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BEHAVIOR OF SOLUTIONS OF NONLINEAR FUNCTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. The authors consider the nonlinear functional Volterra integro-differential equation with multiple delays

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s) f_i(x(s))ds.$$

They give sufficient conditions so that solutions are bounded, belong to L^1, or belong to L^2. They also prove the stability and global asymptotic stability of the zero solution. Their technique of proof involves defining appropriate Lyapunov functionals.

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1. Introduction

In 2009, Becker [4] considered the scalar linear Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_{0}^{t} b(t,s)x(s)ds$$

for $t \geq 0$, where a and b are real-valued functions that are continuous on their respective domains. He discussed the asymptotic behavior of solutions of this equation by using Lyapunov functionals.

In this paper, we consider the scalar nonlinear functional Volterra integro-differential equation with multiple delays

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s) f_i\left(x(s)\right) ds, \tag{1.1}$$

where a, f and b_i are real-valued and continuous functions on their respective domains. We will investigate various questions about the behavior of solutions of (1.1) including their boundedness, integrability, and the stability and global asymptotic stability

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of the zero solution. Our results differ from those currently in the literature (see, [3–12, 14, 16–20] and the references therein).

We let $L^1[0,\infty)$ denote the set of all real-valued functions g for which $\int_0^\infty |g(s)|ds < \infty$ and $L^2[0,\infty)$ denote the set of all real-valued functions h that are square integrable on $[0,\infty)$, i.e., $\int_0^\infty |h(s)|^2 ds < \infty$.

2. Main Results

First, we establish sufficient conditions for all solutions of Eq. (1.1) to be bounded and belong to $L^2[0,\infty)$. Then, under additional conditions, we show that the solutions tend to zero as $t \to \infty$ as well. Our covering assumptions on the functions in equation (1.1) are as follows. Let $\Omega := \{(t,s) : 0 \le s \le t < \infty\}$ and let $\tau = \max_{1\le i\le n} \tau_i$. Assume that:

(H1) The functions $a : [0,\infty) \to [0,\infty)$, $f_i : \mathbb{R} \to \mathbb{R}$, and $b_i : \Omega \to \mathbb{R}$ are continuous for each $i = 1, 2, \dots, n$;

(H2) there exists a constant $\alpha > 0$ such that $|f_i(x)| \le \alpha|x|$ for all $x \in \mathbb{R}$ and $i = 1, 2, \dots, n$.

Our first result shows that solutions of equation (1.1) are bounded and the zero solution is stable.

Theorem 2.1. *Assume conditions (H1)–(H2) hold. If*

$$a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \ge 0 \tag{2.1}$$

for all $t \ge t_0 - \tau \ge 0$ *and*

$$a(t) - \sum_{i=1}^{n} \int_{0}^{t} \alpha |b_i(t,s)| ds \ge 0 \tag{2.2}$$

for all $t \ge 0$*, then all solutions of equation* (1.1) *are bounded and the zero solution of* (1.1) *is stable.*

Proof. For any $t_0 \ge 0$ and initial function $\varphi \in C[t_0-\tau, t_0]$, let $x(t) = x(t, t_0, \varphi)$ denote the solution of Eq. (1.1) on $[t_0 - \tau, \infty)$ such that $x(t) = \varphi(t)$ on $[t_0 - \tau, t_0]$. Define the functional

$$V : [0,\infty) \times C[0,\infty) \to [0,\infty)$$

by

$$V(t,\psi(\cdot)) = \psi^2(t) + \int_0^t \left\{ a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \right\} \psi^2(s) ds. \tag{2.3}$$

From (2.1), it is clear that

$$V(t,\psi(\cdot)) \ge \psi^2(t) \quad \text{for all} \quad t \ge t_0 - \tau. \tag{2.4}$$

Differentiating (2.3), we obtain

$$
\begin{aligned}
V'(t) = \frac{d}{dt}V(t,x(t)) &= 2x(t)x'(t) + a(t)x^2(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) \\
&\quad - \int_0^t \sum_{i=1}^{n}\alpha|b_i(t,s)|x^2(s)ds \\
&= 2x(t)\left[-a(t)x(t) + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} b_i(t,s)f_i(x(s))ds\right] + a(t)x^2(t) \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&= -a(t)x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} b_i(t,s)f_i\left(x(s)\right)ds \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&\le -a(t)x^2(t) + 2|x(t)|\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,s)||x(s)|ds \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&\le -a(t)x^2(t) + \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|\left(x^2(t) + x^2(s)\right)ds \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&= -a(t)x^2(t) + \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|ds\,x^2(t) \\
&\quad + \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|x^2(s)ds \\
&= -a(t)x^2(t) + \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|ds\,x^2(t) \\
&\quad - \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\,x^2(t) \\
&= -\left\{a(t) - \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s)|ds + \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u)|du\right\}x^2(t).
\end{aligned}
$$

Hence,

$$V'(t) \le -\left\{ a(t) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s)| ds + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u)| du \right\} x^2(t) \tag{2.5}$$

for all $t \ge 0$. From condition (2.2), we see that

$$V'(t) \le 0 \quad \text{for} \quad t \ge t_0.$$

This, together with (2.4), shows that all solutions of (1.1) are bounded, and in fact,

$$x^2(t) \le V(t) \le V(t_0) \tag{2.6}$$

for all $t \ge t_0$. From the above estimate and the fact that

$$V(t_0) = \varphi^2(t_0) + \int_0^{t_0} \left\{ a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t_0} \alpha |b_i(t,u)| du \right\} \varphi^2(s) ds \le |\varphi|_{t_0}^2 B_0,$$

where

$$B_0 = 1 + \int_0^{t_0} \left\{ a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t_0} \alpha |b_i(t,u)| du \right\} ds,$$

we obtain

$$|x(t)| \le |\varphi|_{t_0} \sqrt{B_0} \tag{2.7}$$

for all $t \ge t_0 \ge s - \tau$. It immediately follows that the zero solution of (1.1) is stable, i.e., for any $\varepsilon > 0$, choose $\delta = \varepsilon / \sqrt{B_0}$ and so for $\varphi \in C[t_0 - \tau, t_0]$ with $|\varphi|_{t_0} < \delta$, we have

$$|x(t)| \le \delta \sqrt{B_0} = \varepsilon. \tag{2.8}$$

□

In our next theorem, we show that solutions are square integrable, i.e., they belong to the class L^2.

Theorem 2.2. *If in addition to conditions (H1)–(H2) and* (2.1)–(2.2)*, there exist* $t_1 \ge t_0$ *and a constant* $k > 0$ *such that either*

$$a(t) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,u)| du \ge k \quad \text{for} \quad t \ge t_1, \tag{2.9}$$

or

$$a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \ge k \quad \text{for} \quad t \ge s - \tau \ge t_0, \tag{2.10}$$

then every solution of Eq. (1.1) *belongs to* $L^2[0, \infty)$.

Proof. From Theorem 2.1, any solution $x(t)$ of (1.1) is bounded and satisfies (2.5) and (2.6). If (2.9) holds, then from (2.5) we have

$$V'(t) \leq -kx^2(t)$$

for $t \geq t_1$. Integrating, we obtain

$$k \int_{t_1}^{t} x^2(s)ds \leq V(t_1) - V(t) \leq V(t_1)$$

so $x \in L^2[0, \infty)$.

If (2.10) holds, then from the definition of V, we have

$$x^2(t) + k \int_{t_1}^{t} x^2(s)ds \leq V(t) \leq V(t_1), \tag{2.11}$$

so again $x \in L^2[0, \infty)$. This proves the theorem. □

In order to show that the zero solution is globally asymptotically stable, we will have to require an additional condition.

Theorem 2.3. *Let conditions (H1)–(H2),* (2.1)–(2.2)*, and either* (2.9) *or* (2.10) *hold. If there is a constant $K > 0$ such that*

$$a(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,s)| ds \leq K \tag{2.12}$$

for all $t \geq t_1$, then the zero solution of (1.1) *is globally asymptotically stable.*

Proof. By Theorem 2.2, we have that every solution belongs to $L^2[0, \infty)$. From (1.1) and (2.7), we have

$$|x'(t)| \leq a(t)|\varphi|_{t_0}\sqrt{B_0} + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,s)| ds |\varphi|_{t_0}\sqrt{B_0} \tag{2.13}$$

$$\leq |\varphi|_{t_0}\sqrt{B_0}K, \tag{2.14}$$

so $x'(t)$ is bounded. This together with the fact that $x \in L^2[0, \infty)$ implies that $x(t) \to 0$ as $t \to \infty$. Thus, the zero solution of (1.1) is globally asymptotically stable. □

As simple examples such as $x(t) = \frac{1}{t+1}$ show, a function belonging to L^2 may not belong to L^1. We next develop conditions under which solutions of (1.1) do in fact belong to the class $L^1[0, \infty)$.

Theorem 2.4. *If* (2.1) *holds, then all solutions of* (1.1) *are bounded and the zero solution of* (1.1) *is stable. If, in addition, there are constants $k_1 > 0$ and $0 \leq \beta < 1$ such that*

$$a(t) \geq k_1 \tag{2.15}$$

and

$$\beta a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \geq 0, \tag{2.16}$$

then all solutions of (1.1) *belong to* $L^1[0,\infty)$*. Moreover, if* (2.12) *also holds, then* $x \in L^2[0,\infty)$ *and the zero solution of* (1.1) *is globally asymptotically stable.*

Proof. Define the functional

$$V_1 : [0,\infty) \times C[0,\infty) \to [0,\infty)$$

by

$$V_1(t,\psi(\cdot)) = |\psi(t)| + \int_0^t \left\{ a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \right\} |\psi(s)| ds. \tag{2.17}$$

As described in [15, p. 26] and pointed out by Becker [4], for a continuously differentiable function $h(t)$, $|h(t)|$ has a right derivative D_r given by

$$D_r|h(t)| = \begin{cases} h'(t) \operatorname{sgn} h(t), & \text{if } h(t) \neq 0 \\ |h'(t)|, & \text{if } h(t) = 0. \end{cases}$$

We then have

$$\begin{aligned} D_r V_1(t) = D_r V_1(t,x(t)) &= -a(t)|x(t)| + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s) f_i(x(s)) ds \\ &\quad + a(t)|x(t)| - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u)| du \, |x(t)| \\ &\quad - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s)| x(s)| ds \\ &\leq \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,s)||x(s)| ds - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u)| du \, |x(t)| \\ &\quad - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s)||x(s)| ds \\ &\leq - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,s)| ds \, |x(t)| \\ &\leq 0. \end{aligned}$$

Now $V_1(t) \geq |x(t)|$ by (2.1), so the boundedness of solutions and the stability of the zero solution follow as in Theorem 2.1.

Next we modify the functional in (2.17), i.e., consider

$$V_\beta(t,\psi(\cdot)) = |\psi(t)| + \int_0^t \left\{ \beta a(s) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t,u)| du \right\} |\psi(s)| ds. \tag{2.18}$$

Then,

$$
\begin{aligned}
D_r V_\beta'(t) &= -a(t)|x(t)| + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s) f_i(x(s)) ds \\
&\quad + \beta a(t)|x(t)| - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u)| du \, |x(t)| - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s)| |x(s)| ds \\
&\le -(1-\beta) a(t)|x(t)| - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u)| du \, |x(t)| \\
&\le -(1-\beta) k_1 |x(t)| \\
&\le 0
\end{aligned}
$$

by (2.15). Integrating, we obtain

$$
(1-\beta) k \int_{t_0}^{t} |x(s)| ds \le V_\beta(t_0) - V_\beta(t)
$$

so $x \in L^1[t_0, \infty)$.

Now if (2.12) also holds, as in the proof of Theorem 2.3 we can easily show that $|x'(t)|$ is bounded. This, together with the fact that $x \in L^1[0,\infty)$ implies $x \in L^2[0,\infty)$, guarantees that $x(t) \to 0$ as $t \to \infty$, and so the zero solution of (1.1) is globally asymptotically stable. □

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, Vol. 55., Government Printing Office, Washington, D.C., 1964.

[2] I. Barbălat, Systèmes d'équations différentielles d'oscillations non linéaires, *Rev. Math. Pures Appl.* 4 (1959), 267–270.

[3] L. C. Becker, Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint, *Electron. J. Qual. Theory Differ. Equ.* 2006, No. 14, 22 pp.

[4] L. C. Becker, Uniformly continuous solutions of Volterra equations and global asymptotic stability, *Cubo* 11 (2009), 1–24.

[5] T. A. Burton, Stability theory for Volterra equations, *J. Differential Equations* 32 (1979), 101–118.

[6] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-differential Equations*, Mathematics in Science and Engineering, Vol. 178, Academic Press, Orlando, 1985.

[7] T. A. Burton, *Volterra Integral and Differential Equations*, Second edition, Mathematics in Science and Engineering, Vol. 202, Elsevier, Amsterdam, 2005.

[8] T. A. Burton, A Liapunov functional for a linear integral equation, *Electron. J. Qual. Theory Differ. Equ.* 2010, No. 10, 10 pp.

[9] T. A. Burton and W. E. Mahfoud, Stability criteria for Volterra equations, *Trans. Amer. Math. Soc.* 279 (1983), 143–174.

[10] T. A. Burton and J. R. Haddock, Qualitative properties of solutions of integral equations, *Nonlinear Anal.* 71 (2009), 5712–5723.

[11] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Mathematics in Science and Engineering, Vol. 104. Academic Press, New York, 1973.

[12] C. Corduneanu, *Principles of Differential and Integral Equations*, Second edition, Chelsea, Bronx York, 1977.

[13] W. Fulks, *Advanced Calculus: An Introduction to Analysis*, Wiley, New York, 1961.

[14] G. Gripenberg, S. Q. Londen, and O. Staffans, Volterra integral and functional equations, in: *Encyclopedia of Mathematics and its Applications*, Vol. 34. Cambridge University Press, Cambridge, 1990.

[15] P. Hartman, *Ordinary Differential Equations*, Reprint of the second edition, Birkhäuser, Boston, 1982.

[16] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications, Vol. I: Ordinary Differential Equations*, Mathematics in Science and Engineering, Vol. 55-I, Academic Press, New York, 1969.

[17] H. Logemann, R. Hartmut and P. Eugene, Asymptotic behaviour of nonlinear systems, *Amer. Math. Monthly* 111 (2004), 864–889.

[18] R. K. Miller, Asymptotic stability properties of linear Volterra integrodifferential equations, *J. Differential Equations* 10 (1971), 485–506.

[19] O. J. Staffans, A direct Lyapunov approach to Volterra integro-differential equations, *SIAM J. Math. Anal.* 19 (1988), 879–901.

[20] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Applied Mathematical Sciences, Vol. 14, Springer-Verlag, New York, 1975.