FIBONACCI Q-TYPE MATRICES AND PROPERTIES OF A CLASS OF NUMBERS RELATED TO THE FIBONACCI, LUCAS AND PELL NUMBERS

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ABSTRACT: A class of numbers that contains the Fibonacci, Lucas and Pell numbers is investigated, and some properties are obtained. Fibonacci Q-type matrices are constructed as a tool to get many properties for Fibonacci and Pell numbers.

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1. INTRODUCTION

In spite of many articles and books that have been published on the Fibonacci, Lucas and Pell numbers and their applications Bong [1], Dubner and Keller [2], Hoggatt [4], Horadam [6], Horadam [7], Philipou [8], Sloane and Plouffe [9], Vorob’ev [10], these numbers are still an interesting subject for investigation. In this article we introduce interesting properties satisfied by a large class of Fibonacci type numbers. Let us note that this class has been studied extensively by Horadam [6] and [7], where several properties has been obtained.

The numbers we are going to consider satisfy the following recurrence relations:

\[ U(n + 2, u_1, u_2, a, b) = aU(n + 1, u_1, u_2, a, b) + bU(n, u_1, u_2, a, b) \] ,  \hspace{1cm} (1)

where

\[ U(1, u_1, u_2, a, b) = u_1 , \quad U(2, u_1, u_2, a, b) = u_2 , \quad ab \neq 0 \] ,  \hspace{1cm} (2)

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\( n \geq 2 \) is an integer and \( u_1, u_2, a, b \) are real. We observe that:

(i) The Fibonacci numbers \( F(n) \) defined by \( F(n+2) = F(n+1) + F(n) \) and \( F(1) = F(2) = 1 \) satisfy

\[
U(n, 1, 1, 1, 1) = F(n) .
\] (3)

(ii) The Lucas numbers \( L(n) \) defined by \( L(n+2) = L(n+1) + L(n) \) and \( L(1) = 1, \ L(2) = 3 \) satisfy

\[
U(n, 1, 3, 1, 1) = L(n) ,
\] (4)

(iii) The Pell numbers \( P(n) \) defined by \( P(n+2) = P(n+1) + P(n) \) and \( P(1) = 1, P(2) = 2 \) satisfy

\[
U(n, 1, 2, 2, 1) = P(n) .
\] (5)

(iv) The numbers \( R(a, n) \) defined by \( R(a, n+2) = aR(a, n+1) + R(a, n) \), where \( R(a, 0) = 0, R(a, 1) = 1 \) and \( a \geq 1 \) is an integer, satisfy

\[
U(n, 1, a, 1) = R(a, n) .
\] (6)

This class of numbers has been introduced by Entringer and Slater [3], while investigating the problem of information dissemination through telegraphs. In 1986 Bong [1] obtained several interesting properties for \( R(a, n) \).

In Sections 2 and 3 we introduce interesting properties that satisfied by \( U(n, u_1, u_2, a, b) \) and consequently by the Fibonacci, Lucas and Pell numbers. Section 5 deals with solutions of quadratic matrix difference equation and a special class of these solutions that leads to the definitions Q-type matrices.

In what follows, for simplicity, we write \( U(n) \) instead of \( U(n, u_1, u_2, a, b) \) when the appearance of the arguments is not necessary.

### 2. PROPERTIES OF \( U(n) \)

The characteristic equation

\[
\lambda^2 - a\lambda - b = 0
\] (7)

of (1) has two roots

\[
\lambda_1 = \frac{a + \varepsilon\sqrt{\Delta}}{2} , \quad \lambda_2 = \frac{a - \varepsilon\sqrt{\Delta}}{2} ,
\] (8)

where \( \Delta = |a^2 + 4b| \), \( \varepsilon = 1 \) when \( a^2 + 4b > 0 \) and \( \varepsilon = \sqrt{-1} \) for \( a^2 + 4b < 0 \).
In the case when (7) has equal roots, then \( b = -\frac{a^2}{4} \) and \( \lambda_1 = \lambda_2 = \frac{a}{2} \). The corresponding solution of Eq. (1) will be denoted by \( D(n, p_1, p_2, a) \), where we have

\[
D(n + 2, p_1, p_2, a) = aD(n + 1, p_1, p_2, a) - \frac{a^2}{4}D(n, p_1, p_2, a)
\]

(9)

and

\[
D(1, p_1, p_2, a) = p_1, \quad D(2, p_1, p_2, a) = p_2.
\]

(10)

For simplicity we write \( D(n) \) instead of \( D(n, p_1, p_2, a) \) when the appearance of the arguments is not necessary.

**Lemma 1.** Assume that \( \lambda_1 \neq \lambda_2 \), then for \( \alpha, \beta, \gamma, \delta \) and \( k \in \mathbb{R} \), the following relations hold true:

\[
U(n, \alpha u_1, \alpha u_2, a, b) = \alpha U(n, u_1, u_2, a, b),
\]

(11)

\[
U(n, u_1, ku_2, ka, k^2b) = k^{n-1}U(n, u_1, u_2, a, b),
\]

(12)

\[
U(n, \alpha v_1 + \beta v_2, \gamma w_1 + \delta w_2, a, b) = U(n, \alpha v_1, \gamma w_1, a, b) + U(n, \beta v_2, \delta w_2, a, b),
\]

(13)

and

\[
U(n, u_1, u_2, a, b) = bu_1 U(n-2, 1, a, a, b) U(n-1, 1, a, a, b).
\]

(14)

**Proof.** The solution of (1) that satisfies \( u_1 = U(1) \) and \( u_2 = U(2) \) can be obtained by determining \( c_1 \) and \( c_2 \) in the solution

\[
U(n) = c_1 \lambda_1^n + c_2 \lambda_2^n.
\]

(15)

Therefore, we get

\[
U(n) = \frac{u_2 - \lambda_2 u_1}{\lambda_1 - \lambda_2} \lambda_1^{n-1} - \frac{u_2 - \lambda_1 u_1}{\lambda_1 - \lambda_2} \lambda_2^{n-1},
\]

(16)

or

\[
U(n, u_1, u_2, a, b) = \frac{1}{\varepsilon \sqrt{\Delta}}[bu_1(\lambda_1^{n-2} - \lambda_2^{n-2}) + u_2(\lambda_1^{n-1} - \lambda_2^{n-1})].
\]

(17)
Since \( U(n, u_1, u_2, a, b) \) is linear and homogeneous function with respect to \( u_1 \) and \( u_2 \), the relations (11) and (13) follow directly. If we notice that \( \lambda_1 = \lambda_1(a, b) \) and \( \lambda_2 = \lambda_2(a, b) \) satisfy
\[
\lambda_i(ka, k^2b) = k\lambda_i(a, b) , \quad i = 1, 2 , \tag{18}
\]
then from Eq.(17 ) we obtain
\[
U(n, ku_2, ka, k^2b) = \frac{1}{\varepsilon k\sqrt{\Delta}}[k^2u_1k^{n-2}(\lambda_1^{n-2} - \lambda_2^{n-2}) + ku_2k^{n-1}(\lambda_1^{n-1} - \lambda_2^{n-1})] ,
\]
and (12) follows directly.

From (7) we infer that \( Q(p) = \lambda_1^p - \lambda_2^p \) satisfies \( Q(p + 2) = aQ(p + 1) + bQ(p) \) with \( Q(1) = \varepsilon\sqrt{\Delta} \) and \( Q(1) = a\varepsilon\sqrt{\Delta} \). Therefore
\[
Q(p) = U(p, \varepsilon\sqrt{\Delta}, a\varepsilon\sqrt{\Delta}, a, b)
\]
and (11) implies that
\[
Q(p) = \varepsilon\sqrt{\Delta}U(p, 1, a, a, b) , \tag{19}
\]
and we obtain (14) by using (19) and noticing that
\[
U(n, u_1, u_2, a, b) = \frac{1}{\varepsilon\sqrt{\Delta}}[bu_1Q(n-2) + u_2Q(n-1)] . \tag{20}
\]

In what follows we introduce the main properties of \( U(n) \).

**Theorem 1.** Assume that \( \lambda_1 \neq \lambda_2 \) and \( n > m \geq 1 \). Then the following relations hold true:
\[
U(n + m + 1)U(n - m) - U(n + m)U(n - m + 1) = (-b)^{n-m-1}(bu_1^2 + au_1u_2 - u_2^2)V(2m) \tag{21}
\]
and
\[
U(n + k + 1)U(n - k) - U(n + 1)U(n) = \frac{(bu_1^2 + au_1u_2 - u_2^2)}{a^2 + 4b}(-b)^{n-k-1}[-a(-b)^k + V(2k + 2) + bV(2k)] , \tag{22}
\]
where
\[
V(n) = U(n, 1, a, a, b) , \quad (n > k) . \tag{23}
\]
Proof. From (15) we have

\[
U(n + m + 1)U(n - m) - U(n + m)U(n - m + 1) = (c_1\lambda_1^{n+m+1} + c_2\lambda_2^{n+m+1}) (c_1\lambda_1^{n-m} + c_2\lambda_2^{n-m}) - (c_1\lambda_1^{n+m} + c_2\lambda_2^{n+m}) (c_1\lambda_1^{n-m+1} + c_2\lambda_2^{n-m+1}) = c_1c_2(\lambda_1\lambda_2)^{n-m} (\lambda_1 - \lambda_2) (\lambda_1^{2m} - \lambda_2^{2m}) .
\] (24)

If we notice that

\[
\lambda_1\lambda_2 = -b , \lambda_1 - \lambda_2 = \varepsilon\sqrt{\Delta} , \lambda_1^{2m} - \lambda_2^{2m} = Q(2m)
\]

and

\[
c_1c_2 = \frac{(bu_1^2 + au_1u_2 - u_2^2)}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2} ,
\] (25)

it follows from (24) that

\[
U(n + m + 1)U(n - m) - U(n + m)U(n - m + 1) = \frac{(-b)^{n-m-1}}{\varepsilon\sqrt{\Delta}} (bu_1^2 + au_1u_2 - u_2^2) U(2m, \varepsilon\sqrt{\Delta}, a\varepsilon\sqrt{\Delta}, a, b) .
\]

Applying (11) we conclude (21).

From (21), we write in succession

\[
U(n + 2)U(n - 1) - U(n + 1)U(n) = h(-b)^{n-2}V(2)
\]

\[
U(n + 3)U(n - 2) - U(n + 2)U(n - 1) = h(-b)^{n-3}V(4)
\]

\[...
\]

\[
U(n + k + 1)U(n - k) - U(n + k)U(n - k + 1) = h(-b)^{n-k-1}V(2k) ,
\]

where \( h = (bu_1^2 + au_1u_2 - u_2^2) \) and \( V(n) \) is defined by (23). Summing these, we obtain

\[
U(n + k + 1)U(n - k) - U(n + 1)U(n) = (bu_1^2 + au_1u_2 - u_2^2)G ,
\] (26)

where

\[
G = \sum_{i=1}^{k} (-b)^{n-i-1}V(2i) = (-b)^{n-1} \sum_{i=1}^{k} (-b)^{-i} \frac{\lambda_1^{2i} - \lambda_2^{2i}}{\lambda_1 - \lambda_2} .
\]

Since

\[
\sum_{i=1}^{k} (-\frac{\lambda_1^2}{b})^i = -\frac{\lambda_1^2}{b} \cdot \frac{1 - (-\frac{\lambda_1^2}{b})^k}{1 + \frac{\lambda_1^2}{b}} ,
\]

we get

\[
G = \frac{(-b)^{n-1}}{\lambda_1 - \lambda_2} \left[ -\lambda_1^2 \left( \frac{1 - q_1}{b + \lambda_1^2} \right) + \lambda_2^2 \left( \frac{1 - q_2}{b + \lambda_2^2} \right) \right] ,
\]
where \( q_1 = (-\frac{\lambda^2}{b})^k \) and \( q_2 = (-\frac{\lambda^2}{b})^k \). Hence

\[
G = \frac{(-b)^{n-1}}{(\lambda_1 - \lambda_2) (b + \lambda_1^2) (b + \lambda_2^2)} \left[-\lambda_1^2 (b + \lambda_1^2) (1 - q_1) + \lambda_2^2 (b + \lambda_1^2) (1 - q_2)\right].
\]

If we notice that

\[
(b + \lambda_1^2) (b + \lambda_2^2) = (a\lambda_1 + 2b) (a\lambda_2 + 2b) = a^2b + 4b^2
\]

and

\[
\lambda_2(b + \lambda_2^2) = b\lambda^2 + b^2 = (a\lambda + b)b + b^2 = ab\lambda + 2b^2
\]

for \( \lambda = \lambda_1, \lambda_2 \), then

\[
G = \frac{(-b)^{n-1}}{(a^2 + 4b)(\lambda_1 - \lambda_2)} [\frac{1}{(-b)^k} \left[a(\lambda_1^{2k+1} - \lambda_2^{2k+1}) + 2b(\lambda_1^{2k} - \lambda_2^{2k})\right]]
\]

\[
\frac{(-b)^{n-k-1}}{(a^2 + 4b)} \left[-a(-b)^k + aV(2k + 1) + 2bV(2k)\right]
\]

\[
= \frac{(-b)^{n-k-1}}{(a^2 + 4b)} \left[-a(-b)^k + V(2k + 2) + bV(2k)\right].
\]

The substitution of \( G \) in (26) completes the proof.

**Theorem 2.** Assume that \( \lambda_1 \neq \lambda_2 \) and \( n > m \geq 2 \). Then the following relations hold true:

\[
U(n + m)U(n - m) - U(n + m - 1)U(n - m + 1) = (-b)^{n-m-1}(bu_1^2 + au_1u_2 - u_2^2)V(2m - 1) \quad (27)
\]

and

\[
U(n + k)U(n - k) - U(n + 1)U(n - 1) = \frac{b(bu_1^2 + au_1u_2 - u_2^2)(-b)^{n-k}}{a^2 + 4b} \times [(a^2 + 2b)(-b)^{k-1} - V(2k + 1) - bV(2k - 1)] \quad (28)
\]

for \( n > k \geq 2 \), where \( V(n) \) is defined by (23).

**Proof.** Following the steps of the proof of (22), we obtain

\[
U(n + k)U(n - k) - U(n + 1)U(n - 1) = (bu_1^2 + au_1u_2 - u_2^2)H, \quad (29)
\]
But
\[
H = \sum_{i=2}^{k} (-b)^{n-i-1} V(2i - 1).
\]

Therefore
\[
H = \frac{(-b)^{n-1}}{(a^2 + 4b)(\lambda_1 - \lambda_2)} \left[ -a(\lambda_1 - \lambda_2) - \frac{1}{(b)^{k-1}}[\alpha(\lambda_1^{2k-1} - \lambda_2^{2k+1}) + \beta(\lambda_1^{2k-2} - \lambda_2^{2k-2})] \right],
\]

where \( r_1 = (-\frac{\lambda_3^2}{\lambda_2})^k \) and \( r_2 = (-\frac{\lambda_3^2}{\lambda_2})^k \). If we notice that
\[
\lambda_3^2(b + \lambda_3^2) = \lambda_1(b\lambda_1^2 + b^2) = \lambda_1(ab\lambda_1 + 2b^2) = (a^2b + 2b^2)\lambda_1 + ab^2
\]
and \( \lambda_3^2(b + \lambda_3^2) = (a^2b + 2b^2)\lambda_2 + ab^2 \), then
\[
H = \frac{(-b)^{n-k+1}}{(a^2 + 4b)} \left[ (a^2 + 2b)(-b)^{k-1} - [(a^2 + 2b)V(2k - 1) + abV(2k - 2)] \right].
\]

But
\[
(a^2 + 2b)V(2k - 1) + abV(2k - 2) = 2bV(2k - 1) + aV(2k) = bV(2k - 1) + V(2k + 1).
\]

Therefore
\[
H = \frac{(-b)^{n-k+1}}{(a^2 + 4b)} \left[ (a^2 + 2b)(-b)^{k-1} - [V(2k + 1) + bV(2k - 1)] \right].
\]

The substitution of \( H \) in (29) completes the proof of (28). The relation (27) can be proved in a similar way to that of (21).

In the case when the equation (7) has equal roots, we have the following:

**Theorem 3.** Assume that Eq.(7) has equal roots. Then the following relations hold true:

\[
D(n + m + 1)D(n - m) - D(n + m)D(n - m + 1) = -\frac{m}{2} \left( \frac{a}{2} \right)^{2n-3} (2p_2 - ap_1)^2, \ (n > m \geq 1)
\]

(30)
and
\[ D(n + m)D(n - m) - D(n + m - 1)D(n - m + 1) = - \frac{2m - 1}{2} \left( \frac{a}{2} \right)^{2n-4}(2p_2 - ap_1)^2, \quad (n > m \geq 2). \] (31)
where \( D(n) \) is defined by (9) and (10).

**Proof.** The solution of (9) and (10) can be written as follows:
\[ D(n) = \lambda_1^n(c_1 + c_2n), \quad D(1) = p_1, \quad D(2) = p_2, \] (32)
where
\[ c_1 = \frac{2p_1 \lambda_1 - p_2}{\lambda_1^2}, \quad c_2 = \frac{p_2 - p_1 \lambda_1}{\lambda_1^2}, \quad \lambda_1 = \frac{a}{2}. \] (33)
Using (32) and (33) we have
\[ D(n + m + 1)D(n - m) - D(n + m)D(n - m + 1) = -2mc_2^2\lambda_1^{2n+1}. \]
The relation (30) follows directly, if we notice that
\[ c_2^2\lambda_1^{2n+1} = \frac{1}{4}(\frac{a}{2})^{2n-3}(2p_2 - ap_1)^2. \] \( \square \)

**Theorem 4.** Assume that \( \lambda_1 = \lambda_2 \). Then the following relations hold true:
\[ D(n + k + 1)D(n - k) - D(n + 1)D(n) = -\frac{k(k + 1)}{2}\left( \frac{a}{2} \right)^{2n-3}(2p_2 - ap_1)^2, \quad (n > k \geq 1) \] (34)
and
\[ D(n + k)D(n - k) - D(n + 1)D(n - 1) = -\frac{k^2 - 1}{2}\left( \frac{a}{2} \right)^{2n-4}(2p_2 - ap_1)^2, \quad (n > k \geq 2) \] (35)
where \( D(n) \) is defined by (9) and (10).

**Proof.** Putting \( m = 1, 2, ..., k \) in (30) and (31) and summing up the resulting relations, we obtain
\[ D(n + k + 1)D(n - k) - D(n + 1)D(n) = -\frac{1}{2}\left( \frac{a}{2} \right)^{2n-3}(2p_2 - ap_1)^2 \sum_{i=1}^{k} i \]
and
\[ D(n + k)D(n - k) - D(n + 1)D(n - 1) = -\frac{1}{4}\left( \frac{a}{2} \right)^{2n-4}(2p_2 - ap_1)^2 \sum_{i=1}^{k} (2i - 1). \]
The substitution of the summations in the previous equations implies (34) and (35). \( \square \)
3. APPLICATIONS

3.1. FIBONACCI NUMBERS

Since $F(n) = U(n, 1, 1, 1, 1)$, we get from Theorems 2 and 3 the following properties:

$$F(n + m + 1)F(n - m) - F(n + m)F(n - m + 1) = (-1)^{n-m-1}F(2m),$$

$$n > m \geq 1, \quad (36)$$

$$F(n + m)F(n - m) - F(n + m - 1)F(n - m + 1) = (-1)^{n-m-1}F(2m - 1),$$

$$n > m \geq 2, \quad (37)$$

$$F(n + k + 1)F(n - k) - F(n + 1)F(n) = \frac{(-1)^{n-k-1}}{5}[-(1)^{k+1} + F(2k + 2) + F(2k)], \quad n > k \geq 1, \quad (38)$$

and

$$F(n + k)F(n - k) - F(n + 1)F(n - 1) = \frac{(-1)^{n-k}}{5}[3(-1)^{k+1} - F(2k + 1) + F(2k - 1)], \quad n > k \geq 2. \quad (39)$$

3.2. LUCAS NUMBERS

Since $L(n) = U(n, 1, 3, 1, 1)$, we get from Theorems 2 and 3 the following properties:

$$L(n + m + 1)L(n - m) - L(n + m)L(n - m + 1) = (-1)^{n-m-1}5F(2m),$$

$$n > m \geq 1, \quad (40)$$

$$L(n + k + 1)L(n - k) - L(n + 1)L(n) = (1)^{n-k}[-(1)^{k+1} + F(2k + 2) + F(2k)], \quad n > k, \quad (41)$$

$$L(n + m)L(n - m) - L(n + m - 1)L(n - m + 1) = (-1)^{n-m-1}5F(2m - 1),$$

$$n > m \geq 2, \quad (42)$$

and

$$L(n + k)L(n - k) - L(n + 1)L(n - 1) = (1)^{n-k+1}[3(-1)^{k-1} - F(2k + 1) - F(2k - 1)], \quad n > k \geq 2. \quad (43)$$
3.3. PELL NUMBERS

Since \( P(n) = U(n, 1, 2, 2, 1) \), we get from Theorems 2 and 3 the following properties:

\[
P(n + m + 1)P(n - m) - P(n + m)P(n - m + 1) = (-1)^{n-m-1}P(2m),
\]
\[
n > m \geq 1, \quad (44)
\]

\[
P(n + k + 1)P(n - k) - P(n + 1)P(n)
= \frac{(-1)^{n-k-1}}{8}[2(-1)^{k+1} + P(2k + 2) + P(2k)], \quad n > k, \quad (45)
\]

\[
P(n + m)P(n - m) - P(n + m - 1)P(n - m + 1) = (-1)^{n-m-1}P(2m - 1),
\]
\[
n > m \geq 2, \quad (46)
\]

and

\[
P(n + k)P(n - k) - P(n + 1)P(n - 1)
= \frac{(-1)^{n-k}}{8}[6(-1)^{k-1} - P(2k + 1) - P(2k - 1)], \quad n > k \geq 2. \quad (47)
\]

3.4. THE CLASS \( R(n, a) \)

Since \( R(n, a) = U(n, 1, a, a, 1) \), we get from Theorems 2 and 3 the following properties:

\[
R(n + m + 1, a)R(n - m, a) - R(n + m, a)R(n - m + 1, a)
= (-1)^{n-m-1}R(2m, a), \quad n > m \geq 1
\]
\[
\quad (48)
\]

and

\[
R(n + m, a)R(n - m, a) - R(n + m - 1, a)R(n - m + 1, a)
= (-1)^{n-m-1}R(2m - 1, a), \quad n > m \geq 2. \quad (49)
\]

Two more properties similar to (38) and (39) can be written for \( R(n, a) \).

4. FIBONACCI \( Q - \) TYPE MATRICES

The Fibonacci \( Q \) -matrix had been defined first by Honsberger [5] as follows:

\[
Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix},
\]
\[
\quad (50)
\]
where \( F_n \) is a Fibonacci number. He showed that

\[
Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}
\]

(51)

and noticed that the Fibonacci \( Q \)-matrix is a good tool for getting a number of important identities related to Fibonacci numbers. It seems that Fibonacci \( Q \)-matrix belongs to a larger class of matrices that satisfies (50) and (51). In this section we construct this class of matrices, where we can define, for example, Pell \( P \)-matrix, which is similar to Fibonacci \( Q \)-matrix. For this purpose, we start by solving the following quadratic difference equation

\[
X(n + 2) = a X(n + 1) + b X(n),
\]

(52)

where \( a \) and \( b \) are nonzero real numbers and \( X(.) \) is a \( 2 \times 2 \) matrix.

**Lemma 2.** The general solution of equation (52) is given by \( X(n) = c_1 A^n + c_2 B^n \), where

\[
A = \begin{pmatrix} a_{11} & q \\ r & a_{22} \end{pmatrix},
B = \begin{pmatrix} b_{11} & q \\ q & b_{22} \end{pmatrix},
\]

(53)

\[
a_{11} = b_{22} = \frac{1}{2} \left( a + \sqrt{a^2 + 4b - 4qr} \right),
a_{22} = b_{11} = \frac{1}{2} \left( a - \sqrt{a^2 + 4b - 4qr} \right)
\]

and \( q, r \) are arbitrary real numbers.

**Proof.** It suffices to show that \( A^n \) is a solution of Eq. (52). If we observe that \( \text{tr} A = a \) and \( \det A = -b \neq 0 \), then

\[
A^2 - aA - b = 0.
\]

(54)

Therefore \( X(n) = A^n \) is a solution of equation (52). \( \square \)

**Definition 1.** \( (Q-) \) class We say that a symmetric matrix \( A = \begin{pmatrix} u & v \\ v & z \end{pmatrix} \) belongs to the class \( Q \) if:

\((Q_1)\) \( A \) satisfies Eq. (54).

\((Q_2)\) \( u = av + bz \), where \( a \) and \( b \) are given numbers.

The following lemma characterizes all matrices that belong to the class \( Q \).

**Lemma 3.** Let \( q_1 \) and \( q_2 \) be solutions of

\[
q^2 + \frac{a^2(b - 1)}{a^2 + (b + 1)^2} q - b = 0.
\]

(55)
Then there are two matrices belong to the class $Q$: $A_i = \begin{pmatrix} p_i & q_i \\ q_i & s_i \end{pmatrix}$, $(i = 1, 2)$ corresponding to $a$ and $b$, where

$$p_i = \frac{a(b + q_i)}{b + 1}, \quad s_i = \frac{a(1 - q_i)}{b + 1}, \quad i = 1, 2.$$  

(56)

**Proof.** The matrix $A = \begin{pmatrix} p & q \\ q & s \end{pmatrix}$ belongs to the class $Q$ if

$$p + s = a, \quad q^2 - ps = b, \quad p = aq + bs.$$  

(57)

The conclusion follows directly by solving equations (57).  

Example 1. Let $a = \frac{7}{2}$ and $b = 6$. Then

$$A_1 = \begin{pmatrix} 3 & -3 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 4 & 2 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

are the corresponding matrices that belong to $Q$.

**Remark 1.** We notice that $A \in Q$ does not imply that $A^n \in Q$ as the following example shows.

Example 2. The direct computations show that $A_2$ in Example 1 does not satisfy both conditions $Q_1$ and $Q_2$.

**Definition 2.** ($S-$ class) We say that a symmetric matrix $A = \begin{pmatrix} u & v \\ v & z \end{pmatrix}$ is $S$, if for given numbers $a$ and $b$, we have $u = av + bz$ and $u_n = av_n + bz_n$, where $A^n = \begin{pmatrix} u_n & v_n \\ v_n & z_n \end{pmatrix}$.

The following theorem characterizes the class $S$.

Theorem 5. (i) A necessary and sufficient condition so that a nonsingular matrix $A = \begin{pmatrix} aq + bs & q \\ q & s \end{pmatrix}$ belongs to the class $S$ is $b = 1$, where $a, q$ and $s$ are arbitrary numbers.

(ii) If $\det A = 0$, then $A \in S$ if and only if it has the form

$$A = \begin{pmatrix} ak + b & k \\ k & 1 \end{pmatrix},$$  

(58)

where $a, b$ are given numbers and $k$ is a solution of the equation $k^2 - ak - b = 0$. 


Proof. Let \( \lambda_1 \neq \lambda_2 \) be the eigenvalues of the matrix \( A = \begin{pmatrix} p & q \\ q & s \end{pmatrix} \) with \( p = aq + bs \).

It is known that \( A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} \), where

\[
P = \begin{pmatrix} q & q \\ \lambda_1 - p & \lambda_2 - p \end{pmatrix}.
\]

Therefore

\[
A^n = \frac{1}{\det A} \begin{pmatrix} p_n & q_n \\ q_n & s_n \end{pmatrix},
\]

where

\[
p_n = q[\lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - p(\lambda_1^n - \lambda_2^n)], \quad q_n = q(\lambda_2^n - \lambda_1^n),
\]

\[
s_n = pq(\lambda_1^n - \lambda_2^n) - q(\lambda_1^{n+1} - \lambda_2^{n+1}).
\]

The matrix \( A^n \) belongs to the class \( S \) if and only if

\[
p_n - (aq_n + bs_n) = 0.
\]

If we notice that \( \lambda_1 + \lambda_2 = p + s \) and \( \lambda_1 \lambda_2 = ps - q^2 \), then from (61) we obtain for \( q \neq 0 \) that

\[
\frac{1}{q}[p_n - (aq_n + bs_n)] = \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n + (-p + aq - pb)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1})
\]

\[
= \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - b(p + s)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1})
\]

\[
= \lambda_1^n \lambda_2 - \lambda_1 \lambda_2^n - b(\lambda_1 + \lambda_2)(\lambda_1^n - \lambda_2^n) + b(\lambda_1^{n+1} - \lambda_2^{n+1})
\]

\[
= (1 - b)(ps - q^2)(\lambda_1^{n-1} - \lambda_2^{n-1}).
\]

(i) If \( \det A \neq 0 \), then from (63) it follows that (62) is satisfied if and only if \( b = 1 \).

(ii) If \( \det A = 0 \), then (62) is satisfied and we have \( aq + bs = kq \) and \( q = ks \) for some \( k \) and \( s \). Therefore, \( A = \begin{pmatrix} (ak + b)s & ks \\ ks & s \end{pmatrix} \), where \( k^2 - ak - b = 0 \). Without loss of generality we put \( s = 1 \). This completes the proof. \( \square \)

Example 3. Let \( a = 2 \) and \( b = 3 \). Then \( k = 1 \) or \( k = 3 \). Thus we get the two matrices of \( S - \) class:

\[
A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}.
\]

Example 4. Let \( a = 2, \) \( q = 3 \) and \( s = -2 \) with \( b = 1 \) we get the following matrix of \( S - \) class:

\[
A = \begin{pmatrix} 5 & 3 \\ 3 & -1 \end{pmatrix}.
\]
It is clear that \( A \notin Q \) because \( A^2 - 2A - I \neq 0 \).

It is natural to look for solutions of equation (52) that belong to the class \( S \). In fact, these solutions belong to \( Q \cap S \). The following theorem gives the answer.

**Theorem 6.** The only two types of matrices that belong to \( S \cap Q \) are given by

\[
H(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad K(a) = \begin{pmatrix} 0 & -1 \\ -1 & a \end{pmatrix} = -H^{-1}(a),
\]

provided that \( b = 1 \) in equation (52) and \( a \) is a real number.

**Proof.** For any given numbers \( a \) and \( b \), it follows from Lemma 8 that there are two matrices \( A_1 \) and \( A_2 \) belong to \( Q \). Theorem 12 implies that \( A_1 \) and \( A_2 \) belong to \( S \), whenever \( b = 1 \). Using Eq. (55) and (56) we obtain \( A_1 = H(a) \) and \( A_2 = K(a) \).

**Corollary 1.** Let \( U_0(a) = 0 \), \( U_1(a) = 1 \) and \( U_2(a) = a \). Then

\[
H^n(a) = \begin{pmatrix} U_{n+1}(a) & U_n(a) \\ U_n(a) & U_{n-1}(a) \end{pmatrix}
\]

and \( K^n(a) = (-1)^n(H^n(a))^{-1} = (-1)^nH^{-n}(a) \), where

\[
U_{n+1}(a) = aU_n(a) + U_{n-1}(a).
\]

Notice that the Fibonacci \( Q - \) matrix defined by (50) is a particular case of \( H(a) \), where \( a = 1 \). Also, if we let \( a = 2 \), we come to a new matrix that we call \( P - \) matrix

\[
H(2) = P = \begin{pmatrix} P_3 & P_2 \\ P_2 & P_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( P_n \) is a Pell number. Then by Corollary 1, we have

\[
P^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix},
\]

where \( P_{n+1} = aP_n + P_{n-1} \) for \( n \geq 1 \). The \( P - \) matrices also give a number of identities, including \( \det P^n = (\det P)^n \), which gives

\[
P_{n+1}P_{n-1} - P_n^2 = (-1)^n,
\]

\[
P_{n+1}P_{n-1} = P^{2n+1}, \quad \text{which gives}
\]

\[
\begin{pmatrix} P_{n+2} & P_{n+1} \\ P_{n+1} & P_n \end{pmatrix} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} P_{2n+2} & P_{2n+1} \\ P_{2n+1} & P_{2n} \end{pmatrix},
\]

\[
P_{n+1}P_{n-1} = P^{n+m-1}, \quad \text{which gives}
\]

\[
\begin{pmatrix} P_{m+1} & P_m \\ P_m & P_{m-1} \end{pmatrix} \begin{pmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{pmatrix} = \begin{pmatrix} P_{m+n} & P_{m+n-1} \\ P_{m+n-1} & P_{m+n-2} \end{pmatrix},
\]

\[
\quad \text{(71)}
\]
and \( P^n = P^{n-m} P^m \) which gives

\[
\begin{pmatrix}
P_{n+1} & P_n \\
P_n & P_{n-1}
\end{pmatrix} = \begin{pmatrix}
P_{n-m+1} & P_{n-m} \\
P_{n-m} & P_{n-m-1}
\end{pmatrix} \begin{pmatrix}
P_{m+1} & P_m \\
P_m & P_{m-1}
\end{pmatrix}.
\]

(72)

Similar identities for the numbers \( H_n(a) \) defined by (66) can be constructed.

REFERENCES


[7] A.F. Horadam, Special properties of the sequence \( w(n, a, b; p, q) \), *Fibonacci Quarterly*, 5 (1967), 424-434.


