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INFINITELY MANY SOLUTIONS FOR PERTURBED FOURTH-ORDER KIRCHHOFF-TYPE PROBLEMS

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ABSTRACT. Existence results of infinitely many solutions for perturbed fourth-order Kirchhoff-type problems are established. No symmetric condition on the nonlinear term is assumed. The main tool is an infinitely many critical points theorem.

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1. **Introduction**

The aim of the present paper is to investigate the existence of infinitely many solutions for the following perturbed fourth-order Kirchhoff-type problem

$$\text{(1.1)} \qquad \begin{cases} T(u) = \lambda f(x,u) + \mu g(x,u) + h(u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

where λ is a positive parameter, μ is a non-negative parameter,

$$T(u) = u^{iv} + K\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu)$$

in which $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \le K(t) \le m_1$ for all $t \ge 0$, and A and B are two real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory functions and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|h(t_1) - h(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$.

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The problem (1.1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

for $0<x<L$, $t\geq 0$, where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [19]. The equation (1.2) is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. Some interesting results can be found, for example in [3, 9, 27]. On the other hand, nonlocal boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [2, 12, 15, 16, 22, 24, 26] for some related works.

Owing to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, many researchers have studied the existence and multiplicity of solutions for fourth-order two-point boundary value problems, we refer the reader to [1, 4, 5, 6, 20, 23]. Moreover, since fourth order equations of Kirchhoff type arise in the theory of bending extensible elastic beams on nonlinear elastic foundations, in [29, 30, 14, 21] the existence and multiplicity of solutions for nonlinear fourth order equation of Kirchhoff type was studied.

In the present paper, using a smooth version of [8, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [25, Theorem 2.5], requiring that the nonlinear term f has a suitable oscillating behavior at infinity, we establish the existence of a precise interval Λ such that for every $\lambda\in\Lambda$ and for every L^2-Carathéodory function g satisfying a certain growth at infinity, choosing μ sufficiently small, the problem (1.1) admits a sequence of generalized solutions which is unbounded in the space E which will be introduced later (Theorem 3.1). Replacing the conditions at infinity of the nonlinear terms, by a similar one at zero, we obtain a sequence of generalized solutions strongly converging to zero; see Remark 3.7. In our results neither symmetric nor monotonic condition on the nonlinear term f is assumed. We require that f has a suitable oscillating behaviour either at infinity or at zero. Ricceri's Variational Principle and its variants have been successfully used to ensure the existence of infinitely many solutions for boundary value problems in the papers [5, 7, 13, 16]. We also refer to [11] in which the authors obtained a type of a three critical point theorem and applied the theorem to investigate the multiplicity of solutions to discrete anisotropic problems with two parameters.

2. PRELIMINARIES

Our main tool to investigate the existence of infinitely many solutions for the problem (1.1) is a smooth version of Theorem 2.1 of [8] which is a more precise version of Ricceri's Variational Principle [25] that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

(b_1) *I_λ possesses a global minimum,*

or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

Assume that

$$\max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}\right\} < 1.$$

Set

$$\sigma := \max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}, 0\right\}$$

and

$$\delta := \sqrt{1-\sigma}.$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the norm

$$||u|| = \left(\int_0^1 (|u''(x)|^2 - A|u'(x)|^2 + B|u(x)|^2)dx\right)^{1/2}$$

which is equivalent to the usual one and, in particular, one has

$$||u||_\infty \leq \frac{1}{2\pi\delta}||u||, \tag{2.1}$$

(see [4, Proposition 2.1]).

We suppose that the Lipschitz constant $L > 0$ of the function h satisfies

$$\min\{1, m_0\} > \frac{L}{4\pi^2\delta^2}.$$

A function $u : [0,1] \to \mathbb{R}$ is a generalized solution to the problem (1.1) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = 0$, $u''(0) = u''(1) = 0$, and $u^{iv} + K\left(\int_0^1(-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu) = \lambda f(x, u(x)) + \mu g(x, u(x)) + h(u(x))$ for almost every $x \in [0,1]$, and it is a weak solution to the problem (1.1) if $u \in X$ and

$$\begin{aligned}\int_0^1 u''(x)v''(x)dx + K&\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)\\ &\times \int_0^1 (-Au'(x)v'(x) + Bu(x)v(x))dx - \lambda\int_0^1 f(x, u(x))v(x)dx\\ &- \mu\int_0^1 g(x, u(x))v(x)dx - \int_0^1 h(u(x))v(x)dx = 0\end{aligned}$$

for every $v \in X$. Each weak solution to the problem (1.1) is a generalized one (see [4, Proposition 2.2]). If f, g are continuous, then each generalized solution u of the problem (1.1) is a classical solution.

A special case of our main result is the following theorem.

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\liminf_{\xi\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} = 0 \text{ and } \limsup_{\xi\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} = +\infty.$$

Then, the problem

$$\begin{cases} T(u) = f(u) + h(u), & x \in (0,1),\\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has an unbounded sequence of classical solutions.

3. **Main Results**

Let

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$

$$\widetilde{K}(t) = \int_0^t K(\xi)d\xi \quad \text{for all } t > 0$$

and

$$H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

Moreover, set

$$k = 2\delta^2\pi^2 \left(\frac{2048}{27} - \frac{32}{9}A + \frac{13}{40}B\right)^{-1}.$$

Then, $0 < k < 1/2$ (see [4] page 1168).

Put

$$\tau := \frac{\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}}{\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2}},$$

$$C := \liminf_{\xi \to +\infty} \frac{\int_0^1 \sup_{|t| \le \xi} F(x,t)dx}{\xi^2}$$

and

$$D := \limsup_{\xi \to +\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\xi)dx}{\xi^2}$$

We formulate our main result as follows.

Theorem 3.1. *Assume that*

(A_1) $F(x,\xi) \ge 0$ *for all* $(x,\xi) \in ([0, \frac{3}{8}] \cup [\frac{5}{8}, 1]) \times \mathbb{R}$*;*
(A_2) $C < k\tau D$.

Then, setting

$$\lambda_1 = \frac{2\pi^2\delta^2}{kD}\left(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2}\right) \text{ and } \lambda_2 = \frac{2\pi^2\delta^2}{C}\left(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\right)$$

for each $\lambda \in (\lambda_1, \lambda_2)$*, for every arbitrary* L^2*-Carathéodory function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ *whose potential* $G(x,t) = \int_0^t g(x,\xi)d\xi$ *for all* $(x,t) \in [0,1] \times \mathbb{R}$*, is a nonnegative function satisfying the condition*

$$G_\infty := \frac{1}{2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})} \lim_{\xi \to +\infty} \frac{\int_0^1 \sup_{|t| \le \xi} G(x,t)dx}{\xi^2} < +\infty \tag{3.1}$$

and for every $\mu \in [0, \mu_{g,\lambda}[$ *where* $\mu_{g,\lambda} = \frac{1}{G_\infty}(1 - \frac{\lambda}{\lambda_2})$*, the problem* (1.1) *has unbounded sequence of generalized solution in* X*.*

Proof. In order to apply Theorem 2.1, fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$ and let g be a function satisfies the condition (3.1). Since $\bar{\lambda} < \lambda_2$, we have $\mu_{g,\bar{\lambda}} = \frac{1}{G_\infty}(1-\frac{\bar{\lambda}}{\lambda_2}) > 0$. Now fix $\bar{\mu} \in (0, \mu_{g,\bar{\lambda}})$ and put $z_1 = \lambda_1$ and $z_2 = \frac{\lambda_2}{1+\frac{\bar{\mu}}{\bar{\lambda}}\lambda_2 G_\infty}$. If $G_\infty = 0$, clearly, $z_1 = \lambda_1$, $z_2 = \lambda_2$ and $\bar{\lambda} \in]z_1, z_2[$. If $G_\infty \neq 0$, since $\bar{\mu} < \mu_{g,\bar{\lambda}}$, we obtain $\frac{\bar{\lambda}}{\lambda_2} + \bar{\mu} G_\infty < 1$, and so $\frac{\lambda_2}{1+\frac{\bar{\mu}}{\bar{\lambda}}\lambda_2 G_\infty} > \bar{\lambda}$, namely, $\bar{\lambda} < z_2$. Hence, since $\bar{\lambda} > \lambda_1 = z_1$, one has $\bar{\lambda} \in]z_1, z_2[$. For each $u \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined

$$\Phi(u) = \frac{1}{2}\int_0^1 |u''(x)|^2 dx + \frac{1}{2}\widetilde{K}\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right) - \int_0^1 H(u(x))dx,$$

$$\Psi(u) = \int_0^1 F(x, u(x))dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_0^1 G(x, u(x))dx$$

and put

$$I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda}\Psi(u).$$

It is well known Φ is a differentiable functional whose differential at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\begin{aligned}\Phi'(u)(v) = &\int_0^1 u''(x)v''(x)dx + K\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)\\ &\times \int_0^1 (-Au'(x)v'(x) + Bu(x)v(x))dx - \int_0^1 h(u(x))v(x)dx\end{aligned}$$

for every $v \in X$. Moreover, Φ is a weakly sequentially weakly lower semicontinuous on X. Indeed, consider an arbitrary $u \in X$ and $\{u_n\}_{n=1}^{\infty} \subset X$ such that $u_n \rightharpoonup u$ in X. Due to the compact embedding X into $C([0,1])$, we have that $u_n \to u$ in $C([0,1])$. This implies

$$\tilde{K}\left(\int_0^1 (-A|u_n'(x)|^2 + B|u_n(x)|^2)dx\right) \to \tilde{K}\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right) \tag{3.2}$$

and

$$\int_0^1 H(u_n(x))dx \to \int_0^1 H(u(x))dx. \tag{3.3}$$

Moreover,the weakly sequentially lower semicontinuous property of the $\|.\|$ implies

$$\liminf_{n\to+\infty} \|u_n\|^2 \geq \|u\|^2. \tag{3.4}$$

From (3.2)–(3.4) we have

$$\liminf_{n\to+\infty} \Phi(u_n) \geq \Phi(u).$$

Since

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_0^1 |u''(x)|^2dx + \frac{1}{2}\widetilde{K}\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right) - \int_0^1 H(u(x))dx \\ &\geq \frac{1}{2}\min\{1, m_0\}\|u\|^2 - L\int_0^1 |u(x)|^2dx \\ &\geq \frac{1}{2}\left(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\right)\|u\|^2,\end{aligned}$$

taking the condition $\min\{1, m_0\} > \frac{L}{4\pi^2\delta^2}$ into account we observe Φ is coercive. Moreover, Φ is strongly continuous. The functional Ψ is also a differentiable functional whose differential at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_0^1 g(x, u(x))v(x)dx,$$

for every $v \in X$. On the other hand, the fact that X is compact embedding X into $C[0, 1]$ implies that the functional Ψ is continuously differentiable with compact derivative. Hence Ψ sequentially weakly (upper) continuous (see [31, Corollary 41.9]). Now set

$$Q(x, \xi) = F(x, \xi) + \frac{\bar{\mu}}{\bar{\lambda}}G(x, \xi)$$

for all $(x, \xi) \in [0, 1] \times \mathbb{R}$. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|t|\leq\xi_n} Q(x, t)dx}{\xi_n^2} = \lim_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\leq\xi} Q(x, t)dx}{\xi^2}. \tag{3.5}$$

For all $n \in \mathbb{N}$, put $r_n = 2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})\xi_n^2$. Since

$$\frac{1}{2}\left(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\right)\|u\|^2 \leq \Phi(u)$$

for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned}\Phi^{-1}(-\infty, r_n) &= \{u \in X;\ \Phi(u) < r_n\} \\ &= \left\{u \in X;\ \frac{1}{2}\left(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\right)\|u\|^2 < r_n\right\} \\ &\subseteq \{u \in X;\ |u(x)| \leq \xi_n \text{ for each } x \in [0, 1]\}.\end{aligned}$$

Note that $\Phi(0) = \Psi(0) = 0$. Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\ &\leq \frac{1}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})} \frac{\int_0^1 \sup_{|t|\leq\xi_n} Q(x,t)dx}{\xi_n^2} \\ &\leq \frac{1}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})} \Bigg[\frac{\int_0^1 \sup_{|t|\leq\xi_n} F(x,t)dx}{\xi_n^2} \\ &\quad + \frac{\bar{\mu}}{\bar{\lambda}} \frac{\int_0^1 \sup_{|t|\leq\xi_n} G(x,t)dx}{\xi_n^2} \Bigg]\end{aligned}$$

Moreover, Assumption (A_2) follows that

$$\liminf_{\xi\to+\infty} \frac{\displaystyle\int_0^1 \max_{|t|\leq\xi} F(x,t)dx}{\xi^2} < +\infty,$$

so we have

$$\lim_{n\to\infty} \frac{\displaystyle\int_0^1 \max_{|t|\leq\xi_n} F(x,t)dx}{\xi_n^2} < +\infty. \tag{3.6}$$

Then, (3.1) together with (3.6) ensures

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|t|\leq\xi_n} F(x,t)dx}{\xi_n^2} + \lim_{n\to\infty} \frac{\bar{\mu}}{\bar{\lambda}} \frac{\int_0^1 \max_{|t|\leq\xi_n} G(x,t)dx}{\xi_n^2} < +\infty,$$

which follows

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|t|\leq\xi_n}[F(x,t) + \frac{\bar{\mu}}{\bar{\lambda}}G(x,t)]dx}{\xi_n^2} < +\infty.$$

Therefore,

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \lim_{n\to\infty} \frac{\int_0^1 \max_{|t|\leq\xi_n}[F(x,t) + \frac{\bar{\mu}}{\bar{\lambda}}G(x,t)]dx}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})} < +\infty. \tag{3.7}$$

Since

$$\frac{\int_0^1 \max_{|t|\leq\xi_n}[F(x,t) + \frac{\bar{\mu}}{\bar{\lambda}}G(x,t)]dx}{\xi_n^2} \leq \frac{\int_0^1 \max_{|t|\leq\xi_n} F(x,t)dx}{\xi_n^2} + \frac{\bar{\mu}}{\bar{\lambda}} \frac{\int_0^1 \max_{|t|\leq\xi_n} G(x,t)dx}{\xi_n^2},$$

bearing (3.1) in mind, one has

$$\begin{aligned}&\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|t|\leq\xi} Q(x,t)dx}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\xi^2} \\ &\leq \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|t|\leq\xi} F(x,t)dx}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\xi^2} + \frac{\bar{\mu}}{\bar{\lambda}} G_\infty.\end{aligned} \tag{3.8}$$

Moreover, since G is non-negative, we have

$$\limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}}Q(x,\xi)dx}{\xi^2}\geq\limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\xi)dx}{\xi^2}. \tag{3.9}$$

Therefore, from (3.8) and (3.9), and from Assumption (A_2) and (3.7) we have

$$\bar\lambda\in]z_1,z_2[\subseteq\left]\frac{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})}{\limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}}Q(x,\xi)dx}{\xi^2}},\frac{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{\liminf_{\xi\to+\infty}\frac{\int_0^1\sup_{|t|\leq\xi}Q(x,t)dx}{\xi^2}}\right[\subseteq\left]0,\frac{1}{\gamma}\right[.$$

For the fixed $\overline{\lambda}$, the inequality (3.7) concludes that the condition (b) of Theorem 2.1 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n\to+\infty}\|u\|=+\infty$.

The other step is to prove that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us show that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\frac{1}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})}\limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\xi)dx}{\xi^2}$$

there exists a sequence $\{\eta_n\}$ of positive numbers and a constant θ such that $\eta_n\to\infty$ as $n\to\infty$ and

$$\frac{1}{\overline{\lambda}}<\theta\leq\frac{1}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta_n)dx}{\eta_n^2} \tag{3.10}$$

for each $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$ define

$$w_n(x):=\begin{cases}-\frac{64\eta_n}{9}(x^2-\frac{3}{4}x) & \text{if } x\in[0,\frac{3}{8}]\\ \eta_n & \text{if } x\in]\frac{3}{8},\frac{5}{8}]\\ -\frac{64\eta_n}{9}(x^2-\frac{5}{4}x+\frac{1}{4}) & \text{if } x\in]\frac{5}{8},1].\end{cases}$$

We clearly observe that $w_n\in X$ and $\|w_n\|^2=\frac{4\delta^2\pi^2}{k}\eta_n^2$ and so

$$\Phi(w_n)\leq\frac{2\pi^2\delta^2}{k}\left(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}\right)\eta_n^2. \tag{3.11}$$

On the other hand, bearing (A_1) in mind and since G is nonnegative, we have

$$\Psi(w_n)\geq\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta_n)dx. \tag{3.12}$$

It follows from (3.10)–(3.12) that

$$\begin{aligned}I_{\bar\lambda}(w_n)&=\Phi(w_n)-\bar\lambda\Psi(w_n)\\&\leq\frac{2\pi^2\delta^2}{k}\left(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}\right)\eta_n^2-\bar\lambda\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta_n)dx\\&<\frac{2\pi^2\delta^2}{k}\left(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}\right)\eta_n^2(1-\bar\lambda\theta)\end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\bar{\lambda}\theta > 1$ and $\eta_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} I_{\bar{\lambda}}(w_n) = -\infty$$

and it follows that $I_{\bar{\lambda}}$ has no global minimum. Therefore, taking the fact

$$\Phi(u) \leq \frac{1}{2}\left(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2}\right)\|u\|^2$$

into account, by Theorem 2.1 (b), there exist a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda}}$ such that $\lim_{n\to\infty}\|u_n\| = +\infty$. Since the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_{\bar{\lambda}}(u) = 0$ and they are also generalized solutions, the conclusion is achieved. □

Now we present the following example to illustrate the result.

Example 3.2. Let $K(t) = 2 + \sin t$ for all $t \in [0, +\infty)$,

$$f(x,t) = \begin{cases} e^{x^2}(5t^4(1-\cos(\frac{e^t}{t})) + t^3(t-1)e^t\sin(\frac{e^t}{t})) & \text{if } (x,t) \in [0,1]\times(\mathbb{R}-\{0\}) \\ 0 & \text{if } (x,t) \in [0,1]\times\{0\} \end{cases}$$

$g(x,t) = g(t) = t+1$ for all $x \in [0,1]$ and $t \in \mathbb{R}$, and $h(t) = \arctan t$ for all $t \in \mathbb{R}$ and $A = B = 1$. Clearly, $m_0 = 1$, $m_1 = 3$, $\delta = \sqrt{1 - \frac{1}{\pi^2}}$ and $L = 1$. A simple calculation shows that

$$F(x,t) := \begin{cases} e^{x^2}t^5(1-\cos(\frac{e^t}{t})) & \text{if } (x,t) \in [0,1]\times(\mathbb{R}-\{0\}) \\ 0 & \text{if } (x,t) \in [0,1]\times\{0\}. \end{cases}$$

It is clear that

$$C := \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\leq\xi} F(x,t)dx}{\xi^2} = 0,$$

$$D := \limsup_{\xi\to+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\xi)dx}{\xi^2} = +\infty$$

and

$$G_\infty = \frac{1}{2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})} \lim_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\leq\xi} G(x,t)dx}{\xi^2} = \frac{1}{4\pi^2 - 5}.$$

Hence, by Theorem 3.1, for every $\lambda \in (0, +\infty)$ and $\mu \in [0, 4\pi^2 - 5)$ the problem

$$\begin{cases} u^{iv} + (2 + \sin\left(\int_0^1(-|u'(x)|^2 + |u(x)|^2)dx\right)(u'' + u)) \\ \qquad = \lambda f(x,u) + \mu(u+1) + \arctan u, & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has a sequence of generalized solutions which is unbounded in X.

Remark 3.3. In Theorem 3.1, if we assume that the function f is nonnegative, the assumption (A_2) can be written as

$$\liminf_{\xi\to+\infty}\frac{\int_0^1 F(x,\xi)dx}{\xi^2} < k\tau \limsup_{\xi\to+\infty}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\xi)dx}{\xi^2}$$

as well as $\mu_{g,\lambda} = \frac{1}{G_\infty}(1 - \frac{\lambda}{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}\liminf_{\xi\to+\infty}\frac{\int_0^1 F(x,\xi)dx}{\xi^2})$. Moreover, in the autonomous case, putting $F(t) = \int_0^t f(\xi)d\xi$ for all $t\in\mathbb{R}$, the assumption (A_2) assumes the form

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} < \frac{k\tau}{4}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2},$$

and in this case, we have

$$\lambda_1 = \frac{8\pi^2\delta^2(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})}{k\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}} \text{ and } \lambda_2 = \frac{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}}$$

and $\mu_{g,\lambda} = \frac{1}{G_\infty}(1 - \frac{\lambda}{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2})$.

Here we point out the following consequence of Theorem 3.1 with $\mu = 0$.

Corollary 3.4. *Assume that Assumption (A_1) in Theorem 3.1 holds and*

$$D > \frac{2\pi^2\delta^2}{k}\left(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}\right)$$

and

$$C < 2\pi^2\delta^2\left(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2}\right).$$

Then, the problem

$$\begin{cases} T(u) = f(x,u) + h(u), & x\in(0,1),\\ u(0)=u(1)=u''(0)=u''(1)=0 \end{cases}$$

has an unbounded sequence of generalized solution in X.

Remark 3.5. Theorem 2.2 is an immediately consequence of Corollary 3.4.

Now, we give the following consequence of the main result.

Corollary 3.6. *Let $f_1 : [0,1]\to\mathbb{R}$ be a nonnegative continuous function, and put $F_1(t) = \int_0^t f_1(s)ds$ for all $t\in\mathbb{R}$. Assume that*

(B_1) $\liminf_{\xi\to+\infty}\frac{F_1(\xi)}{\xi^2} < +\infty$,
(B_2) $\limsup_{\xi\to+\infty}\frac{F_1(\xi)}{\xi^2} = +\infty$.

Then for every $\alpha, \beta \in L^2([0,1])$ *with* $\min_{x\in[0,1]}\{\alpha(x), \beta(x)\} \geq 0$ *and* $\alpha \neq 0$*, and for every nonnegative continuous function* $f_2 : \mathbb{R} \to \mathbb{R}$ *satisfying* $\sup_{\xi\in\mathbb{R}} F_2(\xi) \geq 0$ *and* $\liminf_{\xi\to+\infty} \frac{F_2(\xi)}{\xi^2} > -\infty$ *where* $F_2(t) = \int_0^t f_2(\xi)d\xi$ *for all* $t \in \mathbb{R}$*, for each*

$$\lambda \in \left(0, \frac{2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})}{(\int_0^1 \beta(x)dx) \liminf_{\xi\to+\infty} \frac{F_1(\xi)}{\xi^2}}\right)$$

the problem

$$\begin{cases} T(u) = \lambda(\alpha(x)f_1(u) + \beta(x)f_2(u)) + h(u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has an unbounded sequence of generalized solution in X*.*

Proof. Put $f(x,t) = \alpha(x)f_1(u) + \beta(x)f_2(u)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. From the assumption (B_2) and the condition $\liminf_{\xi\to+\infty} \frac{F_2(\xi)}{\xi^2} > -\infty$ we have

$$\limsup_{\xi\to+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\xi)dx}{\xi^2} = \limsup_{\xi\to+\infty} \frac{F_1(\xi)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx + F_2(\xi)\int_{\frac{3}{8}}^{\frac{5}{8}} \beta(x)dx}{\xi^2} = +\infty.$$

Moreover, from the assumption (B_1) and the condition $\sup_{\xi\in\mathbb{R}} F_2(\xi) \geq 0$ we obtain

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|t|\leq\xi} F(x,t)dx}{\xi^2} \leq \left(\int_0^1 \alpha(x)dx\right) \liminf_{\xi\to+\infty} \frac{F_1(\xi)}{\xi^2} < +\infty.$$

Hence, the conclusion follows from Theorem 3.1 with $\mu = 0$. □

Remark 3.7. We point out that the same statements of the above given results can be obtained by considering

$$K(t) = a_1t + a_2 \text{ for } t \in [\alpha, \beta]$$

where a_1, a_2, α and β are positive numbers. In fact, in this special case we have

$$\widetilde{K}(t) = \int_0^t [a_1s + a_2]\,ds = \frac{(a_1t + a_2)^2}{2a_1} - \frac{a_2^2}{2a_1} \quad \text{for } t \geq 0,$$

$$m_0 = a_1\alpha + a_2 \text{ and } m_1 = a_1\beta + a_2.$$

Remark 3.8. Replacing $\xi \to +\infty$ with $\xi \to 0^+$ in Theorem 3.1, by the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 3.1 instead of (b), we can obtain a sequence of pairwise distinct generalized solutions to the problem (1.1) which converges uniformly to zero.

We finally present the following example to illustrate the result.

Example 3.9. Let $K(t) = 2 + \cos t$ for all $t \in [0, \infty)$,

$$f(t) = \begin{cases} 2t + 2\alpha t\sin^2(\ln t) + 2\alpha t\cos(\ln t)\sin(\ln t) & \text{if } t \in]0, +\infty[\\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

where $\alpha > 51$ is a real number, $h(t) = \sqrt{t^2+1} - 1$ for all $t \in \mathbb{R}$, $A = B = 1$, $\mu = 0$. In this case, we have $\delta = \sqrt{1 - \frac{1}{\pi^2}}$, $L = 1$, $\tau = \frac{4\pi^2-5}{12\pi^2-11}$, $k = (2\pi^2 - 2)\frac{1080}{78431}$. Putting $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$, we have

$$F(t) = \begin{cases} t^2(1 + \alpha \sin^2(\ln t)) & \text{if } t \in]0, +\infty[\\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

Setting $a_n = e^{-n\pi}$, $b_n = e^{-(\frac{2n+1}{2})\pi}$ for every $n \in \mathbb{N}$, one has

$$C := \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \le \lim_{n\to\infty} \frac{F(a_n)}{a_n^2} = 1 \tag{3.13}$$

and

$$D := \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} \geqslant \lim_{n\to\infty} \frac{F(b_n)}{b_n^2} = \alpha + 1. \tag{3.14}$$

By (3.13) and (3.14) we obtain

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} < \frac{k\tau}{4} \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2},$$

as well as

$$\left(\frac{2(12\pi^2 - 11)}{(\alpha+1)k}, \frac{4\pi^2 - 5}{2} \right) \subset \left(\frac{8\pi^2\delta^2(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2})}{k \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2}}, \frac{2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})}{\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2}} \right).$$

Therefore, by applying Remark 3.3 and Remark 3.8, for every $\lambda \in \left(\frac{2(12\pi^2-11)}{(\alpha+1)k}, \frac{4\pi^2-5}{2}\right)$ the problem

$$\begin{cases} u^{iv} + (2 + \cos\left(\int_0^1 (-|u'(x)|^2 + |u(x)|^2)dx\right)(u'' + u) \\ \qquad = \lambda f(u) + \sqrt{u^2+1} - 1, & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has a sequence of pairwise distinct generalized solutions which converges uniformly to zero.

Remark 3.10. If f, g are non-negative functions, arguing as given in the proof of [17, Lemma 3.4.] one has, the generalized solutions ensured by the previous theorems are non-negative. In addition, if either $f(x,0) \neq 0$ for all $x \in (0,1)$ or $g(x,0) \neq 0$ for all $x \in (0,1)$, or both are true, the solutions are positive.

REFERENCES

[1] G. A. Afrouzi, S. Heidarkhani and D. O'Regan, Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem, *Taiwanese J. Math.* 15:201–210, 2011.

[2] C. O. Alves, F. S. J. A. Corrêa and T. F. Ma, Positive solutions for a quasilinear elliptic equations of Kirchhoff type, *Comput. Math. Appl.* 49:85–93, 2005.

[3] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* 348:305–330, 1996.

[4] G. Bonanno and B. Di Bella, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.* 343:1166–1176, 2008.

[5] G. Bonanno and B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equations, *Nonl. Differ. Equ. Appl.* 18:357–368, 2011.

[6] G. Bonanno, B. Di Bella and D. O'Regan, Non-trivial solutions for nonlinear fourth-order elastic beam equations, *Comput. Math. Appl.* 62:1862–1869, 2011.

[7] G. Bonanno and G. Molica Bisci, A remark on perturbed elliptic Neumann problems, *Studia Univ. "Babeş-Bolyai", Mathematica*, Volume **LV**, Number 4, December 2010.

[8] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* 2009:1–20, 2009.

[9] M. Chipot and B. Lovat, Some remarks on non local elliptic and parabolic problems, *Nonlinear Anal.* 30:4619–4627, 1997.

[10] G. D'Agui, S. Heidarkhani and G. Molica Bisci, Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian, *Electron. J. Qual. Theory Diff. Eqns.* No. 24:1–14, 2013.

[11] M. Galewski and P. Kowalski, Three solutions to discrete anisotropic problems with two parameters, *Cent. Eur. J. Math.* 12(10):1403–1415, 2014.

[12] J.R. Graef, S. Heidarkhani and L. Kong, A variational approach to a Kirchhoff-type problem involving two parameters, *Results. Math.* 63:877–889, 2013.

[13] J. R. Graef, S. Heidarkhani and L. Kong, Infinitely many solutions for systems of Sturm-Liouville boundary value problems, *Results. Math.* 66:327–341, 2014.

[14] M. Ferrara, S. Khademloo and S. Heidarkhani, Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems, *Appl. Math. Comput.* 234:316–325, 2014.

[15] X. He and W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, *Nonlinear Anal.* 70:1407–1414, 2009.

[16] S. Heidarkhani, Infinitely many solutions for systems of n two-point Kirchhoff-type boundary value problems, *Ann. Polon. Math.* 107.2:133–152, 2013.

[17] S. Heidarkhani, M. Ferrara and S. Khademloo, Nontrivial solutions for one-dimensional fourth-order Kirchhoff-type equations, *Mediterr. J. Math.* DOI 10.1007/s00009-014-0471-5.

[18] S. Heidarkhani, S. Khademloo and A. Solimaninia, Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem, *Portugal. Math.* (N.S.) Vol. 71, Fasc. 1:39–61, 2014.

[19] G. Kirchhoff, *Vorlesungen uber mathematische Physik: Mechanik,* Teubner, Leipzig, 1883.

[20] A. C. Lazer and P. J. Mckenna, Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, *SIAM Rev.* 32:537–578, 1990.

[21] T. F. Ma, Existence results and numerical solutions for a beam equation with nonlinear boundary conditions, *Appl. Numer. Math.* 47:189–196, 2003.

[22] A. Mao and Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, *Nonlinear Anal.* 70:1275–1287, 2009.

[23] L. A. Peletier, W. C. Troy and R. C. A. M. Van der Vorst, Stationary solutions of a fourth order nonlinear diffusion equation, (Russian) Translated from the English by V. V. Kurt. *Differentsialnye Uravneniya* 31:327–337, 1995. English translation in *Differential Equations* 31:301–314, 1995.

[24] K. Perera and Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differ. Eqs.* 221:246–255, 2006.

[25] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* 113:401–410, 2000.

[26] B. Ricceri, On an elliptic Kirchhoff-type problem depending on two parameters, *J. Global Optimization* 46:543–549, 2010.

[27] S. M. Shahruz and S. A. Parasurama, Suppression of vibration in the axially moving Kirchhoff string by boundary control, *J. Sound and Vibration* 214:567–575, 1998.

[28] J. Simon, *Regularitè de la solution d'une equation non lineaire dans* R^N, in: Journes d'Analyse Non Linaire (Proc. Conf., Besanon, 1977), (P. Bénilan, J. Robert, eds.), Lecture Notes in Math., 665, pp. 205-227, Springer, Berlin-Heidelberg-New York, 1978.

[29] F. Wang and Y. An, Existence and multiplicity of solutions for a fourth-order elliptic equation, *Bound. Value Probl.* 2012(6):1–9, 2012.

[30] F. Wang, M. Avci and Y. An, Existence of solutions for fourth order elliptic equations of Kirchhoff-type, *J. Math. Anal. Appl.* 409:140–146, 2014.

[31] E. Zeidler, *Nonlinear functional analysis and its applications,* Springer, Vol. II. Berlin-Heidelberg-New York, 1985.