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FIBONACCI Q-TYPE MATRICES AND PROPERTIES OF A CLASS OF NUMBERS RELATED TO THE FIBONACCI, LUCAS AND PELL NUMBERS

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ABSTRACT: A class of numbers that contains the Fibonacci, Lucas and Pell numbers is investigated, and some properties are obtained. Fibonacci Q-type matrices are constructed as a tool to get many properties for Fibonacci and Pell numbers.

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1. INTRODUCTION

In spite of many articles and books that have been published on the Fibonacci, Lucas and Pell numbers and their applications Bong [1], Dubner and Keller [2], Hoggatt [4], Horadam [6], Horadam [7], Philipou [8], Sloane and Plouffe [9], Vorob'ev [10], these numbers are still an interesting subject for investigation. In this article we introduce interesting properties satisfied by a large class of Fibonacci type numbers. Let us note that this class has been studied extensively by Horadam [6] and [7], where several properties has been obtained.

The numbers we are going to consider satisfy the following recurrence relations:

$$U(n+2,u_1,u_2,a,b)=aU(n+1,u_1,u_2,a,b)+bU(n,u_1,u_2,a,b)\ , \qquad (1)$$

where

$$U(1,u_1,u_2,a,b)=u_1\ ,\quad U(2,u_1,u_2,a,b)=u_2\ ,\qquad ab\neq 0\ , \qquad (2)$$

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$n \geq 2$ is an integer and u_1, u_2, a, b are real. We observe that:

(i) The Fibonacci numbers $F(n)$ defined by $F(n+2) = F(n+1) + F(n)$ and $F(1) = F(2) = 1$ satisfy

$$U(n,1,1,1,1) = F(n) \ . \tag{3}$$

(ii) The Lucas numbers $L(n)$ defined by $L(n+2) = L(n+1) + L(n)$ and $L(1) = 1,\ L(2) = 3$ satisfy

$$U(n,1,3,1,1) \quad = \quad L(n)\,, \tag{4}$$

(iii) The Pell numbers $P(n)$ defined by $P(n+2) = P(n+1) + P(n)$ and $P(1) = 1,\ P(2) = 2$ satisfy

$$U(n,1,2,2,1) = \ P(n)\,, \tag{5}$$

(iv) The numbers $R(a,n)$ defined by $R(a,n+2) = aR(a,n+1) + R(a,n)$, where $R(a,0) = 0,\ R(a,1) = 1$ and $a \geq 1$ is an integer, satisfy

$$U(n,1,a,a,1) \quad = \quad R(a,n)\,. \tag{6}$$

This class of numbers has been introduced by Entringer and Slater [3], while investigating the problem of information dissemination through telegraphs. In 1986 Bong [1] obtained several interesting properties for $R(a,n)$.

In Sections 2 and 3 we introduce interesting properties that satisfied by $U(n, u_1, u_2, a, b)$ and consequently by the Fibonacci, Lucas and Pell numbers. Section 5 deals with solutions of quadratic matrix difference equation and a special class of these solutions that leads to the definitions Q-type matrices.

In what follows, for simplicity, we write $U(n)$ instead of $U(n, u_1, u_2, a, b)$ when the appearance of the arguments is not necessary.

2. PROPERTIES OF $U(n)$

The characteristic equation

$$\lambda^2 - a\lambda - b = 0 \tag{7}$$

of (1) has two roots

$$\lambda_1 = \frac{a + \varepsilon\sqrt{\Delta}}{2} \quad , \quad \lambda_2 = \frac{a + \varepsilon\sqrt{\Delta}}{2} \quad , \tag{8}$$

where $\Delta = |a^2 + 4b|$, $\varepsilon = 1$ when $a^2 + 4b > 0$ and $\varepsilon = \sqrt{-1}$ for $a^2 + 4b < 0$.

In the case when (7) has equal roots, then $b = -a^2/4$ and $\lambda_1 = \lambda_2 = a/2$. The corresponding solution of Eq. (1) will be denoted by $D(n, p_1, p_2, a)$, where we have

$$D(n+2, p_1, p_2, a) = aD(n+1, p_1, p_2, a) - \frac{a^2}{4} D(n, p_1, p_2, a) \tag{9}$$

and

$$D(1, p_1, p_2, a) = p_1 \quad , \quad D(2, p_1, p_2, a) = p_2 \quad . \tag{10}$$

For simplicity we write $D(n)$ instead of $D(n, p_1, p_2, a)$ when the appearance of the arguments is not necessary.

Lemma 1. *Assume that* $\lambda_1 \neq \lambda_2$, *then for* $\alpha, \beta, \gamma, \delta$ *and* $k \in \mathbb{R}$, *the following relations hold true :*

$$U(n, \alpha u_1, \alpha u_2, a, b) \;=\; \alpha U(n, u_1, u_2, a, b) \quad , \tag{11}$$

$$U(n, u_1, k u_2, ka, k^2 b) = k^{n-1} U(n, u_1, u_2, a, b) \quad , \tag{12}$$

$$U(n, \alpha v_1 + \beta v_2, \gamma w_1 + \delta w_2, a, b) = U(n, \alpha v_1, \gamma w_1, a, b) + U(n, \beta v_2, \delta w_2, a, b) \, , \tag{13}$$

and

$$U(n, u_1, u_2, a, b) \;= b u_1 \, U(n-2, 1, a, a, b) \; U(n-1, 1, a, a, b) \quad . \tag{14}$$

Proof. The solution of (1) that satisfies $u_1 = U(1)$ and $u_2 = U(2)$ can be obtained by determining c_1 and c_2 in the solution

$$U(n) = c_1 \lambda_1^n + c_2 \lambda_2^n \quad . \tag{15}$$

Therefore, we get

$$U(n) = \frac{u_2 - \lambda_2 u_1}{\lambda_1 - \lambda_2} \lambda_1^{n-1} \; - \; \frac{u_2 - \lambda_1 u_1}{\lambda_1 - \lambda_2} \lambda_2^{n-1} \, , \tag{16}$$

or

$$U(n, u_1, u_2, a, b) = \frac{1}{\varepsilon \sqrt{\Delta}} [b u_1 (\lambda_1^{n-2} - \lambda_2^{n-2}) + u_2 (\lambda_1^{n-1} - \lambda_2^{n-1})] \quad . \tag{17}$$

Since $U(n, u_1, u_2, a, b)$ is linear and homogeneous function with respect to u_1 and u_2, the relations (11) and (13) follow directly. If we notice that $\lambda_1 = \lambda_1(a, b)$ and $\lambda_2 = \lambda_2(a, b)$ satisfy

$$\lambda_i(ka, k^2b) = k\lambda_i(a, b) \quad , \quad i = 1, 2 \, , \tag{18}$$

then from Eq.(17) we obtain

$$U(n, u_1, ku_2, ka, k^2b) = \frac{1}{\varepsilon k\sqrt{\Delta}}[k^2u_1k^{n-2}(\lambda_1^{n-2} - \lambda_2^{n-2}) + ku_2k^{n-1}(\lambda_1^{n-1} - \lambda_2^{n-1})] \quad ,$$

and (12) follows directly.

From (7) we infer that $Q(p) = \lambda_1^p - \lambda_2^p$ satisfies $Q(p + 2) = aQ(p + 1) + bQ(p)$ with $Q(1) = \varepsilon\sqrt{\Delta}$ and $Q(1) = a\varepsilon\sqrt{\Delta}$. Therefore

$$Q(p) = U(p, \varepsilon\sqrt{\Delta}, a\varepsilon\sqrt{\Delta}, a, b)$$

and (11) implies that

$$Q(p) = \varepsilon\sqrt{\Delta}U(p, 1, a, a, b) \, , \tag{19}$$

and we obtain (14) by using (19) and noticing that

$$U(n, u_1, u_2, a, b) = \frac{1}{\varepsilon\sqrt{\Delta}} \left[bu_1Q(n - 2) + u_2Q(n - 1)\right] . \tag{20}$$

In what follows we introduce the main properties of $U(n)$.

Theorem 1. *Assume that* $\lambda_1 \neq \lambda_2$ *and* $n > m \geq 1$ *. Then the following relations hold true:*

$$\begin{aligned} & U(n + m + 1)U(n - m) - U(n + m)U(n - m + 1) \\ = \; & (-b)^{n-m-1}(bu_1^2 + au_1u_2 - u_2^2)V(2m) \end{aligned} \tag{21}$$

and

$$\begin{aligned} & U(n + k + 1)U(n - k) - U(n + 1)U(n) \\ = \; & \frac{(bu_1^2 + au_1u_2 - u_2^2)}{a^2 + 4b}(-b)^{n-k-1}[-a(-b)^k + V(2k + 2) + bV(2k)] \, , \end{aligned} \tag{22}$$

where

$$V(n) = U(n, 1, a, a, b) \qquad , \quad (\, n > k \,) \quad . \tag{23}$$

Proof. From (15) we have

$$\begin{aligned} & U(n+m+1)U(n-m) - U(n+m)U(n-m+1) \\ = & \ (c_1\lambda_1^{n+m+1} + c_2\lambda_2^{n+m+1})(c_1\lambda_1^{n-m} + c_2\lambda_2^{n-m}) \\ & -(c_1\lambda_1^{n+m} + c_2\lambda_2^{n+m})(c_1\lambda_1^{n-m+1} + c_2\lambda_2^{n-m+1}) \\ = & \ c_1c_2(\lambda_1\lambda_2)^{n-m}(\lambda_1 - \lambda_2)(\lambda_1^{2m} - \lambda_2^{2m}) \quad . \end{aligned} \tag{24}$$

If we notice that

$$\lambda_1\lambda_2 = -b \ , \ \lambda_1 - \lambda_2 = \varepsilon\sqrt{\Delta} \ , \ \lambda_1^{2m} - \lambda_2^{2m} = Q(2m)$$

and

$$c_1c_2 = \frac{(bu_1^2 + au_1u_2 - u_2^2)}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2} \quad , \tag{25}$$

it follows from (24) that

$$\begin{aligned} & U(n+m+1)U(n-m) - U(n+m)U(n-m+1) \\ = & \ \frac{(-b)^{n-m-1}}{\varepsilon\sqrt{\Delta}}(bu_1^2 + au_1u_2 - u_2^2)\, U(2m, \varepsilon\sqrt{\Delta}, a\varepsilon\sqrt{\Delta}, a, b) \quad . \end{aligned}$$

Applying (11) we conclude (21).

From (21), we write in succession

$$U(n+2)U(n-1) - U(n+1)U(n) = h(-b)^{n-2}V(2)$$

$$U(n+3)U(n-2) - U(n+2)U(n-1) = h(-b)^{n-3}V(4)$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$U(n+k+1)U(n-k) - U(n+k)U(n-k+1) = h(-b)^{n-k-1}V(2k) \quad ,$$

where $h = (bu_1^2 + au_1u_2 - u_2^2)$ and $V(n)$ is defined by (23). Summing these, we obtain

$$U(n+k+1)U(n-k) - U(n+1)U(n) = (bu_1^2 + au_1u_2 - u_2^2)G \quad , \tag{26}$$

where

$$G = \sum_{i=1}^{k}(-b)^{n-i-1}V(2i) = (-b)^{n-1}\sum_{i=1}^{k}(-b)^{-i}\frac{\lambda_1^{2i} - \lambda_2^{2i}}{\lambda_1 - \lambda_2} \quad .$$

Since

$$\sum_{i=1}^{k}\left(-\frac{\lambda^2}{b}\right)^i = -\frac{\lambda^2}{b} \cdot \frac{1 - (-\frac{\lambda^2}{b})^k}{1 + \frac{\lambda^2}{b}} \quad ,$$

we get

$$G = \frac{(-b)^{n-1}}{\lambda_1 - \lambda_2}\left[-\lambda_1^2\left(\frac{1 - q_1}{b + \lambda_1^2}\right) + \lambda_2^2\left(\frac{1 - q_2}{b + \lambda_2^2}\right)\right] \quad ,$$

where $q_1 = (-\frac{\lambda_1^2}{b})^k$ and $q_2 = (-\frac{\lambda_2^2}{b})^k$. Hence

$$G = \frac{(-b)^{n-1}}{(\lambda_1 - \lambda_2)\,(b+\lambda_1^2)\,(\,b+\lambda_2^2)}\,[-\lambda_1^2\,(\,b+\lambda_2^2)\,(1-q_1) + \lambda_2^2\,(\,b+\lambda_1^2)\,(1-q_2)]\,.$$

If we notice that

$$(b+\lambda_1^2)\,(\,b+\lambda_2^2) = (a\lambda_1 + 2b)\,(a\lambda_2 + 2b) = a^2 b + 4b^2$$

and

$$\lambda^2(b+\lambda^2) = b\lambda^2 + b^2 = (a\lambda + b)b + b^2 = ab\lambda + 2b^2$$

for $\lambda = \lambda_1\,,\ \lambda_2\,,$ then

$$G = \frac{(-b)^{n-1}}{(a^2+4b)(\lambda_1-\lambda_2)}\,[\,-a(\lambda_1-\lambda_2) + a(\lambda_1 q_1 - \lambda_2 q_2) + 2b(q_1 - q_2)\,]$$

$$\begin{aligned} &= \frac{(-b)^{n-1}}{(a^2+4b)(\lambda_1-\lambda_2)} \\ &\quad \times \left[\,-a(\lambda_1-\lambda_2) + \frac{1}{(-b)^k}[a(\lambda_1^{2k+1} - \lambda_2^{2k+1}) + 2b(\lambda_1^{2k} - \lambda_2^{2k})]\,\right] \\ &= \frac{(-b)^{n-k-1}}{(a^2+4b)}[-a(-b)^k + aV(2k+1) + 2bV(2k)] \\ &= \frac{(-b)^{n-k-1}}{(a^2+4b)}[-a(-b)^k + V(2k+2) + bV(2k)]\ . \end{aligned}$$

The substitution of G in (26) completes the proof. □

Theorem 2. *Assume that* $\lambda_1 \neq \lambda_2$ *and* $n > m \geq 2$. *Then the following relations hold true :*

$$\begin{aligned} &U(n+m)U(n-m) - U(n+m-1)U(n-m+1) \\ = \ &(-b)^{n-m-1}(bu_1^2 + au_1u_2 - u_2^2)V(2m-1) \end{aligned} \tag{27}$$

and

$$\begin{aligned} &U(n+k)U(n-k) - U(n+1)U(n-1) \\ = \ &\frac{b(bu_1^2 + au_1u_2 - u_2^2)}{a^2+4b}(-b)^{n-k} \\ &\times[(a^2+2b)(-b)^{k-1} - V(2k+1) - bV(2k-1)] \end{aligned} \tag{28}$$

for $n > k \geq 2$, *where* $V(n)$ *is defined by (23).*

Proof. Following the steps of the proof of (22), we obtain

$$U(n+k)U(n-k) - U(n+1)U(n-1) = (bu_1^2 + au_1u_2 - u_2^2)H\ , \tag{29}$$

where

$$H = \sum_{i=2}^{k} (-b)^{n-i-1} V(2i-1) \ .$$

But

$$\begin{aligned} H &= \sum_{i=2}^{k} (-b)^{n-i-1} \frac{\lambda_1^{2i-1} - \lambda_2^{2i-1}}{\lambda_1 - \lambda_2} \\ &= \frac{(-b)^{n-1}\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \sum_{i=2}^{k} [\lambda_2(-\frac{\lambda_1^2}{b})^i - \lambda_1(-\frac{\lambda_2^2}{b})^i] \\ &= \frac{(-b)^{n-1}}{(a^2+4b)(\lambda_1-\lambda_2)} \left[\lambda_1^3(b+\lambda_2^2)(1-r_1) - \lambda_2^3(b+\lambda_1^2)(1-r_2) \right] \ , \end{aligned}$$

where $r_1 = (-\frac{\lambda_1^2}{b})^{k-1}$ and $r_2 = (-\frac{\lambda_2^2}{b})^{k-1}$. If we notice that

$$\lambda_1^3(b+\lambda_2^2) = \lambda_1(b\lambda_1^2 + b^2) = \lambda_1(ab\lambda_1 + 2b^2) = (a^2b + 2b^2)\lambda_1 + ab^2$$

and $\lambda_2^3(b+\lambda_1^2) = (a^2b+2b^2)\lambda_2 + ab^2$, then

$$\begin{aligned} H &= \frac{(-b)^{n-1}}{(a^2+4b)(\lambda_1-\lambda_2)} \\ &\times \left[-a(\lambda_1-\lambda_2) - \frac{1}{(-b)^{k-1}}[\alpha(\lambda_1^{2k-1} - \lambda_2^{2k-1}) + \beta(\lambda_1^{2k-2} - \lambda_2^{2k-2})] \right] \ , \end{aligned}$$

where $\alpha = a^2b + 2b^2$ and $\beta = ab^2$. Thus

$$H = \frac{(-b)^{n-k+1}}{(a^2+4b)} \left[(a^2+2b)(-b)^{k-1} - [(a^2+2b)V(2k-1) + abV(2k-2)] \right] \ .$$

But

$$(a^2+2b)V(2k-1) + abV(2k-2) = 2bV(2k-1) + aV(2k) = bV(2k-1) + V(2k+1) \ .$$

Therefore

$$H = \frac{(-b)^{n-k+1}}{(a^2+4b)} \left[(a^2+2b)(-b)^{k-1} - [V(2k+1) \ + bV(2k-1)] \right] \ .$$

The substitution of H in (29) completes the proof of (28). The relation (27) can be proved in a similar way to that of (21). □

In the case when the equation (7) has equal roots, we have the following:

Theorem 3. *Assume that Eq.(7) has equal roots. Then the following relations hold true :*

$$\begin{aligned} & D(n+m+1)D(n-m) - D(n+m)D(n-m+1) \\ = \ & -\frac{m}{2}(\frac{a}{2})^{2n-3}(2p_2 - ap_1)^2 \ , \ (n > m \geq 1) \end{aligned} \tag{30}$$

and

$$\begin{aligned} & D(n+m)D(n-m) - D(n+m-1)D(n-m+1) \\ = & -\frac{2m-1}{2}(\frac{a}{2})^{2n-4}(2p_2 - ap_1)^2 \ , \ (n > m \geq 2). \end{aligned} \tag{31}$$

where $D(n)$ is defined by (9) and (10).

Proof. The solution of (9) and (10) can be written as follows:

$$D(n) = \lambda_1^n(c_1 + c_2 n) \ , \ D(1) = p_1 \ , \ D(2) = p_2 \ , \tag{32}$$

where

$$c_1 = \frac{2p_1\lambda_1 - p_2}{\lambda_1^2} \ , \ c_2 = \frac{p_2 - p_1\lambda_1}{\lambda_1^2} \ , \ \lambda_1 = \frac{a}{2} \ . \tag{33}$$

Using (32) and (33) we have

$$D(n+m+1)D(n-m) - D(n+m)D(n-m+1) = -2mc_2^2\lambda_1^{2n+1} \ .$$

The relation (30) follows directly, if we notice that

$$c_2^2\lambda_1^{2n+1} = \frac{1}{4}(\frac{a}{2})^{2n-3}(2p_2 - ap_1)^2 \ . \qquad \square$$

Theorem 4. *Assume that $\lambda_1 = \lambda_2$. Then the following relations hold true:*

$$\begin{aligned} & D(n+k+1)D(n-k) - D(n+1)D(n) \\ = & -\frac{k(k+1)}{2}(\frac{a}{2})^{2n-3}(2p_2 - ap_1)^2 \ , \ (n > k \geq 1) \end{aligned} \tag{34}$$

and

$$\begin{aligned} & D(n+k)D(n-k) - D(n+1)D(n-1) \\ = & -\frac{k^2-1}{2}(\frac{a}{2})^{2n-4}(2p_2 - ap_1)^2 \ , \ (n > k \geq 2) \end{aligned} \tag{35}$$

where $D(n)$ is defined by (9) and (10).

Proof. Putting $m = 1, 2, ..., k$ in (30) and (31) and summing up the resulting relations, we obtain

$$\begin{aligned} & D(n+k+1)D(n-k) - D(n+1)D(n) \\ = & -\frac{1}{2}(\frac{a}{2})^{2n-3}(2p_2 - ap_1)^2 \sum_{i=1}^{k} i \end{aligned}$$

and

$$\begin{aligned} & D(n+k)D(n-k) - D(n+1)D(n-1) \\ & -\frac{1}{4}(\frac{a}{2})^{2n-4}(2p_2 - ap_1)^2 \sum_{i=1}^{k}(2i-1) \ . \end{aligned}$$

The substitution of the summations in the previous equations implies (34) and (35). $\square$

3. APPLICATIONS

3.1. FIBONACCI NUMBERS

Since $F(n) = U(n,1,1,1,1)$, we get from Theorems 2 and 3 the following properties :

$$F(n+m+1)F(n-m) - F(n+m)F(n-m+1) = (-1)^{n-m-1}F(2m), \quad n > m \geq 1, \tag{36}$$

$$F(n+m)F(n-m) - F(n+m-1)F(n-m+1) = (-1)^{n-m-1}F(2m-1), \quad n > m \geq 2, \tag{37}$$

$$\begin{aligned} & F(n+k+1)F(n-k) - F(n+1)F(n) \\ = & \frac{(-1)^{n-k-1}}{5}[(-1)^{k+1} + F(2k+2) + F(2k)],\ n > k \geq 1, \end{aligned} \tag{38}$$

and

$$\begin{aligned} & F(n+k)F(n-k) - F(n+1)F(n-1) \\ = & \frac{(-1)^{n-k}}{5}[3(-1)^{k+1} - F(2k+1) + F(2k-1)],\ n > k \geq 2. \end{aligned} \tag{39}$$

3.2. LUCAS NUMBERS

Since $L(n) = U(n,1,3,1,1)$, we get from Theorems 2 and 3 the following properties :

$$L(n+m+1)L(n-m) - L(n+m)L(n-m+1) = (-1)^{n-m}5F(2m), \quad n > m \geq 1, \tag{40}$$

$$\begin{aligned} & L(n+k+1)L(n-k) - L(n+1)L(n) \\ = & (-1)^{n-k}[(-1)^{k+1} + F(2k+2) + F(2k)],\ n > k, \end{aligned} \tag{41}$$

$$L(n+m)L(n-m) - L(n+m-1)L(n-m+1) = (-1)^{n-m}5F(2m-1), \quad n > m \geq 2, \tag{42}$$

and

$$\begin{aligned} & L(n+k)L(n-k) - L(n+1)L(n-1) \\ = & (-1)^{n-k+1}[3(-1)^{k-1} - F(2k+1) - F(2k-1)],\ n > k \geq 2\ . \end{aligned} \tag{43}$$

3.3. PELL NUMBERS

Since $P(n) = U(n,1,2,2,1)$, we get from Theorems 2 and 3 the following properties :

$$P(n+m+1)P(n-m) - P(n+m)P(n-m+1) = (-1)^{n-m-1}P(2m), \quad n > m \geq 1, \quad (44)$$

$$\begin{aligned} & P(n+k+1)P(n-k) - P(n+1)P(n) \\ = \; & \frac{(-1)^{n-k-1}}{8}[2(-1)^{k+1} + P(2k+2) + P(2k)], \; n > k, \end{aligned} \quad (45)$$

$$P(n+m)P(n-m) - P(n+m-1)P(n-m+1) = (-1)^{n-m-1}P(2m-1), \quad n > m \geq 2, \quad (46)$$

and

$$\begin{aligned} & P(n+k)P(n-k) - P(n+1)P(n-1) \\ = \; & \frac{(-1)^{n-k}}{8}[6(-1)^{k-1} - P(2k+1) - P(2k-1)], \; n > k \geq 2 \, . \end{aligned} \quad (47)$$

3.4. THE CLASS $R(n,a)$

Since $R(n,a) = U(n,1,a,a,1)$, we get from Theorems 2 and 3 the following properties:

$$\begin{aligned} & R(n+m+1,a)R(n-m,a) - R(n+m,a)R(n-m+1,a) \\ = \; & (-1)^{n-m-1}R(2m,a), \; n > m \geq 1 \end{aligned} \quad (48)$$

and

$$\begin{aligned} & R(n+m,a)R(n-m,a) - R(n+m-1,a)R(n-m+1,a) \\ = \; & (-1)^{n-m-1}R(2m-1,a), \; n > m \geq 2 \, . \end{aligned} \quad (49)$$

Two more properties similar to (38) and(39) can be written for $R(n,a)$.

4. FIBONACCI Q – TYPE MATRICES

The Fibonacci Q -matrix had been defined first by Honsberger [5] as follows:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} , \quad (50)$$

where F_n is a Fibonacci number. He showed that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{51}$$

and noticed that the Fibonacci Q -matrix is a good tool for getting a number of important identities related to Fibonacci numbers. It seems that Fibonacci Q -matrix belongs to a larger class of matrices that satisfies (50) and (51). In this section we construct this class of matrices, where we can define, for example, Pell P -matrix, which is similar to Fibonacci Q -matrix. For this purpose, we start by solving the following quadratic difference equation

$$X(n+2) = a\,X(n+1) + b\,X(n) \ , \tag{52}$$

where a and b are nonzero real numbers and $X(.)$ is 2×2 matrix.

Lemma 2. *The general solution of equation (52) is given by* $X(n) = c_1 A^n + c_2 B^n$, *where*

$$A = \begin{pmatrix} a_{11} & q \\ r & a_{22} \end{pmatrix} \ , \ B = \begin{pmatrix} b_{11} & q \\ q & b_{22} \end{pmatrix} \ , \tag{53}$$

$a_{11} = b_{22} = \frac{1}{2}\left(a + \sqrt{a^2 + 4b - 4qr}\right) \ , \ a_{22} = b_{11} = \frac{1}{2}\left(a - \sqrt{a^2 + 4b - 4qr}\right)$
and q, r *are arbitrary real numbers.*

Proof. It suffices to show that A^n is a solution of Eq. (52). If we observe that $trA = a$ and $\det A = -b \neq 0$, then

$$A^2 - aA - b = 0 \ . \tag{54}$$

Therefore $X(n) = A^n$ is a solution of equation (52). □

Definition 1. ($Q-$ class) We say that a symmetric matrix $A = \begin{pmatrix} u & v \\ v & z \end{pmatrix}$ belongs to the class Q if:

(Q_1) A satisfies Eq. (54).

(Q_2) $u = av + bz$, where a and b are given numbers.

The following lemma characterizes all matrices that belong to the class Q.

Lemma 3. *Let* q_1 *and* q_2 *be solutions of*

$$q^2 + \frac{a^2(b-1)}{a^2 + (b+1)^2} q - b = 0 \ . \tag{55}$$

Then there are two matrices belong to the class Q : $A_i = \begin{pmatrix} p_i & q_i \\ q_i & s_i \end{pmatrix}$, $(i = 1, 2)$ *corresponding to* a *and* b , *where*

$$p_i = \frac{a(b+q_i)}{b+1} \ , \quad s_i = \frac{a(1-q_i)}{b+1}, \quad i = 1, 2 \ . \tag{56}$$

Proof. The matrix $A = \begin{pmatrix} p & q \\ q & s \end{pmatrix}$ belongs to the class Q if

$$p + s = a \ , \quad q^2 - ps = b \ , \quad p = aq + bs \ . \tag{57}$$

The conclusion follows directly by solving equations (57). □

Example 1. Let $a = \frac{7}{2}$ and $b = 6$. Then

$$A_1 = \begin{pmatrix} \frac{3}{2} & -3 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 4 & 2 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

are the corresponding matrices that belong to Q.

Remark 1. We notice that $A \in Q$ does not imply that $A^n \in Q$ as the following example shows.

Example 2. The direct computations show that A_2 in Example 1 does not satisfy both conditions Q_1 and Q_2 .

Definition 2. ($S-$ class) We say that a symmetric matrix $A = \begin{pmatrix} u & v \\ v & z \end{pmatrix} \in S$, if for given numbers a and b, we have $u = av + bz$ and $u_n = av_n + bz_n$, where $A^n = \begin{pmatrix} u_n & v_n \\ v_n & z_n \end{pmatrix}$.

Thee following theorem characterizes the class S.

Theorem 5. *(i) A necessary and sufficient condition so that a nonsingular matrix* $A = \begin{pmatrix} aq + bs & q \\ q & s \end{pmatrix}$ *belongs to the class* S *is* $b = 1$, *where* a , q *and* s *are arbitrary numbers.*

(ii)If $\det A = 0$, *then* $A \in S$ *if and only if it has the form*

$$A = \begin{pmatrix} ak + b & k \\ k & 1 \end{pmatrix} \ , \tag{58}$$

where a, b *are given numbers and* k *is a solution of the equation* $k^2 - ak - b = 0$.

Proof. Let $\lambda_1 \neq \lambda_2$ be the eigenvalues of the matrix $A = \begin{pmatrix} p & q \\ q & s \end{pmatrix}$ with $p = aq+bs$. It is known that $A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$, where

$$P = \begin{pmatrix} q & q \\ \lambda_1 - p & \lambda_2 - p \end{pmatrix} . \tag{59}$$

Therefore

$$A^n = \frac{1}{\det A} \begin{pmatrix} p_n & q_n \\ q_n & s_n \end{pmatrix} , \tag{60}$$

where

$$\begin{aligned} p_n &= q[\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - p(\lambda_1^n - \lambda_2^n)] , \; q_n = q(\lambda_2^n - \lambda_1^n) , \\ s_n &= pq(\lambda_1^n - \lambda_2^n) - q(\lambda_1^{n+1} - \lambda_2^{n+1}) . \end{aligned} \tag{61}$$

The matrix A^n belongs to the class S if and only if

$$p_n - (aq_n + bs_n) = 0 . \tag{62}$$

If we notice that $\lambda_1 + \lambda_2 = p + s$ and $\lambda_1 \lambda_2 = ps - q^2$, then from (61) we obtain for $q \neq 0$ that

$$\begin{aligned} & \frac{1}{q}[p_n - (aq_n + bs_n)] \\ = \; & \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n + (-p + aq - pb)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1}) \\ = \; & \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - b(p + s)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1}) \\ = \; & \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - b(\lambda_1 + \lambda_2)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1}) \\ = \; & (1 - b)(ps - q^2)(\lambda_1^{n-1} - \lambda_2^{n-1}) . \end{aligned} \tag{63}$$

(i) If $\det A \neq 0$, then from (63) it follows that (62) is satisfied if and only if $b = 1$.

(ii) If $\det A = 0$, then (62) is satisfied and we have $aq + bs = kq$ and $q = ks$ for some k and s. Therefore, $A = \begin{pmatrix} (ak+b)s & ks \\ ks & s \end{pmatrix}$, where $k^2 - ak - b = 0$. Without loss of generality we put $s = 1$. This completes the proof. □

Example 3. Let $a = 2$ and $b = 3$. Then $k = 1$ or $k = 3$. Thus we get the two matrices of $S-$ class :

$$A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} , \; A = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} .$$

Example 4. Let $a = 2$, $q = 3$ and $s = -2$ with $b = 1$ we get the following matrix of S - class :

$$A = \begin{pmatrix} 5 & 3 \\ 3 & -1 \end{pmatrix} .$$

It is clear that $A \notin Q$ because $A^2 - 2A - I \neq 0$.

It is natural to look for solutions of equation (52) that belong to the class S. In fact, these solutions belong to $Q \cap S$. The following theorem gives the answer.

Theorem 6. *The only two types of matrices that belong to* $S \cap Q$ *are given by*

$$H(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad K(a) = \begin{pmatrix} 0 & -1 \\ -1 & a \end{pmatrix} = -H^{-1}(a) \, , \tag{64}$$

provided that $b = 1$ *in equation (52) and* a *is a real number.*

Proof. For any given numbers a and b, it follows from Lemma 8 that there are two matrices A_1 and A_2 belong to Q . Theorem 12 implies that A_1 and A_2 belong to S , whenever $b = 1$. Using Eq. (55) and (56) we obtain $A_1 = H(a)$ and $A_2 = K(a)$.

Corollary 1. *Let* $U_0(a) = 0$, $U_1(a) = 1$ *and* $U_2(a) = a$. *Then*

$$H^n(a) = \begin{pmatrix} U_{n+1}(a) & U_n(a) \\ U_n(a) & U_{n-1}(a) \end{pmatrix} \tag{65}$$

and $K^n(a) = (-1)^n (H^n(a))^{-1} = (-1)^n H^{-n}(a)$, *where*

$$U_{n+1}(a) = aU_n(a) + U_{n-1}(a). \tag{66}$$

Notice that the Fibonacci $Q-$ matrix defined by (50)is a particular case of $H(a)$, where $a = 1$ .Also, if we let $a = 2$, we come to a new matrix that we call $P-$ matrix

$$H(2) = P = \begin{pmatrix} P_2 & P_1 \\ P_1 & P_0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \, , \tag{67}$$

where P_n is a Pell number. Then by Corollary 1, we have

$$P^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \, , \tag{68}$$

where $P_{n+1} = aP_n + P_{n-1}$ for $n \geq 1$. The $P-$matrices also give a number of identities, including $\det P^n = (\det P)^n$, which gives

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \quad , \tag{69}$$

$P^{n+1}P^n = P^{2n+1}$, which gives

$$\begin{pmatrix} P_{n+2} & P_{n+1} \\ P_{n+1} & P_n \end{pmatrix} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} P_{2n+2} & P_{2n+1} \\ P_{2n+1} & P_{2n} \end{pmatrix} \, , \tag{70}$$

$P^m P^{n-1} = P^{n+m-1}$, which gives

$$\begin{pmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{pmatrix} \begin{pmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{pmatrix} = \begin{pmatrix} P_{m+n} & P_{m+n-1} \\ P_{m+n-1} & P_{m+n-2} \end{pmatrix} \, , \tag{71}$$

and $P^n = P^{n-m}P^m$ which gives

$$\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-m+1} & P_{n-m} \\ P_{n-m} & P_{n-m-1} \end{pmatrix} \begin{pmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{pmatrix} . \tag{72}$$

Similar identities for the numbers $H_n(a)$ defined by (66) can be constructed.

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