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A MINIMAX PRINCIPLE WITH A GENERAL PALAIS-SMALE CONDITION

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ABSTRACT: In this paper we establish a general deformation lemma and a subsequent minimax principle for locally Lipschitz functionals with a new condition of Palais-Smale type.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

In this paper we extend the non-smooth critical point theories of Chang [4] and Kourogenis and Papageorgiou [6] relaxing their compactness conditions: the Palais-Smale condition ((PS)-condition) of Chang [4] for a locally Lipschitz functional and the weakened (PS)-condition in Kourogenis and Papageorgiou [6], extending the Cerami condition ((C)-condition) of Cerami [3].

Let X be a Banach space with the topological dual X^* and the duality pairing $(\cdot,\cdot)$ between X and X^*. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional. The set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \le f^0(x; h) \text{ for all } h \in X\},$$

called the generalized gradient (in the sense of Clarke [5]) of f at x, is nonempty, convex and w^*-compact, where $f^0(x; h)$ denotes the generalized directional derivative at the point $x \in X$ in the direction $h \in X$ (see Clarke [5]). In particular, if $f \in C^1(X, \mathbb{R})$ then we have $\partial f(x) = \{df(x)\}$. An element $x \in X$ is said to be a critical

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point of f if $0 \in \partial f(x)$. We define $\lambda_f(x) = \inf\{\|x^*\| : x^* \in \partial f(x)\}$ (in short, $\lambda(x)$). The infimum is attained since the set $\partial f(x)$ is w^*-compact.

Throughout the paper we use the following notations for the locally Lipschitz function $f : X \to \mathbb{R}$ and a number $c \in \mathbb{R}$:

$$f^c = \{u \in X \ : \ f(u) \leq c\};$$

$$K_c = \{u \in X \ : \ \lambda(u) = 0, \ f(u) = c\};$$
$$(K_c)_\delta = \{u \in X \ : \ dist(u, K_c) < \delta\};$$
$$(K_c)^c_\delta = X \setminus (K_c)_\delta.$$

We say that:

a) *f satisfies the (PS)-condition at level c* (in short, $(PS)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$ has a convergent subsequence;

b) *f satisfies the (C)-condition at level c* (in short, $(C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $(1 + \|x_n\|)\lambda(x_n) \to 0$ has a convergent subsequence.

It is clear that $(PS)_c$ implies $(C)_c$.

Our approach is based on the following idea. We consider a globally Lipschitz functional $\varphi : X \to \mathbb{R}$ such that $\varphi(x) \geq 1$, $\forall x \in X$ (or, generally, $\varphi(x) \geq \alpha$, $\forall x \in X$, for some $\alpha > 0$). We say that

c) *f satisfies the $(\varphi - C)$-condition at level c* (in short, $(\varphi - C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $\varphi(x_n)\lambda(x_n) \to 0$ has a convergent subsequence.

The compactness $(\varphi - C)_c$-condition in c) contains the assertions a) and b) in the sense that if $\varphi \equiv 1$ we get the $(PS)_c$-condition and if $\varphi(x) = 1 + \|x\|$ we have the $(C)_c$-condition.

We will prove that the $(\varphi - C)_c$-condition suffices to give rise to a deformation lemma. Our result in this direction is the following.

Theorem 1. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on the Banach space X satisfying the $(\varphi - C)_c$-condition, with $c \in \mathbb{R}$ and a globally Lipschitz function $\varphi : X \to \mathbb{R}$ with Lipschitz constant $L > 0$ and $\varphi(x) \geq 1$, $\forall x \in X$. Then for every $\varepsilon_0 > 0$ and every neighborhood U of K_c there exist a number $0 < \varepsilon < \varepsilon_0$ and a continuous function $\eta : X \times [0, 1] \to X$ such that*

(a) $\|\eta(x, t) - x\| \leq \varphi(x)te^{Lt}$, $\forall (x, t) \in X \times [0, 1]$;

(b) $\eta(x, t) = x$, $\forall x \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$, $\forall t \in [0, 1]$;

(c) $f(\eta(x, t)) \leq f(x)$, $\forall (x, t) \in X \times [0, 1]$;

(d) $\eta(x, t) \neq x \Rightarrow f(\eta(x, t)) < f(x)$;

(e) $\eta(f^{c+\varepsilon}, 1) \subset f^{c-\varepsilon} \cup U$;

(f) $\eta(f^{c+\varepsilon} \setminus U, 1) \subset f^{c-\varepsilon}$.

The proof will be given in Section 2.

Remark 1. If $\varphi \equiv 1$, Theorem 1 represents the deformation lemma of Chang (see Chang [4]). If $\varphi(x) = 1 + \|x\|$, Theorem 1 reduces to the deformation lemma of Kourogenis-Papageorgiou (see Kourogenis and Papageorgiou [6]).

From the above deformation result we deduce the following general version of Mountain Pass Theorem.

Theorem 2. (Mountain Pass Theorem) *Let X be a Banach space, $f : X \to R$ be a locally Lipschitz function with $f(0) \leq 0$ and $\varphi : X \to R$ a globally Lipschitz function such that $\varphi(x) \geq 1$, $\forall x \in X$. Suppose that there exist a point $x_1 \in X$ and constants $\rho, \alpha > 0$ such that*

(i) *$f(x) \geq \alpha$, $\forall x \in X$ with $\|x\| = \rho$,*

(ii) *$\|x_1\| > \rho$ and $f(x_1) < \alpha$,*

(iii) *the function f satisfies the $(\varphi - C)_c$-condition, where*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = x_1\}.$$

Then the minimax value c in (iii) *is a critical value of f, i.e. K_c is nonempty, and, in addition, $c \geq \alpha$.*

The proof will be given in Section 2.

Remark 2. Following the lines of Mountain Pass Theorem in Theorem 2, we can obtain in our general setting other important minimax principles, like Saddle Point Theorem and Linking Theorem. For other different versions we refer to Ambrosetti and Rabinowitz [1], Bartolo et al [2], Kourogenis and Papageorgiou [6]-Rabinowitz [13].

Section 2 contains the proofs of Theorems 1 and 2.

2. PROOFS

We need the following result for obtaining the existence of a suitable locally Lipschitz vector field.

Lemma 1. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the $(\varphi - C)_c$-condition, where $\varphi : X \to \mathbb{R}$ is a globally Lipschitz function such that $\varphi(x) \geq 1$, $\forall x \in X$, and $c \in \mathbb{R}$. Then for each $\delta > 0$ there exist constants $\gamma, \varepsilon > 0$ and a locally Lipschitz vector field*

$$v : f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta \to X$$

such that for each $x \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta$ one has

$$\|v(x)\| \leq \varphi(x), \tag{1}$$

$$(y^*, v(x)) \geq \frac{\gamma}{2}, \quad \forall y^* \in \partial f(x). \tag{2}$$

Proof. From the $(\varphi - C)_c$-condition we get $\gamma, \varepsilon > 0$ such that

$$\varphi(x)\lambda(x) > \gamma, \quad \forall x \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta. \tag{3}$$

If not, we would find a sequence $\{x_n\} \subset (K_c)^c_\delta$ such that $f(x_n) \to c$ and $\varphi(x_n)\lambda(x_n) \to 0$. Using the $(\varphi - C)_c$-condition we obtain a convergent subsequence of $\{x_n\}$ (denoted again by $\{x_n\}$), say $x_n \to x^0 \in (K_c)^c_\delta$. Since f is continuous and λ is lower semicontinuous we obtain that $f(x^0) = c$ and $\varphi(x^0)\lambda(x^0) = 0$. This implies $x^0 \in K_c$, which is a contradiction. Thus (3) is true.

Let $x_0 \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta$ and $x^* \in \partial f(x_0)$ such that $\lambda(x_0) = \|x^*\|$. Then we have $B(0, \|x^*\|) \cap \partial f(x_0) = \emptyset$, where $B(0, r) = \{z^* \in X^* : \|z^*\| < r\}$, $r > 0$. Using the separation theorem in X^* endowed with the w^*-topology we obtain that there exists some $h_0 \in X$ such that $\|h_0\| = 1$ and $(z^*, h_0) \leq (y^*, h_0)$ for each $z^* \in B(0, \|x^*\|)$ and $y^* \in \partial f(x_0)$.

From Hahn-Banach theorem and (3) we get

$$\sup_{z^* \in B(0, \|x^*\|)} (z^*, h_0) = \|x^*\| > \frac{\gamma}{\varphi(x_0)}.$$

Therefore $(y^*, h_0) \geq \|x^*\| > \frac{\gamma}{2\varphi(x_0)}$ for each $y^* \in \partial f(x_0)$.

The multifunction $x \mapsto \partial f(x)$ is weakly$*$ upper semicontinuous, so for each $x_0 \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta$ there exists $\eta_0 > 0$ and $h_0 \in X$ such that for every $y \in B(x_0, \eta_0) = \{y \in X : \|y - x_0\| < \eta_0\}$ and every $y^* \in \partial f(y)$ we have $(y^*, h_0) > \frac{\gamma}{2\varphi(y)}$.

The set of all such balls $\{B(x_0, \eta_0)\}$ covers $f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta$. By paracompactness there is a locally finite open covering $\{V_i\}_{i \in I}$ subordinated to it. If we denote $\rho_i(x) = dist(x, X \setminus V_i)$, $\forall x \in X$, then the functions ρ_i are Lipschitz continuous and $\rho_i\big|_{X\setminus V_i} = 0$.

For every $x \in \bigcup_{i\in I} V_i$, let $\beta_i(x) = \frac{\rho_i(x)}{\sum\limits_{j\in I}\rho_j(x)}$ and $v(x) = \varphi(x)(\sum\limits_{i\in I}\beta_i(x)h_i)$, where h_i plays the same role for x_i as h_0 for x_0. It follows that the function $v : f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta \to X$ is locally Lipschitz and for every $x \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap (K_c)^c_\delta$ we have

$$\|v(x)\| \le \varphi(x)\left(\sum_{i\in I}\beta_i(x)\|h_i\|\right) = \varphi(x),$$

$$(y^*, v(x)) = \varphi(x)\left(\sum_{i\in I}\beta_i(x)(y^*, h_i)\right) > \frac{\gamma}{2}, \quad \forall y^* \in \partial f(x).$$

Properties (1) and (2) are satisfied. □

Remark 3. If we analyse the proof of above lemma, we notice that the globally Lipschitz property of φ is not used. The globally Lipschitz property of φ is essential in the proof below for Theorem 1.

Proof of Theorem 1. Fix $\varepsilon_0 > 0$ and a neighborhood U of K_c. From the compactness of K_c we can find $\delta > 0$ such that $(K_c)_{3\delta} \subset U$. Moreover, from (3) in the proof of Lemma 1 we can guarantee the existence of $\gamma > 0$ and $0 < \bar\varepsilon < \varepsilon_0$ such that $\varphi(x)\lambda(x) \ge \gamma$ for all $x \in f^{-1}([c-\bar\varepsilon, c+\bar\varepsilon]) \cap (K_c)^c_\delta$.

We consider the following two closed sets:

$$A = \{x \in X : |f(x) - c| \ge \bar\varepsilon\} \cup \overline{(K_c)_\delta}\,, \tag{4}$$

$$B = \{x \in X : |f(x) - c| \le \frac{\bar\varepsilon}{2}\} \cap (K_c)^c_{2\delta}\,. \tag{5}$$

Since $A \cap B = \emptyset$ there exists a locally Lipschitz function $\psi : X \to [0,1]$ such that $\psi = 0$ on a closed neighborhood of A, say $\tilde A$, disjoint of B, $\psi\big|_B = 1$ and $0 \le \psi \le 1$. For instance, we can take $\psi(x) = \frac{d(x,\tilde A)}{d(x,\tilde A)+d(x,B)}$, $\forall x \in X$.

Let $V : X \to X$ be defined by

$$V(x) = \begin{cases} -\psi(x)v(x), & x \in f^{-1}([c-\bar\varepsilon, c+\bar\varepsilon]) \cap (K_c)^c_\delta \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

where $v(x)$ is constructed in Lemma 1. The vector field V is locally Lipschitz and by Lemma 1, for $x \in f^{-1}([c-\bar\varepsilon, c+\bar\varepsilon]) \cap (K_c)^c_\delta$ we have

$$\|V(x)\| = \psi(x)\|v(x)\| \le \varphi(x) \tag{7}$$

and

$$(x^*, V(x)) = -\psi(x)(x^*, v(x)) \le -\psi(x)\frac{\gamma}{2}, \ \forall x^* \in \partial f(x). \tag{8}$$

Since V is locally Lipschitz and $\|V(x)\| \leq \varphi(0)+L\|x\|$ (cf. (7)), the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\eta(x,t) = V(\eta(x,t)) \\ \eta(x,0) = x \end{cases} \tag{9}$$

has a unique solution $\eta(x,\cdot)$ on $\mathbb{R}$, for each $x \in X$.

By (7) we have that

$$\|\eta(x,t) - x\| \leq \int_0^t \|V(\eta(x,s))\| ds \leq \int_0^t \varphi(\eta(x,s)) ds$$

$$= \int_0^t [\varphi(\eta(x,s)) - \varphi(x)] ds + \int_0^t \varphi(x) ds \leq L \int_0^t \|\eta(x,s) - x\| ds + \varphi(x)t.$$

Using Gronwall's inequality we get assertion (a).

If $x \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}])$, then $x \in A$, so $\psi(x) = 0$. By (6) it follows that $V(x) = 0$ and from (9) we obtain that $\eta(x,t) = x$, for each $t \in [0,1]$. This yields (b).

Next, for a fixed $x \in X$, let us consider the function $h_x : [0,1] \to \mathbb{R}$ given by $h_x(t) = f(\eta(x,t))$. Using the chain rule we have

$$\frac{d}{dt}h_x(t) \leq \max\Big\{ (x^*, \frac{d}{dt}\eta(x,t)) \ : \ x^* \in \partial f(\eta(x,t)) \Big\}$$

$$= \max\Big\{ (x^*, V(\eta(x,t))) \ : \ x^* \in \partial f(\eta(x,t)) \Big\} \text{ a.e. on } [0,1].$$

Therefore, taking into account (8), we infer that

$$\frac{d}{dt}h_x(t) \leq -\psi(\eta(x,t))\frac{\gamma}{2} \leq 0 \ \text{ if } \ \eta(x,t) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta \tag{10}$$

and, clearly, by (6),

$$\frac{d}{dt}h_x(t) \leq 0 \ \text{ if } \ \eta(x,t) \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta.$$

Hence property (c) holds true.

In order to prove property (d), suppose that $\eta(x,t) \neq x$. First, we show that

$$\eta(x,s) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta, \quad \forall s \in [0,t]. \tag{11}$$

On the contrary, there would exist $s_0 \in [0,t]$ such that $\eta(x,s_0) \in A$ (cf. (4)). This implies that $V(\eta(x,s_0)) = 0$. Using the uniqueness of solution to the Cauchy problem formed by the equation in (9) and the initial condition with the initial value $\eta(x,s_0)$, we see that

$$\eta(x,\tau+s_0) = \eta(x,s_0), \quad \forall \tau \in \mathbb{R}.$$

Letting $\tau = t - s_0$ and $\tau = -s_0$ one obtains $\eta(x,t) = x$, which contradicts our assumption. Thus the claim in (11) is true.

Using (10) and (11) it follows that

$$f(x) - f(\eta(x,t)) = -\int_0^t \frac{d}{ds} h_x(s)\, ds \geq \frac{\gamma}{2}\int_0^t \psi(\eta(x,s))ds. \tag{12}$$

We show that there is $s \in [0,t]$ such that

$$\psi(\eta(x,s)) \neq 0. \tag{13}$$

For, otherwise, if $\psi(\eta(x,s)) = 0$, $\forall s \in [0,t]$, then $V(\eta(x,s)) = 0$, $\forall s \in [0,t]$. By (9), we get that $\eta(x,\cdot)$ is constant on $[0,t]$, which contradicts $\eta(x,t) \neq x$. It results that (13) is valid. Since $\psi \geq 0$, from (12) and (13) we infer that $f(\eta(x,t)) < f(x)$, which proves assertion (d).

We show now assertion (e). Let $\rho > 0$ such that $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$. We choose

$$0 < \varepsilon \leq \min\left\{\frac{\bar{\varepsilon}}{2}, \frac{\gamma}{4}, \frac{\delta\gamma}{8}\mathrm{e}^{-\mathrm{L}}(\varphi(0) + \mathrm{L}\rho)^{-1}\right\}. \tag{14}$$

We argue by contradiction. Let $x \in f^{c+\varepsilon}$ such that $f(\eta(x,1)) > c-\varepsilon$ and $\eta(x,1) \notin U$. Since, by (c), $f(\eta(x,t)) \leq f(x) \leq c+\varepsilon$ and $f(\eta(x,t)) \geq f(\eta(x,1))$ for each $t \in [0,1]$, we get

$$c - \varepsilon < f(\eta(x,t)) \leq c + \varepsilon, \quad \forall t \in [0,1]. \tag{15}$$

We claim that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} \neq \emptyset. \tag{16}$$

Suppose that (16) does not hold. This means that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} = \emptyset. \tag{17}$$

First, we show that

$$\eta(x,t) \in B, \quad \forall t \in [0,1]. \tag{18}$$

The fact that $\eta(x,t) \in f^{-1}([c - \frac{\bar{\varepsilon}}{2}, c + \frac{\bar{\varepsilon}}{2}])$ follows from (14) and (15). By (17) one has that $\eta(x,t) \in (K_c)^c_{2\delta}$. Consequently, from (5) we conclude that (18) is established. On the basis of (18) and (10) we may write

$$f(x) - f(\eta(x,1)) = h_x(0) - h_x(1) = -\int_0^1 \frac{d}{dt}h_x(t)dt \geq \int_0^1 \frac{\gamma}{2}\psi(\eta(x,t))dt.$$

Then, combining (18) and the definition of ψ it is clear that

$$f(x) - f(\eta(x,1)) \geq \frac{\gamma}{2}. \tag{19}$$

On the other hand, from (15) we obtain that

$$f(x) - f(\eta(x,1)) < 2\varepsilon. \tag{20}$$

From (19) and (20) we get $\frac{\gamma}{2} < 2\varepsilon$, which contradicts (14). This justifies (16).

The next step in the proof is to show that there exist $0 \le t_1 < t_2 \le 1$ such that

$$dist(\eta(x,t_1), K_c) = 2\delta, \quad dist(\eta(x,t_2), K_c) = 3\delta \tag{21}$$

and

$$2\delta < dist(\eta(x,t), K_c) < 3\delta, \quad \forall t_1 < t < t_2. \tag{22}$$

Denote $g(t) = dist(\eta(x,t), K_c)$, $\forall t \in [0,1]$. In view of (16) we have that $\{t \in [0,1] : g(t) \le 2\delta\} \ne \emptyset$. Thus it is permitted to consider

$$t_1 = \sup\{t \in [0,1] \, : \, g(t) \le 2\delta\}.$$

Since it is known that $(K_c)_{3\delta} \subset U$ and $\eta(x,1) \notin U$, we derive that $\eta(x,1) \notin (K_c)_{3\delta}$. This means that $g(1) \ge 3\delta$. Since $g(t_1) \le 2\delta$ it is necessary to have $t_1 < 1$. The definition of t_1 implies $g(t) > 2\delta$ for all $t \in (t_1, 1]$ (which is the first inequality in (22)). Letting $t \downarrow t_1$ we deduce that $g(t_1) \ge 2\delta$. We obtain that $g(t_1) = 2\delta$, so the first part in (21) is proved. Taking into account that $g(1) \ge 3\delta$, we see that $\{t \in [t_1, 1] \, : \, g(t) \ge 3\delta\}$ is nonempty. Then we can define

$$t_2 = \inf\{t \in [t_1, 1] \, : \, g(t) \ge 3\delta\}.$$

Since $g(t_2) \ge 3\delta$ and $g(t_1) = 2\delta$ it is clear that $t_1 < t_2$. By the definition of t_2 we have that $g(t) < 3\delta$ for all $t_1 \le t < t_2$, so (22) holds. In addition, letting $t \uparrow t_2$, we get $g(t_2) = 3\delta$, so (21) holds, too.

Let us show that

$$t_2 - t_1 < \frac{4\varepsilon}{\gamma}. \tag{23}$$

From (22) it follows that $\eta(x,t) \notin (K_c)_{2\delta}$, $\forall t \in [t_1, t_2]$, while (15) and (14) imply $\eta(x,t) \in f^{-1}([c - \frac{\bar{\varepsilon}}{2}, c + \frac{\bar{\varepsilon}}{2}])$, $\forall t \in [t_1, t_2]$. The definition of the set B in (5) yields

$$\eta(x,t) \in B, \quad \forall t \in [t_1, t_2].$$

Using the definition of ψ, (10) and (15) we see that

$$\frac{\gamma}{2}(t_2 - t_1) = \frac{\gamma}{2} \int_{t_1}^{t_2} \psi(\eta(x,t)) dt \le - \int_{t_1}^{t_2} \frac{d}{dt} h_x(t) dt$$
$$= h_x(t_1) - h_x(t_2) = f(\eta(x,t_1)) - f(\eta(x,t_2)) < 2\varepsilon.$$

Therefore (23) is proved.

We need the following inequality

$$\|\eta(x,t_2) - \eta(x,t_1)\| \ge \delta. \tag{24}$$

To check (24) consider a point $v \in K_c$ so that

$$dist(\eta(x,t_1), K_c) = \|\eta(x,t_1) - v\| = 2\delta.$$

Here the compactness of K_c and the first part in (21) have been used. Then, on the basis of the second part in (21) we can write

$$\|\eta(x,t_2)-\eta(x,t_1)\| \geq \|\eta(x,t_2)-v\| - \|\eta(x,t_1)-v\| \geq 3\delta - 2\delta = \delta.$$

Therefore (24) holds.

Using (9), (7) and the Lipschitzianess of φ we can write

$$\|\eta(x,t_2)-\eta(x,t_1)\| \leq \int_{t_1}^{t_2} \|V(\eta(x,s))\|ds \leq \int_{t_1}^{t_2} \varphi(\eta(x,s))ds$$

$$= \int_{t_1}^{t_2} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_1))]ds + \varphi(\eta(x,t_1))(t_2-t_1)$$

$$\leq \int_{t_1}^{t_2} L\|\eta(x,s)-\eta(x,t_1)\|ds + \varphi(\eta(x,t_1))(t_2-t_1). \tag{25}$$

By (25) and Gronwall's inequality we get

$$\|\eta(x,t_2)-\eta(x,t_1)\| \leq \varphi(\eta(x,t_1))(t_2-t_1)e^{L(t_2-t_1)}. \tag{26}$$

From (24), (26), (23) and the Lipschitzianess of φ we deduce that

$$\delta \leq \|\eta(x,t_2)-\eta(x,t_1)\| < \frac{4\varepsilon}{\gamma} e^L \varphi(\eta(x,t_1))$$

$$\leq \frac{4\varepsilon}{\gamma} e^L (\varphi(0) + L\|\eta(x,t_1)\|). \tag{27}$$

In view of (21) and the choice of ρ to satisfy $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ we have $\eta(x,t_1) \in (K_c)_{3\delta} \subset B(0,\rho)$. This property and (14) yield from (27) that

$$\delta \leq \frac{4\varepsilon}{\gamma} e^L (\varphi(0) + L\rho) \leq \frac{\delta}{2},$$

which is a contradiction. This proves (e).

In order to show (f), since $(K_c)_{3\delta} \subset U$ it is enough to prove that

$$\eta(f^{c+\varepsilon} \setminus (K_c)_{3\delta}, 1) \subset f^{c-\varepsilon}. \tag{28}$$

Let us denote

$$C = (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap (K_c)^c_{3\delta}.$$

To check (28), we note that it is sufficient to verify that

$$\eta(x,1) \in f^{c-\varepsilon}, \quad \forall x \in C, \tag{29}$$

because for $x \in f^{c-\varepsilon}$ we have $f(\eta(x,1)) \leq f(x) \leq c - \varepsilon$, due to the nondecreasing monotonicity of $f(\eta(x,\cdot))$.

To show (29), denote

$$D = (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap (K_c)^c_{\frac{5}{2}\delta}.$$

First, we verify that

$$\forall x \in C,\ \exists\, t_x \in (0, \frac{4\varepsilon}{\gamma}] \ \text{ such that } \ \eta(x, t_x) \notin D. \tag{30}$$

To this end, we prove the inclusion below

$$\{t > 0 \ : \ \eta(x, \tau) \in D,\ \forall \tau \in [0, t]\} \subset (0, \frac{4\varepsilon}{\gamma}),\ \forall x \in C. \tag{31}$$

Indeed, if $\eta(x, \tau)$ is in $D \subset B$, $\forall \tau \in [0, t]$, we have $\psi(\eta(x, \tau)) = 1$, $\forall \tau \in [0, t]$. Therefore, by (10), we have $\frac{d}{d\tau} h_x(\tau) \leq -\frac{\gamma}{2}$, $\forall \tau \in [0, t]$. From this and (15) we obtain

$$2\varepsilon > h_x(0) - h_x(t) = -\int_0^t \frac{d}{d\tau} h_x(\tau) d\tau \geq \frac{\gamma}{2} t,$$

so $t < \frac{4\varepsilon}{\gamma}$. Thus (31) is satisfied.

We are now in a position to prove (30). We proceed arguing by contradiction. Assuming that there exist $x \in C$ such that $\eta(x, t) \in D$, $\forall t \in (0, \frac{4\varepsilon}{\gamma}]$, by (31), we arrive at the contradiction

$$\frac{4\varepsilon}{\gamma} \in \{t > 0 \ : \ \eta(x, \tau) \in D,\ \forall \tau \in [0, t]\} \subset (0, \frac{4\varepsilon}{\gamma}),$$

which proves (30).

Let us show that for every $x \in C$, it is true that

$$\eta(\{x\} \times [0, 1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset \Rightarrow \exists\, t_0 \in (0, t_3] \ \text{ such that } \ \eta(x, t_0) \in f^{c-\varepsilon}, \tag{32}$$

with

$$t_3 = \inf\{t \in [0, 1] : \ dist(\eta(x, t), K_c) \leq \frac{5}{2}\delta\},$$

where the set $\{t \in [0, 1] : \ dist(\eta(x, t), K_c) \leq \frac{5}{2}\delta\}$ is nonempty in view of (21). If (32) were not true it would exist $x \in C$ with $\eta(\{x\} \times [0, 1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset$ and $f(\eta(x, t)) > c - \varepsilon$, $\forall t \in [0, t_3]$. Hence $\eta(x, t) \in D$, $\forall t \in [0, t_3]$. This follows from the definition of t_3 and since $x \in C$. The inclusion in (31) implies that

$$t_3 < \frac{4\varepsilon}{\gamma}. \tag{33}$$

Introduce

$$t_4 = \sup\{t \in [0, t_3] : \ dist(\eta(x, t), K_c) \geq 3\delta\}.$$

Since $x \in C$, then $x \in (K_c)^c_{3\delta}$, thus the set $\{t \in [0, t_3] : \ dist(\eta(x, t), K_c) \geq 3\delta\}$ is nonempty. By the definitions of t_3 and t_4 it follows that

$$\eta(x, t) \in (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap ((K_c)_{3\delta} \setminus (K_c)_{\frac{5}{2}\delta}),\ \forall t \in (t_4, t_3].$$

We remark that

$$\|\eta(x,t_3)-\eta(x,t_4)\| \geq \frac{\delta}{2}\,. \tag{34}$$

Indeed, by the definition of t_4 we have

$$\|\eta(x,t_3)-\eta(x,t_4)\| \geq \|\eta(x,t_4)-v\| - \|\eta(x,t_3)-v\|$$

$$\geq 3\delta - \|\eta(x,t_3)-v\|, \quad \forall v \in K_c.$$

This leads to

$$\|\eta(x,t_3)-\eta(x,t_4)\| \geq 3\delta - dist(\eta(x,t_3),K_c) = 3\delta - \frac{5}{2}\delta = \frac{\delta}{2}\,,$$

so (34) is verified.

Using (9), (7) and the Lipschitzianess of φ, we can write

$$\|\eta(x,t_3)-\eta(x,t_4)\| \leq \int_{t_4}^{t_3} \|V(\eta(x,s))\|ds \leq \int_{t_4}^{t_3} \varphi(\eta(x,s))ds$$

$$= \int_{t_4}^{t_3} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_4))]ds + \varphi(\eta(x,t_4))(t_3-t_4)$$

$$\leq \int_{t_4}^{t_3} L\|\eta(x,s)-\eta(x,t_4)\|ds + \varphi(\eta(x,t_4))(t_3-t_4).$$

By Gronwall's inequality we get

$$\|\eta(x,t_3)-\eta(x,t_4)\| \leq \varphi(\eta(x,t_4))(t_3-t_4)e^{L(t_3-t_4)}. \tag{35}$$

Using (34), (35), the Lipschitzianess of φ, the inclusion $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ and (33), we have that

$$\frac{\delta}{2} \leq \|\eta(x,t_3)-\eta(x,t_4)\| \leq e^{L(t_3-t_4)}\varphi(\eta(x,t_4))(t_3-t_4)$$

$$\leq e^{L}(\varphi(0)+L\|\eta(x,t_4)\|)t_3 < e^{L}(\varphi(0)+L\rho)\frac{4\varepsilon}{\gamma}\,.$$

This contradicts the choice of ε in (14), therefore (32) is true.

In order to complete the proof of (f), let $x \in C$. From (30), there exists $t_x \in (0, \frac{4\varepsilon}{\gamma}]$ such that $\eta(x,t_x) \notin D$. This means that

$$\eta(x,t_x) \in (X \setminus f^{c+\varepsilon}) \cup f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}\,.$$

On the other hand, $\eta(x,t_x) \in f^{c+\varepsilon}$ since, as $x \in C$, $f(\eta(x,t_x)) \leq f(x) \leq c+\varepsilon$. Consequently, we deduce that $\eta(x,t_x) \in f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}$. Two cases arise:

1) $\eta(x,t_x) \in f^{c-\varepsilon}$;

2) $\eta(x,t_x) \in (K_c)_{\frac{5}{2}\delta}$.

In case 1) we have directly that

$$f(\eta(x,1)) \leq f(\eta(x,t_x)) \leq c-\varepsilon,$$

which ensures the desired conclusion.

It remains to treat case 2). In this situation, we make use of property (32). Therefore, we find $t_0 \in (0, t_3]$ such that $\eta(x, t_0) \in f^{c-\varepsilon}$. Thus we may write $f(\eta(x, 1)) \leq f(\eta(x, t_0)) \leq c - \varepsilon$. The proof is complete. □

Proof of Theorem 2. We have that $\alpha \leq c$ because every curve $\gamma \in \Gamma$ intersects the sphere $\partial B(0, \rho)$ and condition (i) is imposed.

It remains to prove that c is a critical value. Arguing by contradiction, assume $K_c = \emptyset$. Let $\varepsilon_0 = \frac{1}{2} \min\{\alpha, \alpha - f(x_1)\}$ which is a positive number due to condition (ii). Apply Theorem 1 for ε_0 chosen above and $U = \emptyset$. By Theorem 1 there exist $0 < \varepsilon < \varepsilon_0$ and an homotopy η as in Theorem 1. From the definition of the number c, there exists $\gamma \in \Gamma$ such that

$$\max_{t \in [0,1]} f(\gamma(t)) \leq c + \varepsilon.$$

Let $\gamma_1 : [0, 1] \to X$ defined by $\gamma_1(t) = \eta(\gamma(t), 1)$, $\forall t \in [0, 1]$.

We see that $\gamma_1 \in \Gamma$. Indeed, by the choice of ε_0 we have

$$\varepsilon_0 \leq \frac{\alpha}{2} < \alpha \leq c,$$

so $f(0) \leq 0 < c - \varepsilon_0$. This shows that $0 \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$. Then by Theorem 1 (b) and since $\gamma \in \Gamma$ we obtain

$$\gamma_1(0) = \eta(\gamma(0), 1) = \eta(0, 1) = 0.$$

On the other hand, from the choice of ε_0 one has

$$f(x_1) \leq \alpha - 2\varepsilon_0 < \alpha - \varepsilon_0 \leq c - \varepsilon_0,$$

so $x_1 \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$. Applying Theorem 1 (b) and using $\gamma \in \Gamma$ we infer that

$$\gamma_1(1) = \eta(\gamma(1), 1) = \eta(x_1, 1) = x_1.$$

Therefore, $\gamma_1 \in \Gamma$. Then by Theorem 1 (e) and the definition of γ_1 we have

$$c \leq \max_{t \in [0,1]} f(\gamma_1(t)) \leq c - \varepsilon.$$

The obtained contradiction completes the proof. □

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