QUASILINEAR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT: In this paper, using Aleksandrov-Pucci maximum principle, we prove an $L^\infty$ a priori estimate and also uniqueness for weak solution $u$ of a Dirichlet problem associated to quasilinear strictly elliptic equations with Charatheodory’s coefficients. The results obtained are first step in the study of weak solvability of boundary value problems for quasilinear elliptic equations.

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1. INTRODUCTION AND ASSUMPTIONS

The main goal of the present paper is to establish an a priori bound of $\|u\|_{L^\infty(\Omega)}$ for the weak solutions $u \in W^{1,q}(\Omega)$, $q > n$, of the equation

$$\mathcal{M}u = 0 \text{ almost everywhere in } \Omega,$$

where

$$\mathcal{M} \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x,u) \frac{\partial}{\partial x_j} u) + b(x,u).$$

(1.1)

The coefficients $a_{ij}(x,z), b(x,z) \in \Omega \times \mathbb{R}$ are Charatheodory functions and also $a_{ij}$ are uniformly elliptic and such that is true the local uniform continuity with respect to the second variable $u$, uniformly in $x$.

On the coefficients $b(x,u)$ we point out the characteristic hypotheses that it is less or equal to a function of the Lebesgue space $L^q(\Omega)$, $q > n$.

The second result is uniqueness of the solution of the Dirichlet problem associated to the above equation, then we have extended the uniqueness result obtained in Ragusa [8] from the linear case to the quasilinear one.
Divergence form quasilinear elliptic equations have been studied by Ladyzenskaya and Ural’ceva [5] where they consider the terms $a_{ij}$ to be sufficiently smooth. Also in the book of Gilbarg and Trudinger [3], the authors have supposed regularity assumptions on the coefficients.

We wish to mention the study made in Simon [10] where he supposes that the terms having first partial derivatives are local Lipschitz continuous and also the recent paper of Li and Vogelius [6] where are considered Hölder continuous coefficients for a divergence form elliptic equation to obtain $L^\infty$ estimates for the gradient of the solution.

We point out that our study generalizes those results because in this note the coefficients $a_{ij}$ are discontinuous, precisely belong to the class $VMO$, and it is possible to prove that $C^0 \subset VMO$.

The class $VMO$, first considered by Sarason of [9], has been used by many authors, we want to recall the two papers Chiarenza et al [1] and [2], where the linear elliptic equations in nondivergence form have been very well studied, and the study made by Palagachev [7] for nondivergence form equations.

The vanishing mean oscillation functions made a subset of the set of bounded mean oscillation functions, known as BMO class and defined by John and Nirenberg [4].

Let us now establish the assumptions we will need in the following.

Let us consider $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n > 2$, $a_{ij}(x,z)$ and $b(x,z)$ be Charatheodory functions or equivalently measurable on $x \in \Omega$, $\forall z \in \mathbb{R}$, and continuous respect to $z$ for all almost $x \in \Omega$. Moreover we assume that:

$\exists \lambda$ positive constant such that

$$a_{ij}(x,z)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ a. a. } x \in \Omega, \forall z \in \mathbb{R}, \tag{1.2}$$

and

$$a_{ij}(x,z) = a_{ji}(x,z) \text{ a. a. } x \in \Omega, \forall z \in \mathbb{R}.$$ 

Let $a_{ij}$ be local uniform continuous in $z$, uniformly in $x$:

$$|a_{ij}(x,z) - a_{ij}(x,z')| \leq a(x)\mu_M(|z - z'|) \text{ a.e. } \Omega, \forall z, z' \in [-M,M], \tag{1.3}$$

where $a(x) \in L^\infty(\Omega)$, $\mu_M(t)$ is a non-decreasing function, and $\lim_{t \to 0} \mu_M(t) = 0$, $a_{ij}(x,0) \in L^\infty(\Omega)$.

We set that $a_{ij}(x,z) \in VMO$ with respect to $x$ and are loc. unif. in $z$:

$$\sup_{\rho \leq r} \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} |a_{ij}(x,z) - \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} a_{ij}(y,z)dy| \, dx = \eta_M(r) \tag{1.4}$$
for all \( z \in [-M, M] \) and satisfy the following condition

\[
\lim_{r \to 0} \eta_M(r) = 0,
\]

where \( \Omega_\rho = \Omega \cap B_\rho \) and \( B_\rho \) is in class of the balls with radius \( \rho \) centered at the points of \( \Omega \).

As it concerns \( b(x, z) \) we suppose that:

\[
|b(x, z)| \leq b_1(x) \quad \text{a.e. } \Omega \quad \forall z \in \mathbb{R},
\]

where \( b_1(x) \in L^q(\Omega), q > n \).

### 2. A PRIORI ESTIMATES

**Theorem 2.1.** Let \( u \in W^{1,q}(\Omega), q > n \) be a solution of the quasilinear equation (1.1) and (1.2) and (1.6) be true. Then there exists a constant \( C \) dependent on \( n \) and \( \text{diam } \Omega \) such that

\[
\|u\|_{L^\infty(\Omega)} \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}).
\]

**Proof.** Let us set \( \Omega^+ = \{x \in \Omega : u(x) > 0\} \). Then using Lemma 10.8 in [1] and taking into account that \( u \in W^{1,q}(\Omega) \) we have

\[
\sup_{\Omega^+} u \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}) \quad (u^+ = \max(u(x), 0)).
\]

If we consider the function \( v = -u \) we have

\[
\sup_{\Omega} v \leq C(n, |\Omega|) \cdot (\|v^+\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}),
\]

and then

\[
-\inf_{\Omega} u \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})
\]

\[
\leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})
\]

and

\[
 u(x) \geq \inf_{\Omega} u \geq -C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}).
\]

The last inequality combined with (2.2) gives

\[
\sup_{\Omega} |u| \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}).
\]
3. UNIQUENESS RESULT

Theorem 3.1. Let $a_{ij}$ be bounded and measurable functions independent of $z$, let (1.2) condition be true, $b(x,z)$ non increasing in $z$, a.e. in $\Omega$.

If $u, v \in W^{1,q}(\Omega)$, $q > n$, are solutions of the following Dirichlet problem

$$
\begin{cases}
M\tau = 0 & \text{a.e. in } \Omega \\
\tau = 0 & \text{on } \partial \Omega
\end{cases}
$$

then $u = v$ in $\Omega$.

Proof. The difference $w = u - v \in W^{1,q}(\Omega)$ solves

$$
D_i(a_{ij}(x)D_jw) + b(x,u) - b(x,v) = 0 \text{ a.e. in } \Omega.
$$

Let us set $\Omega^+ = \{x \in \Omega : w(x) > 0\}$, we have $u(x) > v(x)$ a.e. in $\Omega^+$, and from the hypothesis $b(x,u(x)) \leq b(x,v(x))$ a.e. in $\Omega^+$, then

$$
b(x,u(x)) - b(x,v(x)) \leq 0.
$$

Thus (3.2) conducts to the following inequality

$$
D_i(a_{ij}(x)D_jw) \geq 0 \text{ almost everywhere in } \Omega^+.
$$

Applying the maximum principle for divergence form operators

$$
w(x) \leq \sup_{\Omega^+} w \leq \sup_{\partial \Omega^+} w^+ = 0 \quad (w = 0 \text{ on } \partial \Omega),
$$

then $w \leq 0$ in $\overline{\Omega}$.

If we substitute $w$ by $-w$ we have $w \geq 0$, that is equivalent to $u = v$ in $\Omega$.

REFERENCES


