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ANALYTIC SOLUTIONS OF THE PAINLEVÉ EQUATIONS

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ABSTRACT: For each one of the six well-known Painlevé equations, it is proved that there exists a unique analytic solution which together with its first two derivatives converge absolutely in a specified region of the complex plane. Moreover, we give a bound of the solution for all six Painlevé equations and a bound of the first two derivatives of the solution for the last four Painlevé equations. Finally for all of them we give a region, depending on the initial conditions and the parameters of the equations, in which the solution holds.

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1. INTRODUCTION

The well-known six Painlevé equations

$$w''(z) = 6[w(z)]^2 + z\,, \tag{1.1}$$

$$w''(z) = 2[w(z)]^3 + zw(z) + \alpha\,, \tag{1.2}$$

$$w''(z) = \frac{1}{w(z)}[w'(z)]^2 - \frac{1}{z}w'(z) + \frac{1}{z}[\alpha(w(z))^2 + b] + c[w(z)]^3 + \frac{d}{w(z)}\,, \tag{1.3}$$

$$w''(z) = \frac{1}{2w(z)}[w'(z)]^2 + \frac{3}{2}[w(z)]^3 + 4z[w(z)]^2 + 2(z^2-\alpha)w(z) + \frac{b}{w(z)}\,, \tag{1.4}$$

$$\begin{aligned} w''(z) = \left[\frac{1}{2w(z)} + \frac{1}{w(z)-1}\right][w'(z)]^2 - \frac{1}{z}w'(z) + \frac{1}{z^2}[w(z)-1]^2\left[\alpha w(z) + \frac{b}{w(z)}\right] \\ + c\frac{w(z)}{z} + \frac{dw(z)[w(z)+1]}{w(z)-1}\,, \end{aligned} \tag{1.5}$$

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$$w''(z) = \frac{1}{2}\left[\frac{1}{w(z)} + \frac{1}{w(z)-1} + \frac{1}{w(z)-z}\right][w'(z)]^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w(z)-z}\right)w'(z)$$

$$+\frac{w(z)[w(z)-1][w(z)-z]}{z^2(z-1)^2}\left[\alpha + \frac{bz}{(w(z))^2} + c\frac{z-1}{(w(z)-1)^2} + d\frac{z(z-1)}{(w(z)-z)^2}\right], \quad (1.6)$$

where α, b, c and d are in general complex numbers, were discovered by Painlevé [16], Gambier and Fuchs (see Ablowitz and Clarkson [1], p. 353] and the references there in) and have been studied by many authors with various different methods and for different reasons. From the extended literature we refer to Its [11], for a review of the recent developments in the global asymptotic analysis of the classical Painlevé equations and to Its and Novokshenov [12], for a presentation of the isomonodromic deformation method (IDM). Also some of the Painlevé equations were connected in the past (see for example Ablowitz et al [2], Rosales [17]) with well-known non-linear partial differential equations and thus, many information for the solutions of those non-linear partial differential equations can be obtained from the study of the solutions of the Painlevé equations. In McCoy et al [14], a theory was developed for the one-parameter family of bounded solutions of (1.3) (as $z \to \infty$ along the positive real axis) when the parameters α, b, c and d satisfy $\alpha(-d)^{1/2} + bc^{1/2} = 0$. For rational solutions of the Painlevé equations, we refer to Murata [15] and to the references therein. For meromorphic solutions in the complex plane of the first, second and fourth Painlevé equations, we refer to the recent papers of Hinkkanen and Laine [7] and Steinmetz [21]. Also in Fokas et al [4] and Fokas and X. Zhou [5], it was shown that the Cauchy problems for equations (1.1)-(1.5) admit in general global solutions, meromorphic in z. Furthermore, for certain constraints on the monodromic data, it was shown that the solutions of (1.1)-(1.5) are **free of poles** for $t \in \mathbb{R}$ and finite. More precisely in Fokas and X. Zhou [5], a rigorous methodology for studying the Riemann-Hilbert problems associated with certain nonlinear ordinary differential equations was introduced. This method was applied to (1.2) and (1.4) in Fokas and X. Zhou [5] and to (1.1), (1.3) and (1.5) in Fokas et al [4]. However, we have seen only few results concerning analytic solutions of some of the six Painlevé equations (see Hille [6], pp. 441-442], Ifantis [10], Lukaševič [13]). In Lukaševič [13], in particularly, necessary and sufficient conditions were given so as the fifth Painlevé equation to have an analytic solution at $z = 0$. Also, in Ifantis [10], as an example of a more general theorem, necessary conditions were given so as the first Painlevé equation to have a unique analytic solution, which together with its first two derivatives converge absolutely in a specified disk of the complex plane. We formulate this result in the following:

Theorem 1.1. *If*

$$\frac{r^3}{6} + |w(0)| + |w'(0)|r \leq \frac{1}{12r^2}, \quad (1.7)$$

where $z = rx$, $r > 0$, $|z| < r$, then equation (1.1) has a unique analytic solution

$$w(z) = w(0) + w'(0)z + \sum_{n=2}^{\infty} b_n(w(0), w'(0))z^n, \tag{1.8}$$

which together with its first two derivatives converge absolutely in $|z| < r$ and is bounded by:

$$|w(z)| \leq \frac{1}{r^2}. \tag{1.9}$$

Remark 1.1. For the more general equation (than (1.1)):

$$w''(z) = \mu_1[w(z)]^2 + \mu_2 z, \mu_1 \neq 0 \tag{1.10}$$

it was found in Ifantis [10] that if

$$\frac{r^3|\mu_2|}{6} + |w(0)| + |w'(0)|r \leq \frac{1}{2|\mu_1|r^2},$$

where $z = rx$, $r > 0$, $|x| < 1$, equation (1.10) has a unique analytic solution of the form (1.8) which together with its first two derivatives converge absolutely in $|z| < r$ and is bounded by:

$$|w(z)| \leq \frac{1}{|\mu_1|r^2}.$$

Remark 1.2. Equation (1.10) has been studied in Ifantis [10] as a particular case of a general theorem concerning non-linear ordinary differential equations of the form:

$$\frac{d^2 f(z)}{dz^2} + \frac{\alpha}{z}\frac{df(z)}{dz} = \sum_{n=1}^{\infty} c_n(z)[f(z)]^{n-1},$$

where α is in general a complex number and the functions $c_n(z)$ are elements of $H_1(\Delta)$, i.e. $\|c_n(z)\|_{H_1(\Delta)} \leq m_n$, $n = 1, 2, ...$ and the series $\sum_{n=1}^{\infty} m_n w^{n-1}$ has a sufficiently large radius of convergence.

In Section 2, we give conditions so that the rest five Painlevé equations to have a **unique bounded analytic solution** which together with its first two derivatives converge absolutely in a specified region of the complex plane. (It is obvious that, in this region of the complex plane, the solution of the Painlevé equations (1.2)-(1.6), **cannot have any poles**). **These conditions involve only the initial conditions and the parameters of the equations**. These are the main results of this paper, which are presented in Section 2 (Theorems 2.1-2.5). In our approach, we use the Banach space $H_1(\Delta)$ of analytic functions defined as follows:

$$H_1(\Delta) = \{f : \Delta = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}/f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$$

$$\text{analytic in } \Delta \text{ with } \sum_{n=1}^{\infty} |\alpha_n| < +\infty\}. \tag{1.11}$$

The norm of this space is defined by:

$$\|f(z)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |\alpha_n|.$$

In Section 3, we present the method used and give a lemma (Lemma 3.1) necessary for the proofs of Theorems 2.2-2.5. This method is a functional analytic method developed in Ifantis [8] for linear functional-differential equations and functional-differential systems and used also in Siafarikas [18], Siafarikas [19], Siafarikas [20]. Later on, Ifantis extended his method for non-linear ordinary differential equations in Ifantis [9], Ifantis [10]. In Section 4 we give the proofs of Theorems 2.1-2.5. More precisely, we give in details the proofs of Theorems 2.1-2.2. For the proof of Theorem 2.1, we apply the method presented in Ifantis [8], but we give this proof in detail, not only for reasons of completeness, but also in order to point out the differences between the method presented in Ifantis [10] and the method we present here. Unfortunately, due to lack of space, we omit the proof of Theorem 2.5 and we only give the starting point for the proofs of Theorems 2.3 and 2.4, since these proofs (although they involve many non-trivial calculations) are similar to the proof of Theorem 2.2. Finally we point out that **the results regarding equations (1.3)-(1.6) cannot be obtained by applying the theorems presented in Ifantis [9] or Ifantis [10] due to the nonlinearities that they present.**

2. MAIN RESULTS

Theorem 2.1. *If*

$$\frac{|\alpha| r^2}{2} + |w(0)| + |w'(0)| r < \frac{2}{3r}\sqrt{\frac{2-r^3}{6}}, \tag{2.1}$$

where $z = rx$, $0 < r < 2^{1/3}$, $|x| < 1$, then equation (1.2) has a unique analytic solution

$$w(z) = w(0) + w'(0)z + \sum_{n=2}^{\infty} c_n(w(0), w'(0)) z^n, \tag{2.2}$$

which together with its first two derivatives converge absolutely in $|z| < r$ and is bounded by:

$$|w(z)| < \sqrt{\frac{2-r^3}{6r^2}}. \tag{2.3}$$

Remark 2.1. Analogous results hold for the more general equation (than (1.2)):

$$w''(z) = \nu_1 [w(z)]^3 + \nu_2 z w(z) + \alpha, \nu_1 \neq 0. \tag{2.4}$$

In particular we have that if

$$\frac{|\alpha|r^2}{2} + |w(0)| + |w'(0)|r < \frac{2}{3r}\sqrt{\frac{2-|\nu_2|r^3}{3|\nu_1|}},$$

where $z = rx$, $0 < r < \left(\frac{2}{|\nu_2|}\right)^{1/3}$, $|x| < 1$, equation (2.4) has a unique analytic solution of the form (2.2) which together with its first two derivatives converge absolutely in $|z| < r$ and is bounded by:

$$|w(z)| < \sqrt{\frac{2-|\nu_2|r^3}{3|\nu_1|r^2}}.$$

Theorem 2.2. *If*

$$\begin{aligned} &2\,(|w(0)| + r|w'(0)|)\,(3\,|w(0)| + 11\,r|w'(0)|) \\ &\qquad + 4\,(|w(0)| + r|w'(0)|)\,(|b| + |a|\,(|w(0)| + r|w'(0)|)^2)\,r \\ &\qquad\qquad + (|d| + 4\,|c|\,(|w(0)| + r|w'(0)|)^4)\,r^2 < 4(1 - |b|r)P(R_0), \quad (2.5) \end{aligned}$$

where $z = rx$, $r > 0$, $|x| < 1$, $r|b| < 1$, then equation (1.3) has a unique analytic solution which together with its first two derivatives converge absolutely in $|z| < r$ and are bounded by:

$$|w(z)| < R_0 + |w(0)| + r|w'(0)|, |w'(z)| < \frac{R_0}{r} + |w'(0)|, |w''(z)| < \frac{R_0}{r^2}, \qquad (2.6)$$

where $R_0 > 0$ is the point at which the function $P(R) = R - A_2R^2 - A_3R^3 - A_4R^4$ attains its maximum, where

$$\begin{aligned} A_2 = &\\ &\frac{17 + 18\,|w(0)| + 26\,r|w'(0)| + 6\,|a|\,(|w(0)| + r|w'(0)|)\,(1 + |w(0)| + r|w'(0)|)\,r}{2\,(1 - |b|\,r)} \\ &\qquad + \frac{4\,|c|\,(|w(0)| + r|w'(0)|)^2\,(3 + 2(|w(0)| + r|w'(0)|))\,r^2}{2\,(1 - |b|\,r)}, \end{aligned}$$

$$A_3 = \frac{|a|\,r + 4\,|c|\,(|w(0)| + r|w'(0)|)\,r^2}{1 - |b|\,r}, \quad A_4 = \frac{|c|\,r^2}{1 - |b|r},$$

are such that $R_0 > 1$.

Theorem 2.3. *If*

$$\frac{1}{2}(2\,|w(0)|^2 + 7r\,|w(0)|\,|w'(0)| + 5r^2\,|w'(0)|^2$$

$$+\left(2\,b+(|w(0)|+r|w'(0)|)^2\,\left(4\,a+3\,(|w(0)|+r|w'(0)|)^2\right)\right)\,r^2 \tag{2.7}$$
$$+8\,(|w(0)|+r|w'(0)|)^3\,r^3+4\,(|w(0)|+r|w'(0)|)^2\,r^4)<P(R_0),$$

where $z=rx$, $r>0$, $|x|<1$, then equation (1.4) has a unique analytic solution which together with its first two derivatives converge absolutely in $|z|<r$ and are bounded by:

$$|w(z)|<R_0+|w(0)|+r|w'(0)|, |w'(z)|<\frac{R_0}{r}+|w'(0)|, |w''(z)|<\frac{R_0}{r^2}, \tag{2.8}$$

where $R_0>0$ is the point at which the function $P(R)=R-A_2R^2-A_3R^3-A_4R^4$ attains its maximum, where

$$A_2=5+5\,|w(0)|+\frac{13r\,|w'(0)|}{2}+\Big(3\,(|w(0)|+r|w'(0)|)^2\,\left(3+2(|w(0)|+r|w'(0)|)\right)$$
$$+a\,\left(2+4(|w(0)|+r|w'(0)|)\right)\big)\,r^2+12\,\left(|w(0)|+r|w'(0)|\right)\,\left(1+|w(0)|+r|w'(0)|\right)\,r^3$$
$$+2\,\left(1+2(|w(0)|+r|w'(0)|)\right)\,r^4,$$
$$A_3=6\,\left(|w(0)|+r|w'(0)|\right)\,r^2+4\,r^3,\quad A_4=\frac{3\,r^2}{2},$$

are such that $R_0>1$.

Theorem 2.4. *If*

$$(|w(0)|+r|w'(0)|)^4(3+|w(0)|+r|w'(0)|)|a|+|b|$$
$$+\frac{1}{6}(|w(0)|+r|w'(0)|)(8|w(0)|(3+2|w(0)|)+(51+95|w(0)|)r|w'(0)|$$
$$+100r^2|w'(0)|^2+18|b|+6(|w(0)|+r|w'(0)|)((|w(0)|+r|w'(0)|)|3a+b|+|a+3b|$$
$$+(1+|w(0)|+r|w'(0)|)r(|c|+r|d|)))<(1-3|b|)P(R_0), \tag{2.9}$$

where $z=rx$, $r>0$, $|x|<1$, $|b|<1/3$ then equation (1.5) has a unique analytic solution which together with its first two derivatives converge absolutely in $|z|<r$ and are bounded by:

$$|w(z)|<R_0+|w(0)|+r|w'(0)|, |w'(z)|<\frac{R_0}{r}+|w'(0)|, |w''(z)|<\frac{R_0}{r^2}, \tag{2.10}$$

where $R_0>0$ is the point at which the function $P(R)=R-A_2R^2-A_3R^3-A_4R^4-A_5R^5$ attains its maximum, where

$$A_2=\frac{1}{1-3|b|}(|a+3\,b|+\frac{1}{2}(22+44\,|w(0)|^2+r|w'(0)|\,(151+107\,r|w'(0)|)$$
$$+|w(0)|\,(107+144\,r|w'(0)|)+2\,r\,|c|)$$
$$+\left(|w(0)|+r|w'(0)|\right)\,\big((|w(0)|+r|w'(0)|)\,\big(18+5\,|w(0)|^2+2\,|w(0)|\,(11+5\,r|w'(0)|)$$

$$+r|w'(0)|\ (22+5\,r|w'(0)|))\ |a| + 3\,(1+|w(0)|+r|w'(0)|)\ |3\,a+b| + 2\,|a+3\,b|$$

$$+\,(5+3(|w(0)|+r|w'(0))|)\ r\,|c|) + \big(1+3|w(0)|^2$$

$$+r|w'(0)|(5+3r|w'(0)|) + |w(0)|\,(5+6r|w'(0)|)\,)\ r^2|d|,$$

$$A_3 = \frac{1}{1-3|b|}\big(\frac{121}{6} + 12\,|w(0)|\,|a| + 10\,|w(0)|^2\,|a| + 12\,r|w'(0)|\,|a| + 20\,|w(0)|\,r|w'(0)|\,|a|$$

$$+10\,r^2|w'(0)|^2\,|a| + |3\,a+b| + r\,|c| + r^2|d|\big),$$

$$A_4 = \frac{3\,|a| + 5\,|w(0)|\,|a| + 5\,r|w'(0)|\,|a|}{1-3\,|b|}, \quad A_5 = \frac{|a|}{1-3\,|b|},$$

are such that $R_0 > 1$.

Theorem 2.5. *If* $2|b|r^2(1+r) < 1$ *and*

$$(|w(0)|+r|w'(0)|)^5\ (2+|w(0)|+r|w'(0)|+2\,r)\ |a|$$

$$+\frac{1}{48}\,(3\,r^2\ (r+32\,|w(0)|\ (1+r)+32\,r|w'(0)|\ (1+r))\ |b|$$

$$+4\ (|w(0)|+r|w'(0)|)\ \big(3\,|w(0)|^3\ (8+r\ (33+4\,r\ (2+5\,r)))$$

$$+|w(0)|\ (6\,r\ (2+5\,r)\ (6+7\,r)+4\,r|w'(0)|\ (43+r\ (78+31\,r\ (3+4\,r)))$$

$$+9\,r^2|w'(0)|^2\ (56+r\ (153+20\,r\ (4+\ r)))\big)$$

$$+|w(0)|^2\ \big(9\,r|w'(0)|\ \big(26+r\ \big(81+38\,r+20\,r^2\big)\big)+4\ (8+r\ (21+r\ (24+35\,r)))\big)$$

$$+r|w'(0)|\ \big(6\,r\ (24+r\ (74+53\,r))+3\,r^2|w'(0)|^2\ (98+r\ (249+2\,r\ (67+10\,r)))$$

$$+8\,r|w'(0)|\ (22+r\ (33+r\ (39+49\,r))))$$

$$+12\ (|w(0)|+r|w'(0)|)\ \Big((|w(0)|+r|w'(0)|)^2\,|a-c|+|b+c|$$

$$+r\ \Big(|b-d|+2\ (|w(0)|+r|w'(0)|)\ |a+b-c-d|+(|w(0)|+r|w'(0)|)^2$$

$$\times|4\,a+b+c-d|+r\ \Big((|w(0)|+r|w'(0)|)^2\,|a+d|+|a+4\,b-c+d|$$

$$+2\ (|w(0)|+r|w'(0)|)\ |a+b+c+d|)))) < [1-2\ \big(r^2+r^3\big)\ |b|]P(R_0), \qquad (2.11)$$

where $z = rx$, $r > 0$, $|x| < 1$, *then equation (1.6) has a unique analytic solution which together with its first two derivatives converge absolutely in* $|z| < r$ *and are bounded by:*

$$|w(z)| < R_0 + |w(0)| + r|w'(0)|, |w'(z)| < \frac{R_0}{r} + |w'(0)|, |w''(z)| < \frac{R_0}{r^2}, \qquad (2.12)$$

where $R_0 > 0$ *is the point at which the function* $P(R) = R - A_2R^2 - A_3R^3 - A_4R^4 - A_5R^5 - A_6R^6$ *attains its maximum, where*

$$A_2 = \frac{1}{1-2\ (r^2+r^3)\,|b|}(|b+c| + \frac{1}{2}(2+27\,r+80\,r^2+56\,r^3+2\,(|w(0)|+r|w'(0)|)^3$$

$$\times(20+25\,|w(0)|+6\,|w(0)|^2+25\,r|w'(0)|+12r\,|w(0)|\,|w'(0)|+6r^2\,|w'(0)|^2$$
$$+10\,(2+|w(0)|+r|w'(0)|)\,r))\,|a|+2\,r\,|b-d|$$
$$+4\,(|w(0)|+r|w'(0)|)^2\,(3+2(|w(0)|+r|w'(0)|))\,|a-c|$$
$$+2\,|w(0)|^3\,(26+r\,(78+5\,r\,(7+4\,r))+4\,r\,(|4\,a+b+c-d|+r\,|a+d|))$$
$$+r^3|w'(0)|^3\,(205+r\,(516+r\,(277+40\,r))+8\,r\,(|4\,a+b+c-d|+r\,|a+d|))+2\,r^2\,|a+4\,b-c+d|$$
$$+|w(0)|^2\,(190+r\,(477+4\,r\,(72+43\,r))+3r\,|w'(0)|\,(91+r\,(252+r\,(127+40\,r)))$$
$$+12\,r\,(|a+b-c-d|+(1+2r\,|w'(0)|)\,(|4\,a+b+c-d|+r\,|a+d|)+r\,|a+b+c+d|))$$
$$+r|w'(0)|\,(100+r\,(195+r\,(322+317\,r))+4\,|b+c|$$
$$+4\,r\,(|b-d|+3\,|a+b-c-d|+r\,|a+4\,b-c+d|+3\,r\,|a+b+c+d|))$$
$$+r^2|w'(0)|^2\,(415+r\,(939+r\,(591+262\,r))+12\,r\,(|a+b-c-d|+|4\,a+b+c-d|$$
$$+r\,(|a+d|+|a+b+c+d|)))+|w(0)|\,(70+6\,r^2|w'(0)|^2\,(1+2\,r)\,(71+2\,r\,(22+5\,r))$$
$$+r\,(147+r\,(250+251\,r))+r|w'(0)|\,(581+r\,(1374+r\,(855+428\,r)))+4\,|b+c|$$
$$+4\,r\,(|b-d|+(3+6\,r|w'(0)|)\,|a+b-c-d|+6\,r|w'(0)|\,(1+r|w'(0)|)\,|4\,a+b+c-d|$$
$$+r\,(6\,r|w'(0)|\,(1+r|w'(0)|)\,|a+d|+|a+4\,b-c+d|+3\,(1+2\,r|w'(0)|)\,|a+b+c+d|)))),$$

$$A_3=\frac{1}{1-2\,(r^2+r^3)\,|b|}\Big(\frac{1}{6}(106+480\,|w(0)|+669\,r|w'(0)|$$
$$+6\,(25+204\,|w(0)|+276\,r|w'(0)|)\,r+3\,(58+214\,|w(0)|+295\,r|w'(0)|)\,r^2$$
$$+2\,(107+60(|w(0)|+r|w'(0)|))\,r^3)+20\,(|w(0)|+r|w'(0)|)^2\,(1+|w(0)|+r|w'(0)|$$
$$+r)\,|a|+4\,(|w(0)|+r|w'(0)|)\,|a-c|+2\,r\,(|a+b-c-d|$$
$$+2\,(|w(0)|+r|w'(0)|)\,(|4\,a+b+c-d|+r\,|a+d|)+r\,|a+b+c+d|)\,,$$

$$A_4=\frac{1}{1-2\,(r^2+r^3)\,|b|}(29+5\,r^3+5\,(|w(0)|+r|w'(0)|)\,(2+3(|w(0)|+r|w'(0)|)$$
$$+2\,r)\,|a|+|a-c|+r\,(\frac{285}{4}+|4\,a+b+c-d|)+r^2\,(38+|a+d|))\,,$$

$$A_5=\frac{6\,(|w(0)|+r|w'(0)|)\,|a|+2\,(1+r)\,|a|}{1-2\,(r^2+r^3)\,|b|},\quad A_6=\frac{|a|}{1-2\,(r^2+r^3)\,|b|}$$

are such that $R_0>1$.

Remark 2.2. Although conditions (2.5), (2.7), (2.9) and (2.11) seem very complicated, in practice, they are easily verified, since they **only involve the initial conditions, the parameters of the equations and the point R_0 where a specific function $P(R)$ attains its maximum**. The functions $P(R)$ which appear in Theorems 2.2-2.5 are easily calculated and since they are of at most sixth degree, the point at which they attain a maximum can be found easily (analytically or numerically).

3. THE FUNCTIONAL-ANALYTIC METHOD

Denote by H an abstract separable Hilbert space over the complex field, with the orthonormal base $\{e_n\}$, $n = 1, 2, 3, ...$ We use the symbols $(\cdot, \cdot)$ and $\| \cdot \|$ to denote the inner product and norm in H respectively. By H_1 we mean the Banach space consisting of those elements f in H which satisfy the condition $\sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. The norm in H_1 is denoted by $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. Finally by V we mean the shift operator in H, H_1:

$$V : Ve_n = e_{n+1}, n = 1, 2, 3, ...$$

and by V^* its adjoint

$$V^* : V^*e_n = e_{n-1}, n = 2, 3, ..., V^*e_1 = 0.$$

The following statements hold (Ifantis [8], Ifantis [9]):

(i) Every point z in the interior of the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, belongs to the point spectrum of V^* and the set of the proper elements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$, $f_0 = e_1$, forms a complete system in H in the sense that, if f is orthogonal to f_z for every $z : |z| < 1$, then $f = 0$.

(ii) The representation

$$\phi(z) = (f_z, f) = \sum_{n=1}^{\infty} \overline{(f, e_n)} z^{n-1}, |z| < 1 \tag{3.1}$$

is a one-to-one mapping from H_1 onto $H_1(\Delta)$ which preserves the norm. The element f of H_1 defined by (3.1) is called the *abstract form* of $\phi(z)$. In general the abstract form of a function $G(\phi(z)) : H_1(\Delta) \to H_1(\Delta)$ is a mapping $N(f) : H_1 \to H_1$ for which the following relation holds:

$$G(\phi(z)) = (f_z, N(f)), \quad |z| < 1.$$

(iii) H_1 is invariant under the operators V^m, $(V^*)^m$, $m \in \mathbb{N}$.

(iv) Taking into account the representation (3.1) it follows that:

$$z^k f(z) = (f_z, V^k f), k \in \mathbb{N}, |z| < 1, \tag{3.2}$$

$$\frac{d^k f(z)}{dz^k} = (f_z, (C_0 V^*)^k f), k \in \mathbb{N}, |z| < 1, \tag{3.3}$$

where $C_0 e_n = n e_n$, $n = 1, 2, ...$ is a diagonal operator which has a self-adjoint extension with discrete spectrum, i.e. the definition domain of C_0 can be extended to the range of the bounded operator $B_0 e_n = \frac{1}{n} e_n$, $n = 1, 2, ...$ and

$$[f(z)]^k = (f_z, [f_1(V)]^{k-1} f), k \in \mathbb{N}, |z| < 1, \tag{3.4}$$

where $f_1(V)=\overline{(f,e_1)}I+\overline{(f,e_2)}V+\overline{(f,e_3)}V^2+...$ and $\|f_1(V)\|_1=\|f\|_1$.
(v) The following relations hold:

$$(C_0V-VC_0)f=Vf,$$

$$(C_0-I)f=VC_0V^*f,$$

$$(C_0V^*)^kf=C_0(C_0+I)...(C_0+(k-1)I)(V^*)^kf.$$

We can also prove that:

$$[f_1(V)C_0-C_0f_1(V)]f=-Vf_1'(V),$$

$$[f_1'(V)C_0-C_0f_1'(V)]f=-Vf_1''(V),$$

where

$$f_1'(V)=\overline{(f,e_2)}I+2\overline{(f,e_3)}V+3\overline{(f,e_4)}V^2+...,$$

$$f_1''(V)=2\overline{(f,e_3)}I+6\overline{(f,e_4)}V+12\overline{(f,e_5)}V^2+...$$

and

$$\|f_1'(V)\|_1=\|f'\|_1,\quad \|f_1''(V)\|_1=\|f''\|_1,$$

where

$$f'=\sum_{n=2}^{\infty}(n-1)\overline{(f,e_n)}e_n\in H_1, \tag{3.5}$$

$$f''=\sum_{n=3}^{\infty}(n-1)(n-2)\overline{(f,e_n)}e_n\in H_1. \tag{3.6}$$

For the proof of Theorems 2.3-2.5 we shall need the following:

Lemma 3.1. *The following relations hold:*
1) $f(z)f'(z)=\frac{1}{2}(f_z,C_0V^*f_1(V)f)$,
2) $[f'(z)]^2=(f_z,f_1'(V)C_0V^*f)$,
3) $f(z)f''(z)=(f_z,\frac{1}{2}C_0V^*C_0V^*f_1(V)f-f_1'(V)C_0V^*f)$,
4) $[f(z)]^2f'(z)=\frac{1}{3}(f_z,C_0V^*[f_1(V)]^2f)$,
5) $f(z)[f'(z)]^2=(f_z,f_1(V)f_1'(V)C_0V^*f)$,
6) $[f(z)]^2f''(z)=(f_z,\frac{1}{3}C_0V^*C_0V^*[f_1(V)]^2f-2f_1(V)f_1'(V)C_0V^*f))$,
7) $[f(z)]^3f'(z)=\frac{1}{4}(f_z,C_0V^*[f_1(V)]^3f)$,
8) $[f(z)]^2[f'(z)]^2=(f_z,[f_1(V)]^2f_1'(V)C_0V^*f)$,
9) $[f(z)]^3f''(z)=(f_z,\frac{1}{4}C_0V^*C_0V^*[f_1(V)]^3f-3[f_1(V)]^2f_1'(V)C_0V^*f)$.

Proof. 1) From (3.4) we have $[f(z)]^2=(f_z,f_1(V)f)$. Since

$$\frac{d[f(z)]^2}{dz}=2f(z)\frac{df(z)}{dz}$$

it follows that

$$f(z)f'(z) = \frac{1}{2}\frac{d[f(z)]^2}{dz} = \frac{1}{2}(f_z, C_0V^*f_1(V)f),$$

due to (3.3).

2) We have

$$(f_z, f_1'(V)C_0V^*f) = \Big(f_z, [\overline{(f,e_2)}I + 2\overline{(f,e_3)}V + 3\overline{(f,e_4)}V^2 + ...]C_0V^*f\Big)$$

$$= \big([(f,e_2)I + 2(f,e_3)V^* + 3(f,e_4)(V^*)^2 + ...]f_z, C_0V^*f\big)$$

$$= (f,e_2)(f_z, C_0V^*f) + 2(f,e_3)(V^*f_z, C_0V^*f) + 3(f,e_4)((V^*)^2f_z, C_0V^*f) + ...$$

$$= (f,e_2)(f_z, C_0V^*f) + 2(f,e_3)z(f_z, C_0V^*f) + 3(f,e_4)z^2(f_z, C_0V^*f) + ...$$

$$= \big[(f,e_2) + 2(f,e_3)z + 3(f,e_4)z^2 + ...\big]\,(f_z, C_0V^*f) = f'(z)f'(z) = [f'(z)]^2.$$

3) It follows from 1) and 2) if we use (3.3) and the fact that

$$[f(z)f'(z)]' = [f'(z)]^2 + f(z)f''(z).$$

4) From (3.4) we have $[f(z)]^3 = (f_z, [f_1(V)]^2f)$. Since

$$\frac{d[f(z)]^3}{dz} = 3[f(z)]^2\frac{df(z)}{dz},$$

it follows that

$$[f(z)]^2f'(z) = \frac{1}{3}\frac{d[f(z)]^3}{dz} = \frac{1}{3}(f_z, C_0V^*[f_1(V)]^2f),$$

due to (3.3).

5) We have

$$(f_z, f_1(V)f_1'(V)C_0V^*f) = \Big(f_z, [\overline{(f,e_1)}I + \overline{(f,e_2)}V + \overline{(f,e_3)}V^2 + ...]f_1'(V)C_0V^*f\Big)$$

$$= \big([(f,e_1)I + (f,e_2)V^* + (f,e_3)(V^*)^2 + ...]f_z, f_1'(V)C_0V^*f\big)$$

$$= (f,e_1)(f_z, f_1'(V)C_0V^*f) + (f,e_2)(V^*f_z, f_1'(V)C_0V^*f)$$

$$+(f,e_3)((V^*)^2f_z, f_1'(V)C_0V^*f) + ... = (f,e_1)(f_z, f_1'(V)C_0V^*f)$$

$$+(f,e_2)z(f_z, f_1'(V)C_0V^*f) + (f,e_3)z^2(f_z, f_1'(V)C_0V^*f) + ...$$

$$= \big[(f,e_1) + (f,e_2)z + (f,e_3)z^2 + ...\big]\,(f_z, f_1'(V)C_0V^*f)$$

$$= f(z)(f_z, f_1'(V)C_0V^*f) = f(z)[f'(z)]^2,$$

due to 2).

6) It follows from 4) and 5) if we use (3.3) and the fact that

$$[(f(z))^2f'(z)]' = 2f(z)[f'(z)]^2 + [f(z)]^2f''(z).$$

7) From (3.4) we have $[f(z)]^4 = (f_z, [f_1(V)]^3 f)$. Since

$$\frac{d[f(z)]^4}{dz} = 4[f(z)]^3 \frac{df(z)}{dz}$$

it follows that

$$[f(z)]^3 f'(z) = \frac{1}{4}\frac{d[f(z)]^3}{dz} = (f_z, C_0 V^*[f_1(V)]^3 f),$$

due to (3.3).
8) We have

$$(f_z, [f_1(V)]^2 f_1'(V) C_0 V^* f) = (f_z, f_1(V) f_1(V) f_1'(V) C_0 V^* f)$$

$$= \Big(f_z, [\overline{(f,e_1)}I + \overline{(f,e_2)}V + \overline{(f,e_3)}V^2 + ...] f_1(V) f_1'(V) C_0 V^* f\Big)$$

$$= \big([(f,e_1)I + (f,e_2)V^* + (f,e_3)(V^*)^2 + ...] f_z, f_1(V) f_1'(V) C_0 V^* f\big)$$

$$= (f,e_1)(f_z, f_1(V) f_1'(V) C_0 V^* f) + (f,e_2)(V^* f_z, f_1(V) f_1'(V) C_0 V^* f)$$

$$+ (f,e_3)((V^*)^2 f_z, f_1(V) f_1'(V) C_0 V^* f) + ... = (f,e_1)(f_z, f_1(V) f_1'(V) C_0 V^* f)$$

$$+ (f,e_2) z (f_z, f_1(V) f_1'(V) C_0 V^* f) + (f,e_3) z^2 (f_z, f_1(V) f_1'(V) C_0 V^* f) + ...$$

$$= \big[(f,e_1) + (f,e_2)z + (f,e_3)z^2 + ...\big] (f_z, f_1(V) f_1'(V) C_0 V^* f)$$

$$= f(z)(f_z, f_1(V) f_1'(V) C_0 V^* f) = f(z) f(z) [f'(z)]^2 = [f(z)]^2 [f'(z)]^2,$$

due to 5).
9) It follows from 7) and 8) if we use (3.3) and the fact that

$$[(f(z))^3 f'(z)]' = 3[f(z)]^2 [f'(z)]^2 + [f(z)]^3 f''(z).$$

For the proof of Theorems 2.1-2.5 we also need the following fixed point theorem of Earle and Hamilton [3]:

If $F : X \to X$ is a holomorphic map and $F(X)$ lies strictly inside X, then F has a unique fixed point in X, where X is a bounded, connected and open subset of a Banach space B.

By saying holomorphic map, we mean that its Fréchet derivative exists and by saying that a subset X' of X lies strictly inside X we mean that there exists an $\epsilon > 0$ such that $\|x' - y\| > \epsilon$ for all $x' \in X'$ and $y \in B - X$.

4. PROOFS

Proof of Theorem 2.1. First of all we set $z = rx$, $0 < r < 2^{1/3}$, $|x| < 1$, $w(z) = w(rx) = f(x)$ and (1.2) becomes:

$$f''(x) = 2r^2[f(x)]^3 + r^3 x f(x) + \alpha r^2. \tag{4.1}$$

According to the representation presented in Section 3, the abstract form of (4.1) in H_1 is:

$$C_0(C_0 + I)(V^*)^2 f = 2r^2[f_1(V)]^2 f + r^3 V f + \overline{\alpha} r^2 e_1,$$

which together with the initial conditions:

$$(f, e_1) = f(0) = w(0) = \lambda_1$$

$$(f, e_2) = \left.\frac{df(x)}{dx}\right|_{x=0} = r \left.\frac{dw(z)}{dz}\right|_{z=0} = r\lambda_2$$

is equivalent to the following:

$$(I - r^3 V^2 B_1 V) f = \overline{\lambda_1} e_1 + r\overline{\lambda_2} e_2 + \frac{\overline{\alpha} r^2}{2} e_3 + 2r^2 V^2 B_1 [f_1(V)]^2 f, \tag{4.2}$$

where $B_1 e_n = \dfrac{1}{n(n+1)} e_n$. Since $0 < r < 2^{1/3}$, it follows that the operator $I - r^3 V^2 B_1 V$ has a bounded inverse, defined on all H_1 with bound given by:

$$\| \left(I - r^3 V^2 B_1 V\right)^{-1} \|_1 \leq \frac{2}{2 - r^3}.$$

Thus equation (4.2) becomes:

$$f = \left(I - r^3 V^2 B_1 V\right)^{-1} \left[\overline{\lambda_1} e_1 + r\overline{\lambda_2} e_2 + \frac{\overline{\alpha} r^2}{2} e_3 + 2r^2 V^2 B_1 [f_1(V)]^2 f\right] = \phi(f). \tag{4.3}$$

From equation (4.3) it follows that for $\|f\|_1 \leq R < +\infty$, R as large as we want but finite:

$$\|\phi(f)\|_1 \leq \frac{2}{2 - r^3} \left[|\lambda_1| + r|\lambda_2| + \frac{|\alpha| r^2}{2} + r^2 R^3\right].$$

Let $P(R) = R - \dfrac{2r^2}{2 - r^3} R^3$. This function has a maximum at $R_0 = \sqrt{\dfrac{2 - r^3}{6r^2}}$, which is $P(R_0) = P_0 = \dfrac{2 - r^3}{3r} \sqrt{\dfrac{2 - r^3}{6}}$. Thus for $R = R_0$ and

$$\frac{|\alpha| r^2}{2} + |w(0)| + |w'(0)| r < \frac{2}{3r} \sqrt{\frac{2 - r^3}{6}} \tag{4.4}$$

we find that $\|\phi(f)\|_1 < R_0$. Also it can be proved that $\phi(f)$ is Frechét differentiable. Thus from the fixed point theorem of Earle and Hamilton [3] it follows that if condition

(4.4) is satisfied, equation (4.3) has a unique solution in H_1. Moreover, since $\|f\|_1 < R_0$ it follows that

$$|w(z)| < \sqrt{\frac{2-r^3}{6r^2}}.$$

Proof of Theorem 2.2. First of all we set $z = rx$, $|x| < 1$, $r > 0$, $w(z) = w(rx) = f(x)$ and (1.3) becomes:

$$f''(x) = \frac{1}{f(x)}[f'(x)]^2 - \frac{1}{x}f'(x) + \frac{r}{x}[\alpha(f(x))^2 + b] + r^2c[f(x)]^3 + \frac{dr^2}{f(x)}.$$

According to the representation given in Section 3, the abstract form of the above equation in H_1 is:

$$\frac{1}{2}VC_0V^*C_0V^*f_1(V)f - Vf_1'(V)C_0V^*f = Vf_1'(V)C_0V^*f - \frac{1}{2}C_0V^*f_1(V)f$$
$$+r^2\bar{c}V[f_1(V)]^3f + r\overline{\alpha}[f_1(V)]^2f + r\bar{b}f + r^2\overline{d}e_2\,,$$

or

$$(I+r\bar{b}B_0^2)f = -\frac{r^2\overline{d}}{4}e_2 + f + \frac{1}{2}(I-B_0)V^*f_1(V)f + 2B_0^2V^2f_1''(V)V^*f - r\overline{\alpha}B_0^2[f_1(V)]^2f$$
$$-2B_0Vf_1'(V)V^*f + 2B_0^2Vf_1'(V)V^*f + \frac{1}{2}B_0V^*f_1(V)f - r^2\bar{c}B_0^2V[f_1(V)]^3f, \qquad (4.5)$$

where $B_0e_n = \frac{1}{n}e_n$, $n = 1, 2, 3, ...$

Since $r|b| < 1$, it follows that the operator $I + r\bar{b}B_0^2$ has a bounded inverse, defined on all H_1 with bound given by:

$$\|\left(I + r\bar{b}B_0^2\right)^{-1}\|_1 \leq \frac{1}{1-r|b|}.$$

Thus equation (4.5) becomes:

$$f = \left(I + r\bar{b}B_0^2\right)^{-1}[-\frac{r^2\overline{d}}{4}e_2 + f + \frac{1}{2}(I-B_0)V^*f_1(V)f + 2B_0^2V^2f_1''(V)V^*f - r\overline{\alpha}B_0^2[f_1(V)]^2f$$
$$-2B_0Vf_1'(V)V^*f + 2B_0^2Vf_1'(V)V^*f + \frac{1}{2}B_0V^*f_1(V)f - r^2\bar{c}B_0^2V[f_1(V)]^3f] = \phi(f). \qquad (4.6)$$

Since

$$|\alpha_n| \leq (n-1)|\alpha_n| \leq (n-1)(n-2)|\alpha_n|, \quad \forall n \geq 3\,,$$

it follows that

$$\|f\|_1 - |f(0)| - |f'(0)| \leq \|f'\|_1 - |f'(0)| \leq \|f''\|_1,$$

where f' and f'' are defined by (3.5) and (3.6) respectively. Then for $\|f''\|_1 \leq R < +\infty$, R as large as we want but finite, we obtain $\|f'\|_1 \leq R + M$, $\|f\|_1 \leq R + L + M$, where $L = |f(0)|$, $M = |f'(0)|$ and from (4.6):

$$\|\phi(f)\|_1 \leq L + M + \frac{1}{1-r|b|}[\frac{|d|\,r^2}{4} + r|b|\ (L+M)\ + R + 2\,R\ (R+L+M)$$

$$+4\,(R+M)\,(R+L+M)+\frac{3\,(R+L+M)^2}{2}+|a|\,r\,(R+L+M)^3+|c|\,r^2\,(R+L+M)^4]$$

$$\Rightarrow \|\phi(f)\|_1 \leq L+M+A_2R^2+A_3R^3+A_4R^4$$

$$+\frac{2\,(L+M)\,(3\,L+11\,M)+4\,(L+M)\,(|b|+|a|\,(L+M)^2)\,r+(|d|+4\,|c|\,(L+M)^4)\,r^2}{4(1-\,|b|r)},$$

since R is sufficiently large. Let

$$P_1(R) = A_2 + A_3R + A_4R^2\,.$$

Since R is sufficiently large, there exists a $R_1 \in (0,R)$ such that

$$R_1P_1(R_1) > 1.$$

Thus the function

$$P_2(R) = 1 - RP_1(R)$$

has a zero $R_2 \in (0,R_1)$, since $P_2(0) = 1 > 0$ and $P_2(R_1) < 0$. Consequently the function

$$P(R) = RP_2(R)$$

has a maximum $P(R_0) = P_0$ at a point $R_0 \in (0,R_2)$, since $P(0) = P(R_2) = 0$ and $P'(0) > 0$, $P'(R_2) < 0$. Thus $\forall\ \epsilon > 0$, $R = R_0$, $\|f\|_1 < R_0 + L + M$ and

$$2\,(L+M)\,(3\,L+11\,M)+4\,(L+M)\,(|b|+|a|\,(L+M)^2)\,r$$

$$+(|d|+4\,|c|\,(L+M)^4)\,r^2 < P_0 4(1-\,|b|r) \tag{4.7}$$

we find that $\|\phi(f)\|_1 \leq R_0+L+M-\epsilon < R_0+L+M$. Also it can be proved that $\phi(f)$ is Frechét differentiable. Thus from the fixed point theorem of Earle and Hamilton [3] it follows that if condition (4.7) is satisfied, equation (4.6) has a unique solution in H_1. Moreover, since $\|f\|_1 < R_0 + L + M$, $\|f'\|_1 < R_0 + M$, $\|f''\|_1 < R_0$ it follows that

$$|f(z)| < R_0 + L + M,\ \ |f'(z)| < R_0 + M,\ \ |f''(z)| < R_0r^2.$$

The proof for the case where $b = 0$ is similar and thus we omit it. We only mention that in this case, the abstract form of equation (1.4) after setting $z = rx$, $r > 0$, $|x| < 1$ and $w(z) = w(rx) = f(x)$ is:

$$f = \phi(f) = [-\frac{r^2\overline{d}}{4}e_2 + f + \frac{1}{2}(I-B_0)V^*f_1(V)f + 2B_0^2V^2f_1''(V)V^*f - r\overline{a}B_0^2[f_1(V)]^2f$$

$$-2B_0Vf_1'(V)V^*f + 2B_0^2Vf_1'(V)V^*f + \frac{1}{2}B_0V^*f_1(V)f - r^2\overline{c}B_0^2V[f_1(V)]^3f].$$

Therefore Theorem 2.2 is proved.

The proofs of Theorems 2.3-2.5 are similar to the proof of Theorem 2.2 and thus we shall only write down the abstract forms of equations (1.4)-(1.5), after setting $z = rx$, $r > 0$, $|x| < 1$ and $w(z) = w(rx) = f(x)$.

Proof of Theorem 2.3. The abstract form of the resulting equation of (1.4) in H_1 is

$$f = f - (V^*)^2 f_1(V)f + 3B_1 f_1'(V)V^* - 3BVf_1''(V)V^*f + 3r^2B[f_1(V)]^3 f$$
$$+8r^3BV[f_1(V)]^2 f + 4r^4BV^2 f_1(V)f - 4r^2\overline{\alpha}Bf_1(V)f + \overline{b}r^2 e_1,$$

where $B_1 e_n = \dfrac{1}{n+1}e_n$, $Be_n = \dfrac{1}{n(n+1)}e_n$, $n = 1, 2, \ldots$. The rest of the proof is omitted.

Proof of Theorem 2.4. The abstract form of the resulting equation of (1.5) in H_1 is

$$f = (I - 3\overline{b}B_0)^{-1}[-\overline{b}e_1 + f - \frac{1}{3}(I - 3B_0 + 2B_0^2)V^2(V^*)^2[f_1(V)]^2 f$$
$$+\frac{7}{2}(B_0 - 2B_0^2)V^2 f_1(V)f_1'(V)V^*f - \frac{7}{2}B_0^2V^3[f_1'(V)]^2V^*f$$
$$-\frac{7}{2}B_0^2 f_1(V)Vf_1''(V)V^*f + \frac{1}{2}(I - 3B_0 + 2B_0^2)V^2(V^*)^2 f_1(V)f$$
$$-\frac{3}{2}(B_0 - 2B_0^2)V^2 f_1'(V)V^*f + \frac{3}{2}B_0^2V^3 f_1''(V)V^*f - \frac{1}{3}(B_0 - B_0^2)[f_1(V)]^2 f$$
$$+\frac{1}{2}(B_0 - B_0^2)f_1(V)f + \overline{\alpha}B_0^2[f_1(V)]^4 f + (\overline{b} + 3\overline{a})B_0^2[f_1(V)]^2 f - 3\overline{\alpha}B_0^2[f_1(V)]^3 f$$
$$-(3\overline{b} + \overline{a})B_0^2 f_1(V)f + r\overline{c}B_0^2[f_1(V)]^2 f - r\overline{c}B_0^2Vf_1(V)f$$
$$+r^2\overline{d}B_0^2V^2[f_1(V)]^2 f + r^2\overline{d}B_0^2V^2 f_1(V)f],$$

where $b \neq 0$ and $B_0 e_n = \dfrac{1}{n}e_n$, $n = 1, 2, \ldots$.

For the case $b = 0$ the corresponding abstract form is

$$\phi(f) = f - \frac{1}{3}(I - 3B_0 + 2B_0^2)V^2(V^*)^2[f_1(V)]^2 f$$
$$+\frac{7}{2}(B_0 - 2B_0^2)V^2 f_1(V)f_1'(V)V^*f - \frac{7}{2}B_0^2V^3[f_1'(V)]^2V^*f$$
$$-\frac{7}{2}B_0^2 f_1(V)Vf_1''(V)V^*f + \frac{1}{2}(I - 3B_0 + 2B_0^2)V^2(V^*)^2 f_1(V)f$$
$$-\frac{3}{2}(B_0 - 2B_0^2)V^2 f_1'(V)V^*f + \frac{3}{2}B_0^2V^3 f_1''(V)V^*f - \frac{1}{3}(B_0 - B_0^2)[f_1(V)]^2 f$$
$$+\frac{1}{2}(B_0 - B_0^2)f_1(V)f + \overline{\alpha}B_0^2[f_1(V)]^4 f + 3\overline{a}B_0^2[f_1(V)]^2 f - 3\overline{\alpha}B_0^2[f_1(V)]^3 f$$
$$+\overline{a}B_0^2 f_1(V)f + r\overline{c}B_0^2[f_1(V)]^2 f - r\overline{c}B_0^2Vf_1(V)f$$
$$+r^2\overline{d}B_0^2V^2[f_1(V)]^2 f + r^2\overline{d}B_0^2V^2 f_1(V)f,$$

where $B_0 e_n = \dfrac{1}{n}e_n$. The rest of the proof is omitted.

Proof of Theorem 2.5. The proof of Theorem 2.5 is similar to the proof of Theorem 2.2 and since the corresponding abstract form is quite extended to write, we omit it.

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