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LERAY-SCHAUDER AND FURI-PERA TYPE RESULTS BASED ON Φ-EPI MAPS

DONAL O'REGAN

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway Ireland
donal.oregan@nuigalway.ie

ABSTRACT. In this paper using the notion of Φ-epi maps we present new and abstract Leray-Schauder and Furi-Pera type results.

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1. INTRODUCTION

The 0-epi maps were introduced by Furi, Martelli and Vignoli in [1]. These maps were generalized by Gabor, Gorniewicz and Slosarski in [3]. More recently the general notion of Φ-epi maps for a general class of maps was presented by O'Regan in [6] (see also [4]) and these result allow us to study coincidence points (i.e. $F(x) \cap \Phi(x) \neq \emptyset$) of the maps F and Φ. In this paper using the theory in [6] we begin by presenting some new Leray-Schauder alternatives for a very general class of maps. Next we present a very general Furi-Pera type result (see [2, 5]) based on Leray-Schauder type alternatives.

2. LERAY-SCHAUDER AND FURI-PERA RESULTS

We begin this section by recalling the following definitions and results from the literature [6].

Let E be a Hausdorff topological space and U an open subset of E. We will consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2.2. We say $F \in B(\overline{U}, E)$ if $F \in \mathbf{B}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map.

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Now we fix a $\Phi \in B(\overline{U}, E)$.

Definition 2.3. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. We say $F \in B_{\Phi}(\overline{U}, E)$ if $F \in B(\overline{U}, E)$ and $F(x) \subseteq \Phi(x)$ for $x \in \partial U$.

Definition 2.5. A map $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi if for every map $G \in B_{\Phi}(\overline{U}, E)$ there exists $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Remark 2.6. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi. Then there exists $x \in U$ with $F(x) \cap \Phi(x) \neq \emptyset$ (take $G = \Phi$ in Definition 2.5).

In [6] we established the following Leray-Schauder alternative.

Theorem 2.7. *Let E be a normal topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and $G \in B(\overline{U}, E)$ and assume the following condition holds:*

$$\left\{ \begin{array}{l} \mu(\,.\,)\, G(\,.\,) + (1 - \mu(\,.\,))\, \Phi(\,.\,) \in B(\overline{U}, E) \ \ \textit{for any} \\ \textit{continuous map } \mu : \overline{U} \to [0,1] \textit{ with } \mu(\partial U) = 0. \end{array} \right. \tag{2.1}$$

Then either

(A1). there exists $x \in \overline{U}$ with $F(x) \cap G(x) \neq \emptyset$,

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $F(x) \cap [\lambda G(x) + (1-\lambda)\Phi(x)] \neq \emptyset$,

holds.

Remark 2.8. We can remove the assumption that E is normal in the statement of Theorem 2.7 provided we have that (so we need to put conditions on the maps)

$$D = \left\{ x \in \overline{U} : F(x) \cap [tG(x) + (1-t)\Phi(x)] \neq \emptyset \text{ for some } t \in [0,1] \right\}$$

is relatively compact. The existence of the map μ in [6] is then guaranteed since topological vector spaces are completely regular (i.e. in the proof in [6] there exists a map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 0$).

A special case of Theorem 2.7 is the following applicable result of Leray-Schauder type.

Theorem 2.9. *Let E be a normal topological vector space and U an open convex subset of E with $0 \in U$. Suppose $G \in B(\overline{U}, E)$ and (2.1) holds. In addition assume the following conditions hold:*

$$i \in \mathbf{A}(\overline{U}, E) \textit{ where } i \textit{ is the identity map} \tag{2.2}$$

$$\Phi(\partial U) \subseteq U \tag{2.3}$$

(2.4) $\overline{U}$ *is a retract of* E *i.e. there exists a retraction (continuous)* $r : E \to \overline{U}$

$$\text{any map } \Psi \in A(E,E) \text{ has a fixed point} \tag{2.5}$$

and

$$\begin{cases} \text{for any continuous map } \eta : E \to [0,1] \text{ with } \eta(E\backslash U) = 0 \\ \text{and } H \in B_\Phi(\overline{U}, E) \text{ the map } J \in A(E,E) \\ \text{where } J(x) = \eta(x) H(r(x)). \end{cases} \tag{2.6}$$

Then either

(A1). there exists $x \in \overline{U}$ *with* $x \in G(x)$

or

(A2). there exists $x \in \partial U$ *and* $\lambda \in (0,1)$ *with* $x \in \lambda G(x) + (1-\lambda)\Phi(x)$

holds.

Proof. Let $F(x) = i(x)$. Note $F \in A_{\partial U}(\overline{U}, E)$ since if $x \in \partial U$ we have $F(x) \cap \Phi(x) = \emptyset$ (note for $x \in \partial U$ we have $x \notin \Phi(x)$ from (2.3)). The result follows from Theorem 2.7 if we show F is Φ-epi. Let $H \in B_\Phi(\overline{U}, E)$ (i.e. $H \in B(\overline{U}, E)$ with $H(x) \subseteq \Phi(x)$ for $x \in \partial U$). We must show there exists $x \in U$ with $x \in H(x)$. Let

$$\Omega = \left\{ x \in \overline{U} : x \in \lambda H(x) \text{ for some } \lambda \in [0,1] \right\}.$$

Now Ω is closed (since H is upper semicontinuous) and $\Omega \subset U$ since if there exists $x \in \partial U$ and $\lambda \in [0,1]$ with $x \in \lambda H(x)$ then since $H(y) \subseteq \Phi(y)$ for $y \in \partial U$ we have $x \in \lambda \Phi(x)$ and so $x \in U$ (recall $\Phi(\partial U) \subseteq U$, U is convex and $0 \in U$), a contradiction. Now Urysohn's Lemma guarantees that there exists a continuous map $\eta : E \to [0,1]$ with $\eta(\Omega) = 1$ and $\eta(E\backslash U) = 0$. Define a map J by $J(x) = \eta(x) H(r(x))$. Now (2.6) guarantees that $J \in A(E,E)$ and (2.5) guarantees that there exists $x \in E$ with $x \in \eta(x) H(r(x))$. If $x \in E\backslash U$ then $\eta(x) = 0$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \eta(x) H(x)$. As a result $x \in \Omega$ so $\eta(x) = 1$. Thus $x \in H(x)$. □

Remark 2.10. We note from the proof above that we could replace U convex and (2.3) with the condition

$$x \notin \lambda \Phi(x) \text{ for } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.7}$$

Note in (2.7) we have in fact $\lambda \in [0,1]$ since $x \neq 0$ if $x \in \partial U$ (recall $0 \in U$).

Remark 2.11. We can remove the assumption that E is normal in the statement of Theorem 2.9 provided we have that (so we need to put conditions on the maps) D (see Remark 2.8) and Ω (see the proof of Theorem 2.9) are relatively compact (note the existence of the map η in Theorem 2.9 is then guaranteed since topological vector spaces are completely regular).

In our next result E will be a locally convex topological vector space. The more general case when E is a topological vector space will be presented in Remark 2.15.

Theorem 2.12. *Let E be a normal locally convex Hausdorff topological vector space and U an open convex subset of E with $0 \in U$. Suppose $G \in B(\overline{U}, E)$ and (2.1), (2.2) and (2.3) hold. Let $r : E \to \overline{U}$ be given by*

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E,$$

where μ is the Minkowski functional on $\overline{U}$ (i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha\overline{U}\}$). In addition assume the following conditions hold:

$$\text{for any map } H \in B_\Phi(\overline{U}, E) \text{ we have } rH \in A(\overline{U}, \overline{U}) \tag{2.8}$$

and

$$\text{any map } \Psi \in A(\overline{U}, \overline{U}) \text{ has a fixed point.} \tag{2.9}$$

Then either

(A1). there exists $x \in \overline{U}$ with $x \in G(x)$

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $x \in \lambda G(x) + (1 - \lambda)\Phi(x)$

holds.

Proof. Let $F(x) = i(x)$. Note $F \in A_{\partial U}(\overline{U}, E)$ and the result follows from Theorem 2.7 if we show F is Φ-epi. Let $H \in B_\Phi(\overline{U}, E)$ (i.e. $H \in B(\overline{U}, E)$ with $H(x) \subseteq \Phi(x)$ for $x \in \partial U$). We must show there exists $x \in U$ with $x \in H(x)$. Let $\Psi = rH$. Then from (2.8) and (2.9) we see that $\Psi \in A(\overline{U}, \overline{U})$ and there exists $x \in \overline{U}$ with $x \in rH(x)$. Then $x = r(y)$ where $y \in H(x)$; here $x \in \overline{U} = U \cup \partial U$. If we show

$$x \in U \text{ and } r(y) = y \tag{2.10}$$

then $x = y$ so $x \in H(x)$ and we are finished. It remains to show (2.10). Let $x \in \partial U$. Then $\mu(x) = 1$ so

$$1 = \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}},$$

so $\mu(y) \geq 1$. Thus $x = r(y) = \frac{y}{\mu(y)}$ so with $\lambda = \frac{1}{\mu(y)}$ we have $x \in \lambda H(x)$. Then since $H(w) \subseteq \Phi(w)$ for $w \in \partial U$ we have $x \in \lambda\Phi(x)$ and so $x \in U$ (recall $\Phi(\partial U) \subseteq U$, U is convex and $0 \in U$), a contradiction. Thus $x \in U$. Then $\mu(x) < 1$ so

$$1 > \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}},$$

and as a result $\mu(y) < 1$. Thus $r(y) = y$, so (2.10) holds. □

Remark 2.13. We can remove the assumption that E is normal in the statement of Theorem 2.12 provided we have that (so we need to put conditions on the maps) D (see Remark 2.8) is relatively compact.

Remark 2.14. We note from the proof above that we could replace (2.3) with (2.7).

Remark 2.15. Let E be a normal topological vector space and U an open subset of E with $0 \in U$. Suppose $G \in B(\overline{U}, E)$ and (2.1), (2.2) and (2.7) hold. Also assume

$$\text{(2.11)} \qquad \text{there exists a retraction } r : E \to \overline{U} \text{ with } r(w) \in \partial U \text{ if } w \in E \backslash U$$

and

$$\text{(2.12)} \qquad \text{there is no } z \in \partial U \text{ with } z = r(y) \text{ and } y \in \Phi(z).$$

Finally suppose (2.8) (with r in (2.11)) and (2.9) hold. Then the conclusion in Theorem 2.12 holds. To see this let $H \in B_\Phi(\overline{U}, E)$ and exactly the same argument as in Theorem 2.12 guarantees that there exists $x \in \overline{U}$ with $x \in rH(x)$. Then

$$\text{(2.13)} \qquad x = r(y) \text{ with } y \in H(x);$$

here $x \in \overline{U} = U \cup \partial U$. If we show

$$\text{(2.14)} \qquad x \in U \text{ and } r(y) = y$$

then (2.13) implies $x \in H(x)$ and we are finished. It remains to show (2.14). If $x \in \partial U$ then $x = r(y)$ and $y \in H(x) \subseteq \Phi(x)$, so (2.12) yields a contradiction. Thus $x \in U$. As a result since $r(y)(= x) \in U$ we have from (2.11) that $y \in U$ and so $r(y) = y$.

We now present a very general abstract Furi-Pera type result based on Leray-Schauder type results (see (2.17)) below).

Theorem 2.16. *Let E be a metrizable topological vector space and Q a closed subset of E. Let $F : Q \to K(E)$, $\Phi : Q \to K(E)$ and assume the following hold:*

$$\text{(2.15)} \qquad \textit{there exists a retraction } r : E \to Q \textit{ with } r(z) \in \partial Q \textit{ for } z \in E \backslash Q$$

and

$$\text{(2.16)} \qquad Fr \in B(E, E) \textit{ and } Fr \textit{ has a fixed point.}$$

For $i \in \{1, 2, \dots\}$ let $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$; here d is the metric associated with E. Suppose for each $i \in \{1, 2, \dots\}$ we have $Fr \in B(\overline{U_i}, E)$, $\Phi r \in B(\overline{U_i}, E)$ and assume the following conditions hold:

$$\text{(2.17)} \qquad \begin{cases} \textit{either (A1). there exists } x \in \overline{U_i} \textit{ with } x \in Fr(x) \textit{ or (A2). there exists} \\ x \in \partial U_i \textit{ and } \lambda \in (0,1) \textit{ with } x \in \lambda Fr(x) + (1-\lambda)\Phi r(x) \textit{ hold} \end{cases}$$

$$\text{(2.18)} \qquad \begin{cases} \{x \in E : x \in \lambda Fr(x) + (1-\lambda)\Phi r(x) \textit{ for some } \lambda \in [0,1]\} \\ \textit{is relatively compact.} \end{cases}$$

Finally suppose

$$\text{(2.19)} \qquad \begin{cases} \textit{if } \{(x_j, \lambda_j\}_{j=1}^\infty \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging} \\ \textit{to } (x, \lambda) \textit{ with } x \in \lambda F(x) + (1-\lambda)\Phi(x) \textit{ and } 0 \le \lambda < 1, \\ \textit{then } \{\lambda_j F(x_j) + (1-\lambda_j)\Phi(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large.} \end{cases}$$

Then F has a fixed point in Q.

Proof. Let

$$\Omega = \{x \in E : x \in Fr(x)\}.$$

Now $\Omega \neq \emptyset$ (from (2.16)) and Ω is closed since Fr is upper semicontinuous. Now (2.18) guarantees that Ω is compact. We claim $\Omega \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose that $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed there exists $\delta > 0$ with $dist(\Omega, Q) > \delta$. Choose $m \in \{1, 2, \dots\}$ with $1 < \delta m$ and let (as in the statement of the theorem) $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$ for $i \in \{m, m+1, \dots\}$.

Fix $i \in \{m, m+1, \dots\}$. Since $dist(\Omega, Q) > \delta$ we see that $\Omega \cap \overline{U_i} = \emptyset$. Now (2.17) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with $y_i \in \lambda_i Fr(y_i) + (1-\lambda_i)\Phi r(y_i)$. Since $y_i \in \partial U_i$ we have

$$\{\lambda_i Fr(y_i) + (1-\lambda_i)\Phi r(y_i)\} \not\subseteq Q \text{ for } i \in \{m, m+1, \dots\}. \tag{2.20}$$

Now let

$$D = \{x \in E : x \in \lambda Fr(x) + (1-\lambda)\Phi r(x) \text{ for some } \lambda \in [0,1]\}.$$

Now $D \neq \emptyset$ (from (2.16)) is closed so compact from (2.18). This together with

$$d(y_j, Q) = \frac{1}{j} \text{ and } |\lambda_j| \leq 1 \text{ for } j \in \{m, m+1, \dots\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. In addition since Fr and Φr are upper semicontinuous and $y_j \in \lambda_j Fr(y_j) + (1-\lambda_j)\Phi r(y_j)$ we have

$$y^\star \in \lambda^\star Fr(y^\star) + (1-\lambda^\star)\Phi r(y^\star)$$

i.e. $y^\star \in \lambda^\star F(y^\star) + (1-\lambda^\star)\Phi(y^\star)$ since $r(y^\star) = y^\star$. If $\lambda^\star = 1$ then $y^\star \in Fr(y^\star)$ which contradicts $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. Now (2.19) with $x_j = r(y_j)$ (note $y_j \in \partial U_j$ so $r(y_j) \in \partial Q$) and $x = y^\star = r(y^\star)$ implies

$$\{\lambda_j Fr(y_j) + (1-\lambda_j)\Phi r(y_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.}$$

This contradicts (2.20). Thus $\Omega \cap Q \neq \emptyset$ so there exists $x \in Q$ with $x \in Fr(x) = F(x)$. ☐

Remark 2.17. If E is a locally convex Hausdorff topological vector space and Q is convex then Dugundji's extension theorem guarantees that there exists a retraction $r : E \to Q$. If say $0 \in intQ$ then we could take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E,$$

where μ is the Minkowski functional on Q and with this r we have $r(z) \in \partial Q$ for $z \in E \backslash Q$ i.e. (2.15) holds with this r. On the other hand if $intQ = \emptyset$ then $\partial Q = Q$ so (2.15) holds.

Remark 2.18. Note U_i for each $i \in \{1, 2, \dots\}$ (in the statement of Theorem 2.16) is an open subset of E. Also note if $0 \in Q$ then $0 \in U_i$ for each $i \in \{1, 2, \dots\}$.

Remark 2.19. Let E be a metrizable locally convex topological vector space and let Q be convex also. We may choose d to be a translational invariant metric associated with E (see [7 pg 29]) so we see that U_i for each $i \in \{1, 2, \dots\}$ (in the statement of Theorem 2.16) is convex.

Remark 2.20. Let E be a metrizable locally convex topological vector space and Q a closed convex subset of E with $0 \in Q$. Let $r : E \to Q$ be the retraction as in (2.15) (guaranteed from Remark 2.17). For $i \in \{1, 2, \dots\}$ let $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$; here d is the translational invariant metric associated with E (as in Remark 2.19). Suppose for each $i \in \{1, 2, \dots\}$ we have $Fr \in B(\overline{U_i}, E)$ and $\Phi r \in B(\overline{U_i}, E)$. Note for each $i \in \{1, 2, \dots\}$ from Dugundji's extension theorem $\overline{U_i}$ is a retract of E i.e. there exists a retraction $r_i : E \to \overline{U_i}$ (we could take $r_i(x) = \frac{x}{\max\{1, \mu_i(x)\}}$ where μ_i is the Minkowski functional on $\overline{U_i}$).

(i). For each $i \in \{1, 2, \dots\}$ assume the following conditions hold:

$$\left\{ \begin{array}{l} \eta(\,.\,)\, F\, r(\,.\,) + (1 - \eta(\,.\,))\Phi\, r(\,.\,) \in B(\overline{U_i}, E) \text{ for any} \\ \text{continuous map } \eta : \overline{U_i} \to [0, 1] \text{ with } \eta(\partial U_i) = 0 \end{array} \right. \tag{2.21}$$

$$i \in \mathbf{A}(\overline{U_i}, E) \text{ where } i \text{ is the identity map} \tag{2.22}$$

$$x \notin \lambda \Phi r(x) \text{ for } x \in \partial U_i \text{ and } \lambda \in (0, 1] \tag{2.23}$$

$$\text{any map } \Psi \in A(E, E) \text{ has a fixed point} \tag{2.24}$$

and

$$\left\{ \begin{array}{l} \text{for any continuous map } \eta : E \to [0, 1] \text{ with } \eta(E \backslash U_i) = 0 \\ \text{and } H \in B_{\Phi r}(\overline{U_i}, E) \text{ the map } J \in A(E, E) \\ \text{where } J(x) = \eta(x) H(r_i(x)). \end{array} \right. \tag{2.25}$$

Now Theorem 2.9 (with G being Fr, Φ being Φr and U being U_i) guarantees that (2.17) holds.

(ii). For each $i \in \{1, 2, \dots\}$ assume (2.21), (2.22) and (2.23) hold and in addition assume the following conditions hold:

$$\text{for any map } H \in B_{\Phi r}(\overline{U_i}, E) \text{ we have } r_i H \in A(\overline{U_i}, \overline{U_i}) \tag{2.26}$$

and

$$\text{any map } \Psi \in A(\overline{U_i}, \overline{U_i}) \text{ has a fixed point.} \tag{2.27}$$

Now Theorem 2.12 (with G being Fr, Φ being Φr and U being U_i) guarantees that (2.17) holds.

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