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# OSCILLATION OF THE SOLUTIONS OF SECOND ORDER RETARDED IMPULSIVE DIFFERENTIAL EQUATIONS

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**ABSTRACT:** The second order retarded impulsive differential equation of the type

$$\begin{aligned} &(r(t)x'(t))' + F(t, x(\tau_1(t)), \cdots, x(\tau_m(t))) = 0, \qquad t \neq t_k, \\ &\Delta(r(t_k)x'(t_k)) + F_k(x(\tau_1(t_k)), \cdots, x(\tau_m(t_k))) = 0 \end{aligned} \tag{E}$$

is considered in the case, when $\displaystyle\int^{\infty} \frac{dt}{r(t)} = +\infty$.

Various oscillation criteria for equation (E) are obtained, which extend some oscillation results for equations without impulses.

**AMS (MOS) Subject Classification.** 34A37

## 1. INTRODUCTION

In this paper the oscillatory behaviour of the solutions of second order impulsive retarded differential equation of the type

$$\begin{aligned} &(r(t)x'(t))' + F(t, x(\tau_1(t)), \cdots, x(\tau_m(t))) = 0, \qquad t \neq t_k, \\ &\Delta(r(t_k)x'(t_k)) + F_k(x(\tau_1(t_k)), \cdots, x(\tau_m(t_k))) = 0 \end{aligned} \tag{1}$$

is studied in the case, when $\displaystyle\int^{\infty} \frac{dt}{r(t)} = +\infty$.

In Theorems 1, 2 and 3 the oscillatory properties of the solutions of equation (1) are compared with the oscillatory properties of the inequality

$$\begin{aligned} &[(r(t)x'(t))' + F(t, x(\tau_1(t)), \cdots, x(\tau_m(t)))] \operatorname{sgn} x(t) \leq 0, \quad t \neq t_k, \\ &[\Delta(r(t_k)x'(t_k)) + F_k(x(\tau_1(t_k)), \cdots, x(\tau_m(t_k)))] \operatorname{sgn} x(t_k) \leq 0 \end{aligned} \tag{2}$$

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and of the comparison equation

$$\begin{aligned}&(q(t)y'(t))' + G(t, y(\sigma_1(t)), \cdots, y(\sigma_m(t))) = 0, \qquad t \neq t_k,\\ &\Delta(q(t_k)y'(t_k)) + G_k(y(\sigma_1(t_k)), \cdots, y(\sigma_m(t_k))) = 0.\end{aligned} \tag{3}$$

Some oscillation criteria for the solutions of equation (1) are found in Theorems 4 and 5. These results are specified in Theorem 6 for equations of the type

$$\begin{aligned}&(r(t)x'(t))' + p(t)x(\tau(t)) = 0, \qquad t \neq t_k,\\ &\Delta(r(t_k)x'(t_k)) + p_k x(\tau(t_k)) = 0\end{aligned} \tag{4}$$

and

$$\begin{aligned}&(r(t)x'(t))' + p(t)f(x(\tau(t))) = 0, \qquad t \neq t_k,\\ &\Delta(r(t_k)x'(t_k)) + p_k f_0(x(\tau(t_k))) = 0.\end{aligned} \tag{5}$$

Other oscillation criteria for equations (4) and (5) are obtained in Theorems 7–11, while Theorem 12 gives similar oscillation conditions for the equation

$$\begin{aligned}&(r(t)x'(t))' + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \qquad t \neq t_k,\\ &\Delta(r(t_k)x'(t_k)) + \sum_{i=1}^{m} p_{ik}x(\tau_i(t_k)) = 0.\end{aligned} \tag{6}$$

We denote that the oscillatory and asymptotic behaviour of second order impulsive equations of type (1) have been investigated in Bainov and Simeonov [1] and Bainov et al [2].

The results obtained in this paper cover some oscillation theorems from Hamedani [3] and Philos [4]–[6], where equations without impulses are considered.

## 2. PRELIMINARY NOTES

Let $r \in \mathbf{N}$, the interval $D \subseteq \mathbf{R}$ and the sequence $\{t_k\}$ be fixed. We denote by $PC(D, \mathbf{R})$ the set of all functions $\psi : D \to \mathbf{R}$ which are continuous for $t \in D$, $t \neq t_k$, they are continuous from the left for $t \in D$ and have discontinuity of the first kind at the points $t_k \in D$. By $PC^r(D, R)$ we denote the set of functions $\psi : D \to \mathbf{R}$ having derivative $\dfrac{d^r\psi}{dt^r} \in PC(D, \mathbf{R})$.

Introduce the following conditions:

**H1.** The sequence $\{t_k\}_1^\infty$ is such that: $t_0 \equiv 0 < t_1 < t_2 < \cdots$ and $\lim\limits_{k\to+\infty} t_k = +\infty$.

**H2.** $r \in PC^1(\mathbf{R}_+, \mathbf{R}_+)$, $\displaystyle\int^\infty \frac{dt}{r(t)} = +\infty$
and $r(t) > 0$, $r(t_k^+) > 0$ for $t, t_k \in \mathbf{R}_+$.

**H3.** $\tau_i \in C(\mathbf{R}_+, \mathbf{R})$, $\lim_{t\to+\infty} \tau_i(t) = +\infty$ and $\tau_i(t) \le t$ for $t \in \mathbf{R}_+$, $i = 1, \cdots, m$.

**H4.** $F \in C((t_{k-1}, t_k] \times \mathbf{R}^m, \mathbf{R})$, $F_k \in C(\mathbf{R}^m, \mathbf{R})$ for $k \in \mathbf{N}$ and

$$x_1 F(t, x_1, \cdots, x_m) > 0, \quad x_1 F_k(x_1, \cdots, x_m) > 0, \qquad \lim_{\substack{y \to x \\ t \to t_{k-1}^+}} F(t, y) \in \mathbf{R}$$

for $x_1 x_i > 0$, $i = 1, \cdots, m$, $t \in \mathbf{R}$, $k \in \mathbf{N}$, where $y = (y_1, \cdots, y_m)$, $x = (x_1, \cdots, x_m)$.

**H5.** $F(t, x_1, \cdots, x_m)$ and $F_k(x_1, \cdots, x_m)$ are nondecreasing functions with respect to $x_i$, $i = 1, \cdots, m$ for each fixed $t \in \mathbf{R}$ and $k \in \mathbf{N}$.

**H6.** $q \in PC^1(\mathbf{R}_+, \mathbf{R}_+)$, $\displaystyle\int^{\infty} \frac{dt}{q(t)} = +\infty$
and $q(t) > 0$, $q(t_k^+) > 0$ for $t, t_k \in \mathbf{R}_+$.

**H7.** $\sigma_i \in C(\mathbf{R}_+, \mathbf{R})$, $\lim_{t\to+\infty} \sigma_i(t) = +\infty$ and $\sigma_i(t) \le t$ for $t \in \mathbf{R}_+$, $i = 1, \cdots, m$.

**H8.** $G \in C((t_{k-1}, t_k] \times \mathbf{R}^m, \mathbf{R})$, $G_k \in C(\mathbf{R}^m, \mathbf{R})$ for $k \in \mathbf{N}$ and

$$x_1 G(t, x_1, \cdots, x_m) > 0, \quad x_1 G_k(x_1, \cdots, x_m) > 0, \qquad \lim_{\substack{y \to x \\ t \to t_{k-1}^+}} G(t, y) \in \mathbf{R}$$

for $x_1 x_i > 0$, $i = 1, \cdots, m$, $t \in \mathbf{R}$, $k \in \mathbf{N}$, where $y = (y_1, \cdots, y_m)$, $x = (x_1, \cdots, x_m)$.

**H9.** $G(t, x_1, \cdots, x_m)$ and $G_k(x_1, \cdots, x_m)$ are nondecreasing functions with respect to $x_i$, $i = 1, \cdots, m$ for each fixed $t \in \mathbf{R}$ and $k \in \mathbf{N}$.

**H10.** $\tau \in C(\mathbf{R}_+, \mathbf{R})$, $\tau$ is nondecreasing in $\mathbf{R}_+$, $\lim_{t\to+\infty} \tau(t) = +\infty$ and $\tau(t) \le t$ for $t \in \mathbf{R}_+$.

**H11.** $p \in PC(R_+, \mathbf{R}_+)$ and $p_k \ge 0$, $k \in \mathbf{N}$.

**H12.** $f \in C(\mathbf{R}, \mathbf{R})$, $f_0 \in C(\mathbf{R}, \mathbf{R})$ and $xf(x) > 0$, $xf_0(x) > 0$ for $x \ne 0$.

**H13.** $f(x)$ and $f_0(x)$ are nondecreasing in $\mathbf{R}$.

**H14.** $p_i \in PC(\mathbf{R}_+, \mathbf{R}_+)$ and $p_{ik} \ge 0$ for $k \in \mathbf{N}$, $i = 1, \cdots, m$.

Let $T \in \mathbf{R}_+ = [0, +\infty)$. Define $T_{-1} = \min_{1\le i\le m} \{\inf\{\tau_i(t) : t \ge T\}\}$.

**Definition 1.** The function $x(t)$ is said to be a *solution* of equation (1) in the interval $[T, +\infty)$, if:

1. $x \in C([T_{-1}, +\infty), \mathbf{R})$, $rx' \in PC^1([T, +\infty), \mathbf{R})$;
2. $x(t)$ satisfies the differential equation in (1) for $t \ge T$, $t \ne t_k$;
3. $x(t)$ satisfies the jump condition in (1) for $t = t_k \ge T$.

The solution of inequality (2) can be defined analogously.

Let $x(t)$ be a solution of equation (1) (or inequality (2)).

**Definition 2.** The solution $x(t)$ is said to be:

2.1. **proper** if it is defined in some half-line $[T_x, +\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq T_x$;

2.2. **finally positive** if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) > 0$ for $t \geq T$;

2.3. **finally negative** if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) < 0$ for $t \geq T$;

2.4. **nonoscillatory** if it is either finally positive or finally negative;

2.5. **oscillatory** if it is proper and neither finally positive nor finally negative.

**Definition 3.** Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

Further we will apply the following simple lemma.

**Lemma 1.** *Let conditions* **H1** *and* **H2** *hold,* $x \in C([T, +\infty), \mathbf{R})$ *and* $rx' \in PC^1([T, +\infty), \mathbf{R})$. *Then:*

*1. If*

$$x(t) > 0, \quad (r(t)x'(t))' \leq 0, \quad \Delta(r(t_k)x'(t_k)) \leq 0,$$

*for* $t \geq T$, $t \neq t_k \geq T$, *then* $r(t)x'(t)$ *is nonincreasing and nonnegative for* $t \geq T$.

*2. If*

$$x(t) < 0, \quad (r(t)x'(t))' \geq 0, \quad \Delta(r(t_k)x'(t_k)) \geq 0,$$

*for* $t \geq T$, $t \neq t_k \geq T$, *then* $r(t)x'(t)$ *is nondecreasing and nonpositive for* $t \geq T$.

**Proof.** 1. Obviously $r(t)x'(t)$ is nonincreasing for $t \geq T$. Assume that there exists $T_0 \geq T$ such that $r(t)x'(t) \leq -m < 0$ for $t \geq T_0$. Integrating this inequality we obtain a contradiction:

$$0 < x(t) \leq x(T_0) - mR(t) + mR(T_0) \to -\infty \text{ as } t \to +\infty.$$

Hence $r(t)x'(t) \geq 0$ for $t \geq T$.

2. The proof of assertion 2 is analogous. □

## 3. MAIN RESULTS

**Theorem 1.** *Suppose that conditions* **H1**–**H9** *hold and there exists* $T \geq 0$ *such that*

$$\begin{array}{c} r(t) \leq q(t), \quad t \geq T, \\ \tau_i(t) \geq \sigma_i(t), \quad t \geq T, \ i = 1, \cdots, m, \\ F(t, x_1, \cdots, x_m) \operatorname{sgn} x_1 \geq G(t, x_1, \cdots, x_m) \operatorname{sgn} x_1, \\ F_k(x_1, \cdots, x_m) \operatorname{sgn} x_1 \geq G_k(x_1, \cdots, x_m) \operatorname{sgn} x_1 \end{array} \tag{7}$$

*for* $x_1 x_i > 0$, $i = 1, \cdots, m$, $t \geq T$ *and* $k : t_k \geq T$.

*Then equation (3) has a nonoscillatory solution if inequality (2) has a nonoscillatory solution.*

**Proof.** Without loss of generality we assume that the nonoscillatory solution $x(t)$ of inequality (2) is finally positive: $x(t) > 0$, $t \geq T$. It follows from condition **H3** that there exists $T_0 \geq T$ such that $\tau_i(t) \geq T$ for $t \geq T_0$, $i = 1, \cdots, m$. Then $x(\tau_i(t)) > 0$, $t \geq T_0$, $i = 1, \cdots, m$ and keeping in mind **H4** and (2) we conclude that

$$(r(t)x'(t))' \leq 0, \quad \Delta(r(t_k)x'(t_k)) \leq 0, \quad t \geq T_0, \quad t \neq t_k \geq T_0.$$

By Lemma 1, $r(t)x'(t)$ is nonicreasing and $r(t)x'(t) \geq 0$ for $t \geq T_0$. Then it follows from (2) that

$$\begin{aligned} r(t)x'(t) \ &\geq \int_t^\infty F(s, x(\tau_1(s)), \cdots, x(\tau_m(s)))ds \\ &+ \sum_{t \leq t_k} F_k(x(\tau_1(t_k)), \cdots, x(\tau_m(t_k))). \end{aligned}$$

This implies

$$\begin{aligned} x(t) \geq x(T_0) \ &+ \int_{T_0}^t \frac{1}{r(u)} \Big[ \int_u^\infty F(s, x(\tau_1(s)), \cdots, x(\tau_m(s)))ds \\ &+ \sum_{u \leq t_k} F_k(x(\tau_1(t_k)), \cdots, x(\tau_m(t_k))) \Big] du, t \geq T_0. \end{aligned} \tag{8}$$

Since $x(t)$ is nondecreasing, it follows from (8) and conditions (7) that

$$\begin{aligned} x(t) \geq x(T_0) \ &+ \int_{T_0}^t \frac{1}{q(u)} \Big[ \int_u^\infty G(s, x(\sigma_1(s)), \cdots, x(\sigma_m(s)))ds \\ &+ \sum_{u \leq t_k} G_k(x(\sigma_1(t_k)), \cdots, x(\sigma_m(t_k))) \Big] du, t \geq T_0. \end{aligned} \tag{9}$$

Let $T_{-1} = \min\limits_{1 \leq i \leq m} \inf\{\sigma_i(t) : t \geq T_0\}$ and $B$ be the space of continuous functions $y : [T_{-1}, +\infty) \to \mathbf{R}$. In the set

$$Y = \{y \in B : x(T_0) \leq y(t) \leq x(t), \ t > T_0; y(t) = x(t), \ T_{-1} \leq t \leq T_0\}$$

we define the operator $S : Y \to B$:

$$Sy(t) = \begin{cases} x(T_0) + \displaystyle\int_{T_0}^{t} \frac{1}{q(u)} \Big[ \int_u^{\infty} G(s, y(\sigma_1(s)), \cdots, y(\sigma_m(s)))ds \\ \qquad + \displaystyle\sum_{u \le t_k} G_k(y(\sigma_1(t_k)), \cdots, y(\sigma_m(t_k))) \Big] du, \ t > T_0. \\ x(t), \qquad T_{-1} \le t \le T_0. \end{cases}$$

Using (9) we obtain that if $y \in Y$, then

$$x(T_0) \le Sy(t) \le x(T_0) + \int_{T_0}^{t} \frac{1}{q(u)} \Big[ \int_u^{\infty} G(s, x(\sigma_1(s)), \cdots, x(\sigma_m(s)))ds + \sum_{u \le t_k} G_k(x(\sigma_1(t_k)), \cdots, x(\sigma_m(t_k))) \Big] du \le x(t)\,, \quad t > T_0$$

and $y(t) = x(t)$, $T_{-1} \le t \le T_0$. Consequently, $SY \subseteq Y$. Define the functions $y_0(t) \equiv x(t_0)$, $t \ge T_{-1}$ and $y_n(t) = Sy_{n-1}(t)$, $t \ge T_{-1}$, $n \in \mathbf{N}$. By induction in $n \in \mathbf{N}$ we conclude that

$$y_n(t) = x(t)\,,\ T_1 \le t \le T_0 \ \text{ and } \ x(T_0) \le y_n(t) \le y_{n-1}(t) \le x(t), \quad t \ge T_0,\ n \in \mathbf{N}.$$

Therefore there exists the finite limit

$$\lim_{n \to +\infty} y_n(t) = y(t)$$

for each $t \ge T_{-1}$, $y(t) = x(t)$, $T_{-1} \le t \le T_0$ and $x(T_0) \le y(t) \le x(t)$, $t > T_0$.

In view of Lebesgue's monotone convergence theorem we have $y \in Y$ and $y(t) = Sy(t)$, $t \ge T_{-1}$. The direct verification shows that $y(t)$ is a nonoscillatory solution of equation (3). □

The following two theorems are corollaries of Theorem 1.

**Theorem 2.** *Suppose that conditions* **H1**–**H9** *and (7) hold. Then equation (1) is oscillatory if equation (3) is oscillatory.*

**Theorem 3.** *Suppose that conditions* **H1**–**H5** *hold. Then the following assertions are equivalent:*

1. *Equation (1) has a nonoscillatory solution.*

2. *Inequality (2) has a nonoscillatory solution.*

**Theorem 4.** *Suppose that conditions* **H1**–**H5** *hold and the condition*

$$\int^{\infty} |F(t, c, \cdots, c)|dt + \sum_{k}^{\infty} |F_k(c, \cdots, c)| = +\infty \tag{10}$$

*is satisfied for each constant $c \neq 0$.*

*Then equation (1) is oscillatory.*

**Proof.** Assume the opposite, that equation (1) has a nonoscillatory solution $x(t)$ and suppose that this solution is finally positive: $x(t) > 0$, $t \geq T$. It follows from condition **H3** that $\tau_i(t) \geq T$, $i = 1, \cdots, m$ for $t \geq T_0$ and some $T_0 \geq T$.

Then $x(\tau_i(t)) > 0$, $t \geq T_0$, $i = 1, \cdots, m$ and we conclude from (1) and **H4** that

$$(r(t)x'(t))' \leq 0, \quad \Delta(r(t_k)x'(t_k)) \leq 0, \quad t \geq T_0, \quad t \neq t_k \geq T_0.$$

By Lemma 1,

$$r(t)x'(t) \geq 0, \quad t \geq T_0. \tag{11}$$

Let $T_{-1} = \min\limits_{1 \leq i \leq m} \inf\{\tau_i(t) : t \geq T_0\}$. Since $\tau_i(t) \geq T_{-1}$ for $t \geq T_0$, $i = 1, \cdots, m$ and $x(t)$ is a nondecreasing function, then we have for $t \geq T_0$, $i = 1, \cdots, m$ that

$$x(\tau_i(t)) \geq x(T_{-1}) \equiv c > 0.$$

Therefore

$$\begin{array}{ll} (r(t)x'(t))' + F(t, c, \cdots, c) \leq 0, & t \geq T_0, \quad t \neq t_k, \\ \Delta(r(t_k)x'(t_k)) + F_k(c, \cdots, c) \leq 0, & t_k \geq T_0 \end{array}$$

and after integration we obtain

$$r(t)x'(t) \leq r(T_0)x'(T_0) - \int_{T_0}^{t} F(s, c, \cdots, c)ds - \sum_{T_0 \leq t_k < t} F_k(c, \cdots, c),$$

for $t \geq T_0$. Using (10) we get the relation

$$\lim_{t \to +\infty} r(t)x'(t) = -\infty,$$

which contradicts (11).

If we suppose that $x(t)$ is a finally negative solution of equation (1), we reach a contradiction with analogous reasoning. □

**Theorem 5.** *Suppose that conditions* **H1**–**H5** *hold and the conditions*

$$\int^{\infty} |F(s, c, \cdots, c)|ds + \sum_{k}^{\infty} |F_k(c, \cdots, c)| < +\infty \tag{12}$$

*and*

$$\int^{\infty} \frac{1}{r(t)} \Big[ \int_{t}^{\infty} |F(s, c, \cdots, c)|ds + \sum_{t \leq t_k} F_k(c, \cdots, c)| \Big] dt = +\infty \tag{13}$$

*are satisfied for each constant $c \neq 0$.*

*Then every proper bounded solution of equation (1) is oscillatory.*

**Proof.** Assume the opposite, that equation (1) has a bounded nonoscillatory solution $x(t)$. Suppose, without loss of generality, that this solution is finally positive: $x(t) > 0$, $x(\tau_i(t)) > 0$, $t \geq T_0$, $i = 1, \ldots, m$. Since $(r(t)x'(t))' \leq 0$, $\Delta(r(t_k)x'(t_k)) \leq 0$, $t \geq T_0$, $t \neq t_k \geq T_0$, then by Lemma 1, $r(t)x'(t) \geq 0$ for $t \geq T_0$. Hence there exist numbers $c > 0$ and $d > 0$ such that

$$c \leq x(t) \leq d,\ c \leq x(\tau_i(t)) \leq d,\ t \geq T_0,\ i = 1, \ldots, m. \tag{14}$$

It follows from (1), (14) and conditions **H4** and **H5** that

$$\begin{array}{ll} (r(t)x'(t))' + F(t, c, \cdots, c) \leq 0, & t \geq T_0,\ \ t \neq t_k, \\ \Delta(r(t_k)x'(t_k)) + F_k(c, \cdots, c) \leq 0, & t_k \geq T_0. \end{array}$$

Integrating from $t \geq T_0$ to $+\infty$ we obtain

$$r(t)x'(t) \geq \int_t^\infty F(s, c, \cdots, c)ds + \sum_{t \leq t_k} F_k(c, \cdots, c),\ \ t \geq T_0.$$

This implies the inequality

$$\begin{aligned} d - x(T_0) &\geq x(+\infty) - x(T_0) \\ &\geq \int_{T_0}^\infty \frac{1}{r(t)} \Big[ \int_t^\infty F(s, c, \cdots, c)ds + \sum_{t \leq t_k} F_k(c, \cdots, c) \Big] dt, \end{aligned}$$

which contradicts (13). □

The following theorem is a corollary of Theorems 4 and 5.

**Theorem 6.** *Suppose that conditions* **H1**, **H2**, **H10** *and* **H11** *(and* **H12**, **H13***) hold. Then:*

*1. If*

$$\int^\infty p(t)dt + \sum_k^\infty p_k = +\infty, \tag{15}$$

*then equation (4) (and equation (5)) is oscillatory.*

*2. If*

$$\int^\infty p(t)dt + \sum_k^\infty p_k < +\infty \tag{16}$$

*and*

$$\int^\infty \frac{1}{r(t)} \Big[ \int_t^\infty p(s)ds + \sum_{t \leq t_k} p_k \Big] dt = +\infty\,, \tag{17}$$

*then every proper bounded solution of equation (4) (and equation (5)) is oscillatory.*

Set $R(t) = \displaystyle\int_0^t \frac{ds}{r(s)}$.

**Theorem 7.** *Suppose that conditions* **H1**, **H2**, **H10** *and* **H11** *hold and*

$$\int^{\infty} p(t)R(\tau(t))dt + \sum_{k}^{\infty} p_k R(\tau(t_k)) = +\infty. \tag{18}$$

*Then:*

*1. Every nonoscillatory solution $x(t)$ of equation (4) is such that*

$$\lim_{t\to+\infty} r(t)x'(t) = 0. \tag{19}$$

*2. Every proper bounded solution of equation (4) is oscillatory.*

**Proof.** 1. Let $x(t)$ be a finally positive solution of equation (4): $x(t) > 0$, $x(\tau(t)) > 0$, $t \geq T_0$. Then it follows from (4) and condition **H11** that

$$(r(t)x'(t))' \leq 0, \quad \Delta(r(t_k)x'(t_k)) \leq 0, \quad t \geq T_0, \quad t \neq t_k \geq T_0.$$

By Lemma 1, $r(t)x'(t)$ is nonincreasing and $r(t)x'(t) \geq 0$ for $t \geq T_0$. Therefore there exists the limit

$$\lim_{t\to+\infty} r(t)x'(t) = m \in \mathbf{R}_+.$$

Assume that $m > 0$. Then $r(t)x'(t) \geq m$, $t \geq T_0$. This implies that

$$x(t) \geq x(T_0) + mR(t) - mR(T_0) \geq \frac{m}{2}R(t) \tag{20}$$

for $t \geq T_1 \geq T_0$, where $T_1$ is sufficiently large.

Integrating (4) from $T_1$ to $t \geq T_1$ we get

$$0 \leq r(t)x'(t) = r(T_1)x'(T_1) - \int_{T_1}^{t} p(s)x(\tau(s))ds - \sum_{T_1\leq t_k<t} p_k x(\tau(t_k)). \tag{21}$$

Since $x(t)$ is nondecreasing, then in view of (21) and (20) we obtain the inequality

$$r(T_1)x'(T_1) \geq \int_{T_1}^{\infty} p(s)\frac{m}{2}R(\tau(s))ds + \sum_{T_1\leq t_k} p_k \frac{m}{2}R(\tau(t_k))$$

which contradicts (18). Hence, $m = 0$ and (19) is proved.

If $x(t)$ is a finally negative solution of equation (4), then the proof of equality (19) is analogous.

2. The assertion follows from Theorem 6, since condition (18) implies (17) in the case, when (16) holds. □

Introduce the following notations:

$$S = \max\Big\{\limsup_{y\to-\infty}\frac{y}{f(y)}, \limsup_{y\to+\infty}\frac{y}{f(y)}, \limsup_{y\to-\infty}\frac{y}{f_0(y)}, \limsup_{y\to+\infty}\frac{y}{f_0(y)}\Big\},$$

$$\beta(t) = \inf\{\tau(s) : s \geq t\}, \quad \gamma(t) = \sup\{s \geq 0 : \tau(s) \leq t\}.$$

**Remark 1.** *The function $\beta(t)$ is continuous and nondecreasing in $\mathbf{R}_+$, and the function $\tau(s)$ takes all values of the interval $[\beta(t), +\infty)$ for $s \geq t$. Moreover,*

$$\lim_{t\to+\infty} \beta(t) = \lim_{t\to+\infty} \gamma(t) = +\infty,$$

$$\beta(t) \leq t \leq \gamma(t), \quad \tau(\gamma(t)) = t, \quad t \in \mathbf{R}_+. \tag{22}$$

In order to prove Theorem 8 we need the following

**Lemma 2.** *Let conditions* **H1** *and* **H2** *hold, $x \in C([T, +\infty), \mathbf{R})$ and $rx' \in PC^1([T, +\infty), \mathbf{R})$. Then:*

*1. If*

$$(r(t)x'(t))' \leq 0, \quad \Delta(r(t_k)x'(t_k)) \leq 0, \quad x'(t) \geq 0, \quad x(t) > 0,$$

*for $t \geq T$, $t \neq t_k \geq T$, then for each $\lambda \in (0, 1)$ there exists $T_\lambda \geq T$ such that*

$$x(t) \geq \lambda R(t)r(t)x'(t), \quad t \geq T_\lambda. \tag{23}$$

*2. If*

$$(r(t)x'(t))' \geq 0, \quad \Delta(r(t_k)x'(t_k)) \geq 0, \quad x'(t) \leq 0, \quad x(t) < 0,$$

*for $t \geq T$, $t \neq t_k \geq T$, then for each $\lambda \in (0, 1)$ there exists $T_\lambda \geq T$ such that*

$$x(t) \leq \lambda R(t)r(t)x'(t), \quad t \geq T_\lambda. \tag{24}$$

**Proof.** 1. Let $T \in [t_{i-1}, t_i)$ and $t \in (t_{i+k}, t_{i+k+1}]$. Then by the Cauchy Mean-value Theorem

$$x(t) - x(t_{i+k}) = \frac{x'(\xi)}{R'(\xi)}[R(t) - R(t_{i+k})],$$

where $\xi \in (t_{i+k}, t)$. Since $r(t)x'(t)$ is nonincreasing for $t \geq T$, then

$$\frac{x'(\xi)}{R'(\xi)} = r(\xi)x'(\xi) \geq r(t)x'(t)$$

and this implies

$$x(t) - x(t_{i+k}) \geq r(t)x'(t)[R(t) - R(t_{i+k})]. \tag{25}$$

Analogously,

$$\begin{aligned} &x(t_{i+k}) - x(t_{i+k-1}) \geq r(t)x'(t)[R(t_{i+k}) - R(t_{i+k-1})], \\ &\cdots \\ &x(t_{i+1}) - x(t_i) \geq r(t)x'(t)[R(t_{i+1}) - R(t_i)], \\ &x(t_i) - x(T) \geq r(t)x'(t)[R(t_i) - R(T)]. \end{aligned} \tag{26}$$

Summing inequalities (25) and (26) we obtain

$$x(t) - x(T) \geq r(t)x'(t)[R(t) - R(T)]. \tag{27}$$

Let $\lambda(0,1)$ be given. Since $R(t) \to +\infty$ as $t \to +\infty$, then there exists $T_\lambda \geq T$ such that

$$R(t) - R(T) \geq \lambda R(t), \quad t \geq T_\lambda. \tag{28}$$

Then (23) follows from (27) and (28).

2. The proof of (24) is analogous. □

**Theorem 8.** *Suppose that conditions* **H1, H2, H10–H13** *and (16) hold and*

$$\limsup_{t\to+\infty} R(\tau(t))\Big[\int_t^\infty p(s)ds + \sum_{t\leq t_k} p_k\Big] > S. \tag{29}$$

*Then equation (5) is oscillatory.*

**Proof.** Since condition (29) implies condition (17) of Theorem 6, it suffices to prove that equation (5) has no unbounded nonoscillatory solutions.

Assume the opposite, that equation (5) has an unbounded nonoscillatory solution $x(t)$. Suppose, without loss of generality, that this solution is finally positive: $x(t) > 0$, $x(\tau(t)) > 0$, $t \geq T_0 \geq 0$. It follows from (5) and conditions **H11, H12** that $(r(t)x'(t))' \leq 0$, $\Delta(r(t_k)x'(t_k)) \leq 0$ for $t \geq T_0$, $t \neq t_k \geq T_0$. By Lemma 1, $r(t)x'(t)$ is nonincreasing and $r(t)x'(t) \geq 0$ for $t \geq T_0$. Then there exists the limit

$$\lim_{t\to+\infty} r(t)x'(t) \in \mathbf{R}_+.$$

Integrating (5) from $t \geq T_0$ to $+\infty$ we obtain

$$r(t)x'(t) \geq \int_t^\infty p(s)f(x(\tau(s)))ds + \sum_{t\leq t_k} p_k f_0(x(\tau(t_k))), \quad t \geq T_0. \tag{30}$$

By Lemma 2, for each $\lambda \in (0,1)$ there exists a $T_\lambda \geq T_0$ such that

$$x(t) \geq \lambda R(t) r(t)x'(t), \quad t \geq T_\lambda. \tag{31}$$

From (30) and (31) it follows that

$$x(t) \geq \lambda R(t)\Big[\int_t^\infty p(s)f(x(\tau(s)))ds + \sum_{t\leq t_k} p_k f_0(x(\tau(t_k)))\Big], \quad t \geq T_\lambda.$$

Since $\gamma(t) \geq t$, then

$$x(t) \geq \lambda R(t)\Big[\int_{\gamma(t)}^\infty p(s)f(x(\tau(s)))ds + \sum_{\gamma(t)\leq t_k} p_k f_0(x(\tau(t_k)))\Big], \quad t \geq T_\lambda.$$

Since $x(t)$ is nondecreasing and $\tau(s) \geq t$ for $s \geq \gamma(t)$, the above inequality gives

$$x(t) \geq \lambda R(t)x(t)\Big[\int_{\gamma(t)}^\infty p(s)\frac{f(x(\tau(s)))}{x(\tau(s))}ds + \sum_{\gamma(t)\leq t_k} p_k \frac{f_0(x(\tau(t_k)))}{x(\tau(t_k))}\Big].$$

Hence

$$\begin{aligned}\max\Big[\sup_{s\ge t}\frac{x(\tau(s))}{f(x(\tau(s)))},\sup_{s\ge t}\frac{x(\tau(s))}{f_0(x(\tau(s)))}\Big]\\ \ge \lambda R(t)\Big[\int_{\gamma(t)}^{\infty}p(s)ds+\sum_{\gamma(t)\le t_k}p_k\Big],\quad t\ge T_\lambda.\end{aligned}\tag{32}$$

Keeping in mind the relations

$$\begin{gathered}\sup_{s\ge t}\frac{x(\tau(s))}{f(x(\tau(s)))}=\sup_{s\ge\beta(t)}\frac{x(s)}{f(x(s))}=\sup_{x\ge x(\beta(t))}\frac{x}{f(x)},\\ \sup_{s\ge t}\frac{x(\tau(s))}{f_0(x(\tau(s)))}=\sup_{s\ge\beta(t)}\frac{x(s)}{f_0(x(s))}=\sup_{x\ge x(\beta(t))}\frac{x}{f_0(x)},\\ \lim_{t\to+\infty}x(t)=\lim_{t\to+\infty}x(\beta(t))=+\infty\end{gathered}$$

we conclude from (32) that

$$\begin{aligned}S&\ge\max\Big[\limsup_{x\to+\infty}\frac{x}{f(x)},\limsup_{x\to+\infty}\frac{x}{f_0(x)}\Big]\\ &\ge\lambda\limsup_{t\to+\infty}R(t)\Big[\int_{\gamma(t)}^{\infty}p(s)ds+\sum_{\gamma(t)\le t_k}p_k\Big].\end{aligned}\tag{33}$$

Since $\lambda\in(0,1)$ is arbitrary, then it follows from (33) and (22) that

$$\begin{aligned}S&\ge\limsup_{t\to+\infty}R(t)\Big[\int_{\gamma(t)}^{\infty}p(s)ds+\sum_{\gamma(t)\le t_k}p_k\Big]\\ &=\limsup_{t\to+\infty}R(\tau(t))\Big[\int_{t}^{\infty}p(s)ds\sum_{t\le t_k}p_k\Big].\end{aligned}$$

This relation contradicts (29). □

As a particular case of Theorem 8, we obtain

**Theorem 9.** *Suppose that conditions* **H1, H2, H10, H11** *and (16) hold and*

$$\limsup_{t\to+\infty}R(\tau(t))\Big[\int_{t}^{\infty}p(s)ds+\sum_{t\le t_k}p_k\Big]>1.\tag{34}$$

*Then equation (4) is oscillatory.*

**Theorem 10.** *Suppose that conditions* **H1, H2, H10, H11** *and (16) hold, the function $\tau(t)$ is differentiable and finally nondecreasing, and*

$$\liminf_{t\to+\infty}R(\tau(t))\Big[\int_{t}^{\infty}p(s)ds+\sum_{t\le t_k}p_k\Big]>\frac{1}{4}.\tag{35}$$

*Then equation (4) is oscillatory.*

**Proof.** Since condition (35) implies condition (17) of Theorem 6, it suffices to prove that equation (4) has no unbounded nonoscillatory solutions. Assume, without loss of generality, that equation (4) has an unbounded finally positive solution $x(t) : x(t) > 0$, $x(\tau(t)) > 0$, $t \geq T \geq 0$. As in the proof of Theorem 8 we conclude that $r(t)x'(t)$ is nonincreasing and $r(t)x'(t) \geq 0$ for $t \geq T$. We observe that (35) implies (18) and by Theorem 7, $\lim\limits_{t\to+\infty} r(t)x'(t) = 0$. Therefore we obtain from (4) that

$$r(t)x'(t) = \int_t^\infty p(s)x(\tau(s))ds + \sum_{t\leq t_k} p_k x(\tau(t_k)), \quad t \geq T.$$

Let $\Lambda$ be the set of all $\lambda > 0$ for which there exists $T_\lambda \geq T$ such that the function $x(t)/R^\lambda(t)$ is nondecreasing for $t \geq T_\lambda$, or equivalently, such that

$$r(t)x'(t) \geq \lambda \frac{x(t)}{R(t)}, \quad t \geq T_\lambda.$$

We have that $\lambda = 1 \,\overline{\in}\, \Lambda$. Otherwise, $\dfrac{x(t)}{R(t)}$ is nondecreasing,

$$r(t)x'(t) \geq \frac{x(t)}{R(t)}, \quad t \geq T_1,$$

and this leads to a contradiction:

$$0 = \lim_{t\to+\infty} r(t)x'(t) \geq \lim_{t\to+\infty} \frac{x(t)}{R(t)} > 0.$$

Now we show that $\frac{1}{4} \in \Lambda$, i. e., the set $\Lambda$ is not empty. Set

$$P(t) = \int_t^\infty p(s)ds + \sum_{t\leq t_k} p_k.$$

Let the numbers $c$ and $T_0 \geq T$ be chosen so that $\tau(t)$ is nondecreasing for $t \geq T_0$ and

$$\tau(t) \geq T, \quad R(\tau(t))P(t) \geq c > \frac{1}{4} \quad \text{for} \quad t \geq T_0.$$

Then for $t \geq T_0$ we obtain the inequality

$$\begin{aligned} r(\tau(t))x'(\tau(t)) \geq r(t)x'(t) = \int_t^\infty p(s)x(\tau(s))ds + \sum_{t\leq t_k} p_k x(\tau(t_k)) \\ \geq x(\tau(t))P(t) \geq \frac{1}{4}\frac{x(\tau(t))}{R(\tau(t))}, \end{aligned}$$

which means that $\frac{1}{4} \in \Lambda$.

Let $\lambda \in (0,1) \cap \Lambda$. Then for $t \geq T_0$ we have

$$\begin{aligned} r(\tau(t))x'(\tau(t)) \geq r(t)x'(t) &= \int_t^\infty p(s)x(\tau(s))ds + \sum_{t\leq t_k} p_k x(\tau(t_k)) \\ &= \int_t^\infty p(s)R^\lambda(\tau(s))\frac{x(\tau(s))}{R^\lambda(\tau(s))}ds + \sum_{t\leq t_k} p_k R^\lambda(\tau(t_k^+))\frac{x(\tau(t_k))}{R^\lambda(\tau(t_k))} \\ &\geq \frac{x(\tau(t))}{R^\lambda(\tau(t))}\Big[\int_t^\infty p(s)R^\lambda(\tau(s))ds + \sum_{t\leq t_k} p_k R^\lambda(\tau(t_k^+))\Big] \\ &\geq \frac{x(\tau(t))}{R^\lambda(\tau(t))}c^\lambda\Big[\int_t^\infty p(s)P^{-\lambda}(s)ds + \sum_{t\leq t_k} p_k P^{-\lambda}(t_k^+)\Big] \qquad (36) \\ &= \frac{x(\tau(t))}{R^\lambda(\tau(t))}c^\lambda\Big[-\int_t^\infty P^{-\lambda}(s)dP(s) - \sum_{t\leq t_k} P^{-\lambda}(t_k^+)\Delta P(t_k)\Big] \\ &\geq \frac{x(\tau(t))}{R^\lambda(\tau(t))}c^\lambda\Big[-\int_t^\infty d\Big(\frac{P^{1-\lambda}(s)}{1-\lambda}\Big) - \sum_{t\leq t_k}\Delta\Big(\frac{P^{1-\lambda}(t_k)}{1-\lambda}\Big)\Big] \\ &= \frac{x(\tau(t))}{R^\lambda(\tau(t))}c^\lambda\frac{P^{1-\lambda}(t)}{1-\lambda} \geq \frac{c}{1-\lambda}\frac{x(\tau(t))}{R(\tau(t))}. \end{aligned}$$

Since $\lambda \in (0,1)$ and $c > \frac{1}{4}$ we have that

$$\frac{c}{1-\lambda} = \lambda + \Big(\frac{c}{1-\lambda} - \lambda\Big) \geq \lambda + m\,, \qquad (37)$$

where $m = 2\sqrt{c} - 1 > 0$. From (36) and (37) we conclude that $\lambda + m \in \Lambda$. But this leads to a contradiction because $m > 0$ is independent of $\lambda$ and $\lambda$ can be chosen arbitrarily close to $\sup\Lambda$. □

**Theorem 11.** *Suppose that conditions* **H1, H2, H10–H13** *and (16) hold, the function $\tau(t)$ is differentiable and finally nondecreasing, and*

$$\liminf_{t\to+\infty} R(\tau(t))\Big[\int_t^\infty p(s)ds + \sum_{t\leq t_k} p_k\Big] > \frac{1}{4}S. \qquad (38)$$

*Then equation (5) is oscillatory.*

**Proof.** It suffices to prove that equation (5) has no unbounded nonoscillatory solutions. Assume, without loss of generality, that equation (5) has an unbounded finally positive solution $x(t) : x(t) > 0, x(\tau(t)) > 0, t \geq T > 0$. Choose $T_0 \geq T$ so that $\tau(t) \geq T$ and $\tau(s)$ is nondecreasing for $t \geq T_0$. Then $x(t)$ is a positive solution (in the interval $[T_0, +\infty)$) of the equation

$$\begin{aligned} &(r(t)y'(t))' + \tilde{p}(t)y(\tau(t)) = 0, \qquad t \neq t_k, \\ &\Delta(r(t_k)y'(t_k)) + \tilde{p}_k y(\tau(t_k)) = 0\,, \end{aligned}$$

where

$$\tilde{p}(t) = p(t)\frac{f(x(\tau(t)))}{x(\tau(t))}, \quad \tilde{p}_k = p_k\frac{f_0(x(\tau(t_k)))}{x(\tau(t_k))}.$$

Thus, by Theorem 10, we must have

$$\liminf_{t\to+\infty} R(\tau(t))\Big[\int_t^\infty \tilde{p}(s)ds + \sum_{t\le t_k}\tilde{p}_k\Big] \le \frac{1}{4}.$$

Since $x(t)$ is nondecreasing, we obtain

$$\begin{aligned}
P(t) &= \int_t^\infty p(s)ds + \sum_{t\le t_k} p_k \\
&= \int_t^\infty \tilde{p}(s)\frac{x(\tau(s))}{f(x(\tau(s)))}ds + \sum_{t\le t_k}\tilde{p}_k\frac{x(\tau(t_k))}{f_0(x(\tau(t_k)))} \\
&\le \max\Big[\sup_{s\ge t}\frac{x(\tau(s))}{f(x(\tau(s)))}, \sup_{s\ge t}\frac{x(\tau(s))}{f_0(x(\tau(s)))}\Big]\Big[\int_t^\infty \tilde{p}(s)ds + \sum_{t\le t_k}\tilde{p}_k\Big] \\
&\le \max\Big[\sup_{y\ge x(\tau(t))}\frac{y}{f(y)}, \sup_{y\ge x(\tau(t))}\frac{y}{f_0(y)}\Big]\Big[\int_t^\infty \tilde{p}(s)ds + \sum_{t\le t_k}\tilde{p}_k\Big].
\end{aligned}$$

Keeping in mind that $\lim\limits_{t\to+\infty} x(\tau(t)) = +\infty$, we get

$$\begin{aligned}
&\liminf_{t\to+\infty} R(\tau(t))P(t) \\
&\le \max\Big[\limsup_{y\to+\infty}\frac{y}{f(y)}, \limsup_{y\to+\infty}\frac{y}{f_0(y)}\Big]\times \\
&\quad\times \liminf_{t\to+\infty} R(\tau(t))\Big[\int_t^\infty \tilde{p}(s)ds + \sum_{t\le t_k}\tilde{p}_k\Big] \\
&\le \frac{1}{4}\max\Big[\limsup_{y\to+\infty}\frac{y}{f(y)}, \limsup_{y\to+\infty}\frac{y}{f_0(y)}\Big] \le \frac{1}{4}S,
\end{aligned}$$

which contradicts (38). □

**Theorem 12.** *Suppose that conditions* **H1**–**H3** *and* **H14** *hold and the function* $\tau(t) = \min[\tau_1(t), \cdots, \tau_m(t)]$ *is differentiable and finally nondecreasing. Assume that either*

$$\limsup_{t\to+\infty} R(\tau(t))\sum_{i=1}^m\Big[\int_t^\infty p_i(s)ds + \sum_{t\le t_k} p_{ik}\Big] > 1\,, \tag{39}$$

*or*

$$\liminf_{t\to+\infty} R(\tau(t))\sum_{i=1}^m\Big[\int_t^\infty p_i(s)ds + \sum_{t\le t_k} p_{ik}\Big] > \frac{1}{4}. \tag{40}$$

*Then equation (6) is oscillatory.*

**Proof.** Theorem 12 is a corollary of Theorems 2, 9 and 10. □

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