Communications in Applied Analysis **9** (2005), no. 4, 549–553

Completeness Results in Probabilistic Metric Spaces, I

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Communicated by K. Farahmand

ABSTRACT: In this paper, we define probabilistic limit point and consider completeness in probabilistic metric spaces.

AMS (MOS) Subject Classification. 54E70, 54E50

1. INTRODUCTION AND PRELIMINARIES

Menger [5] introduced the notion of a probabilistic metric spaces in 1942 and since then the theory of probabilistic metric spaces has developed in many directions, see Schweizer and Sklar [7], Schweizer et al [8]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. It may also turn out to be relevant to the study of *physical quantities*. In the sequel, we shall adopt usual terminology, notation and conventions of the theory of probabilistic metric spaces, as in Beg [1], Chang et al [2], Pap et al [6], Schweizer and Sklar [7], Schweizer et al [9].

In the sequel, the space of all probability distribution functions (briefly, d.f.'s) is $\Delta^+ = \{F : \mathbf{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] : F$ is left-continuous and non-decreasing on $\mathbf{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}.$$

The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbf{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0 = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Received May 24, 2005

A triangle function is a binary operation on Δ^+ , namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, that is, for all $F, G, H \in \Delta^+$,

$$\begin{aligned} \tau(\tau(F,G),H) &= \tau(F,\tau(G,H)), \\ \tau(F,G) &= \tau(G,F), \\ F \leq G &\Longrightarrow \tau(F,H) \leq \tau(G,H), \\ \tau(F,\varepsilon_0) &= F. \end{aligned}$$

Continuity of a triangle functions means continuity with respect to the topology of weak convergence in Δ^+. Typical continuous triangle functions are

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),$$

and

$$\tau_{T^*}(F,G) = \inf_{s+t=x} T^*(F(s),G(t)).$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity, and T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$.

Definition 1.1. A *Probabilistic Metric* (briefly PM) space is a triple $(X, \mathcal{F}, \tau)$, where X is a nonempty set, τ is a continuous triangle function, and $\mathcal{F}$ is a mapping from $X \times X$ into Δ^+ such that, if $F_{p,q}$ denotes the value of $\mathcal{F}$ at the pair (p,q), the following conditions hold for all p, q, r in X:

(PM1) $F_{p,q} = \varepsilon_0$ if and only if, $p = q$;

(PM2) $F_{p,q} = F_{q,p}$;

(PM3) $F_{p,q} \geq \tau(F_{p,r}, F_{r,q})$ for all $p, q, r \in X$.

Definition 1.2. Let $(X, \mathcal{F}, \tau)$ be a PM space. For each p in X and $\lambda > 0$, the strong $\lambda -$ *neighborhood* of p is the set

$$N_p(\lambda) = \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for X is the union $\bigcup_{p \in V} \mathcal{N}_p$, where $\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Definition 1.3. Let $(X, \mathcal{F}, \tau)$ be a PM space. Then a sequence $\{p_n\}_n$ in X is said to be strongly convergent to p in X if for every $\lambda > 0$, there exists positive integer

N such that $p_n \in N_p(\lambda)$ whenever $n \geq N$. Also the sequence $\{p_n\}_n$ in X is called strongly Cauchy sequence if for every $\lambda > 0$, there exists positive integer N such that $F_{p_n,p_m}(\lambda) > 1 - \lambda$ whenever $m, n \geq N$. A PM space $(X, \mathcal{F}, \tau)$ is said to be strongly complete in the strong topology if and only if every strongly Cauchy sequence in X is strongly convergent to a point in X. Let $S \subseteq X$. Then $p \in X$ is said to be probabilistic limit point of S whenever for every $G \in D^+$ and $G < \varepsilon_0$

$$D_p(G) \cap (S - \{p\}) \neq \emptyset,$$

where

$$D_p(G) = \{q \in X : F_{p,q} > \varepsilon_0 - G\}.$$

A topological space is said to be strongly topologically complete if there exists a strong complete probabilistic metric inducing the given topology on it (see Megginson [4]).

Example 1.4. Let $X = (0, 1]$. If we consider PM space $(X, \mathcal{F}, \tau)$ where $F_{p,q} = \varepsilon_{|p-q|}$ and τ is a triangle function such that $\tau(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$ for all $a, b \geq 0$ (see Lafuerza-Guillén et al [3], Example 1.1), then this PM space is not strong complete because the Cauchy sequence $\{1/n\}$ is not convergent. Now consider the PM space $(X, \mathcal{F}', \tau)$, where the probabilistic metric define by $F'_{p,q} = \varepsilon_{|p-q|+|\frac{1}{p}-\frac{1}{q}|}$ and $\tau(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}, (a \geq 0, b \geq 0)$. Then $(X, \mathcal{F}', \tau)$ is strongly complete PM space. Since p_n tend to p with respect to probabilistic metric $\mathcal{F}$ if and only if $|p_n - p| \longrightarrow 0$, and if and only if p_n tends to p with respect to probabilistic metric $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{F}'$ are equivalent probabilistic metrics. Hence X is strongly topologically complete.

2. MAIN RESULTS

Theorem 2.1. *Let $(X, \mathcal{F}, \tau)$ be a PM space and S be a compact subset of X. Then every infinite subset of S has a limit point in S.*

We omit the proof because it is same as the usual one.

Theorem 2.2. *Let $(X, \mathcal{F}, \tau)$ be a PM space and S be a compact subset of X. Then S is strongly complete.*

Proof. Let $\{p_n\} \subseteq S$ be a strongly Cauchy sequence and $A = \{p_1, \cdots, p_n\}$ be the range of $\{p_n\}$. If A is a finite set, then $\{p_n\}$ is strongly convergent to an element of S. If A is an infinite set, then A has a probabilistic limit point p_0 in S. We show that $p_n \longrightarrow p_0$. For each d.f. G we can find a $H \in D^+$ and $H < \varepsilon_0$ such that $\tau(\varepsilon_0 - H, \varepsilon_0 - H) \geq \varepsilon_0 - G$. Since $\{p_n\}$ is a strongly Cauchy sequence, there exists

$N \in \mathbf{N}$ such that for each $m, n \geq N$, $F_{p_m,p_n} > \varepsilon_0 - H$. Since p_0 is probabilistic limit point of S, we have

$$D_{p_0}(H) \cap (S - \{p_0\}) \neq \emptyset.$$

Consequently there exists $m \geq N$ such that $p_m \in D_{p_0}(H)$. Now for each $n \geq N$, we have,

$$\begin{aligned} F_{p_n,p_0} &\geq \tau(F_{p_n,p_m}, F_{p_m,p_0}) \\ &\geq \tau(\varepsilon_0 - H, \varepsilon_0 - H) \\ &\geq \varepsilon_0 - G. \end{aligned}$$

Since G was arbitrary, it follows that $F_{p_n,p_0} \longrightarrow \varepsilon_0$ or $p_n \longrightarrow p_0$. Hence S is strongly complete. □

Similarly as the proof of the above theorem, we can prove following theorem.

Theorem 2.3. *Let $(X, \mathcal{F}, \tau)$ be a PM space, S be a compact subset of X, and $\{p_n\} \subseteq S$. Then $\{p_n\}$ has a convergent subsequence in X.*

Theorem 2.4. *Let $(Y, \mathcal{F}, \tau)$ be a PM space and X be a strongly topologically complete subset of Y. Then X is a G_δ subset of Y.*

Proof. Let $(X, \mathcal{F}', \tau)$ be strongly complete PM space that induces the same topology for X as does $\mathcal{F}$. For each $p \in X$ and each $n \in \mathbf{N}$, let H_n be a d.f. such that, $H_n < G_n$, G_n decreasingly tend to ε_∞ and $F'_{r,p} > \varepsilon_0 - G_n$, whenever $r \in X$ and $F_{r,p} > \varepsilon_0 - H_n$ for each $n \in \mathbf{N}$. Suppose that $\mathcal{O}_n = \cup_n \{D_p(H_n) : p \in X\}$ for each $n \in \mathbf{N}$, and $\Gamma = \cap_n \{\mathcal{O}_n : n \in \mathbf{N}\}$. Then Γ is a G_δ subset of Y which clearly contains X. It is enough to shown that $\Gamma \subseteq X$. Let $p_0 \in \Gamma$. Then $p_0 \in \mathcal{O}_n$ for each $n \in \mathbf{N}$. Hence for each $n \in \mathbf{N}$, there is $p_n \in X$ such that $p_0 \in D_{p_n}(H_n)$. Therefore $F_{p_0,p_n} > \varepsilon_0 - H_n > \varepsilon_0 - G_n$ for each $n \in \mathbf{N}$. This means that $p_n \longrightarrow p_0$ in Y. Now let $E \in D^+$, $E < \varepsilon_0$ and $N \in \mathbf{N}$ such that $\tau(\varepsilon_0 - G_N, \varepsilon_0 - G_N) > \varepsilon_0 - E$. Let $m \in \mathbf{N}$ be such that

$$\tau(\varepsilon_0 - G_m, F_{p_0,p_N}) > \varepsilon_0 - H_N.$$

Now for every $k \in \mathbf{N}$ and $k > m$, we have

$$\begin{aligned} F_{p_k,p_N} &\geq \tau(F_{p_k,p_0}, F_{p_N,p_0}) \\ &> \tau(\varepsilon_0 - G_k, F_{p_N,p_0}) \\ &\geq \tau(\varepsilon_0 - G_m, F_{p_N,p_0}) \\ &> \varepsilon_0 - H_N. \end{aligned}$$

Therefore $F'_{p_k,p_N} > \varepsilon_0 - G_N$. If $k, l > m$, then

$$\begin{aligned} F'_{p_k,p_l} &\geq \tau(F'_{p_k,p_N}, F'_{p_N,p_l}) \\ &\geq \tau(\varepsilon_0 - G_N, \varepsilon_0 - G_N) \\ &> \varepsilon_0 - E. \end{aligned}$$

Hence the sequence $\{p_n\}$ is a strongly Cauchy sequence in the PM space $(X, \mathcal{F}', \tau)$ and so strong convergent to some member of X. Since $p_n \longrightarrow p_0$ in Y, it follows that $p_0 \in X$, so $\Gamma \subseteq X$. □

ACKNOWLEDGEMENT

The authors are grateful to the referee for his valuable suggestions.

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