082,\; 0.0070$	$0.0008,\; 0.0007$
$t^{(r)} = 0.35$	$0.0094,\; 0.0095$	$0.0009,\; 0.0009$
$t^{(r)} = 0.40$	$0.0114,\; 0.0139$	$0.0011,\; 0.0014$
$t^{(r)} = 0.45$	$0.0145,\; 0.0209$	$0.0014,\; 0.0021$
$t^{(r)} = 0.50$	$0.0191,\; 0.0328$	$0.0019,\; 0.0033$

Methods described in this paper have the potential for applications in the numerical solution of functional differential problems. They have the following property: a large numbers od previous values $(z_h^{(r,m)}, u_{h.0}^{(r,m)}, u_h^{(r,m)})$ must be preserved, because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

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