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THE CAUCHY PROBLEM FOR GENERALIZED ABSTRACT BOUSSINESQ EQUATIONS

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ABSTRACT. In this paper, the existence and uniqueness of solution of the Cauchy problem for abstract Boussinesq equations is obtained. By applying this result, the Wentzell-Robin type mixed problem for Boussinesq equations and the Cauchy problem for finite or infinite systems of Boussinesq equations are studied.

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1. Introduction, Definitions and Background

The subject of this paper is to study the local existence and uniqueness of solution of the Cauchy problem for the following Boussinesq-operator equation

$$u_{tt} - \Delta u_{tt} + Au = \Delta f\left(u\right), \quad x \in R^n,\ t \in \left(0, T\right), \tag{1.1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), \quad u_t\left(x, 0\right) = \psi\left(x\right), \tag{1.2}$$

where A is a linear operator in a Banach space E, $u(x,t)$ denotes the E-valued unknown function, $f(u)$ is the given nonlinear function, $\varphi\left(x\right)$ and $\psi\left(x\right)$ are the given initial value functions, subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x and Δ denotes the Laplace operator in R^n.

This is a first paper that devoted to initial value problem for abstract Boussinesq equations. Since the equation contain generally, unbounded operator in a abstract Banach space E and the problem (1.1) is considered in UMD-valed function space, the methods applied for Boussinesq equations in scalar case does not pass here. By this reason, in this paper, the methods of proofs naturally differs to those used in scalar case. Since the Banach space E is arbitrary and A is a possible linear operator, by choosing E and A we can obtain numerous classis of Boussinesq type equations which have a different applications (see [1–7] and the references therein).

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Here, by inspiring [8] and [9] in this paper, we obtain the local existence and uniqueness of small-amplitude solution of the problem (1.1)–(1.2). Note that, differential operator equations were studied e.g. in [10–31]. The Cauchy problem for abstract hyperbolic equations were treated e.g. in [10–12]. The strategy is to express the abstract Boussinesq equation as an integral equation with operator coefficient, to treat in the nonlinearity as a small perturbation of the linear part of the equation, then use the contraction mapping theorem and utilize an estimate for solutions of the linearized version to obtain a priori estimates on E-valued L^p norms of solutions. The key step is the derivation of the uniform estimate for the solutions of the linearized Boussinesg-operator equation. If we choose the UMD space E as a abstract Hilbert space H, then we obtain the existence and uniqueness results of the Cauchy problem for abstract Boussinesg equation defined in Hilbert space valued classes. For example, if we choose E a concrete Hilbert space, for example $E = L^2(R^n)$ and $A = -\Delta$, we obtain the scalar Cauchy problem for generalized Boussinesq type equation

$$u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u), \quad x \in R^n,\ t \in (0,T), \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{1.4}$$

The equation (1.3) arise in different situations (see [1, 2]). For example, equation (1.3) for $n = 1$ describes a limit of a one-dimensional nonlinear lattice [3], shallow-water waves [4,5] and the propagation of longitudinal deformation waves in an elastic rod [6]. In [8] and [9] the existence of the global classical solutions and the blow-up of the solution for the initial boundary value problem and the Cauchy problem (1.3)–(1.4) are obtained. Moreover, let we choose $E = L_{p_1}(0,1)$ and A to be differential operator with generalized Wentzell-Robin boundary condition defined by

$$D(A) = \left\{ u \in W_{p_1}^2(0,1),\ B_j u = Au(j) + \sum_{i=0}^{1} \alpha_{ji} u^{(i)}(j),\ j = 0,1 \right\}, \tag{1.5}$$

$$Au = au^{(2)} + bu^{(1)} + cu,$$

in (1.1)–(1.2), where α_{ji} are complex numbers, a, b, c are complex-valued functions. Then, we get the following Wentzell-Robin type mixed problem for Boussinesq equation

$$u_{tt} - \Delta_x u_{tt} + a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y} + cu = \Delta_x f(u),\ x \in R^n,\ y \in (0,1),\ t \in (0,T), \tag{1.6}$$

$$B_j u = Au(x,t,j) + \sum_{i=0}^{1} \alpha_{ji} u^{(i)}(x,t,j) = 0,\ j = 0,1, \tag{1.7}$$

$$u(x,y,0) = \varphi(x,y),\ u_t(x,y,0) = \psi(x,y), \tag{1.8}$$

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [31,32] and the references therein. Here $\tilde{\Omega} = R^n \times (0,1)$, $\mathbf{p} = (p_1, p)$ and $L^{\mathbf{p}}\left(\tilde{\Omega}\right)$ denotes the space of all $\mathbf{p}$-summable complex-valued functions with mixed norm i.e., the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L^{\mathbf{p}}(\tilde{\Omega})} = \left(\int_{R^n} \left(\int_0^1 |f(x,y)|^{p_1} dy\right)^{\frac{p}{p_1}} dx\right)^{\frac{1}{p}} < \infty.$$

By applying the general Theorem 2.5 obtained here, we established the local existence and uniqueness of small-amplitude solution of the problem (1.6)–(1.8) in mixed $L^{\mathbf{p}}\left(\tilde{\Omega}\right)$ space.

In order to state our results precisely, we introduce some notations and some function spaces.

Let E be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_E .$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embeds into E. Let $H^{s,p}(R^n; E)$, $-\infty < s < \infty$ denotes the E-valued Sobolev space of order s which is defined as:

$$H^{s,p} = H^{s,p}(R^n; E) = (I - \Delta)^{-\frac{s}{2}} L^p(R^n; E), \quad \|u\|_{H^{s,p}} = \left\|(I-\Delta)^{\frac{s}{2}} u\right\|_{L^p(R^n;E)}.$$

$H^{s,p}(R^n; E_0, E)$ denotes the Sobolev-Lions type space, i.e.,

$$H^{s,p}(R^n; E_0, E) = \Big\{u \in H^{s,p}(R^n; E) \cap L^q(R^n; E_0),$$
$$\|u\|_{H^{s,p}(R^n;E_0,E)} = \|u\|_{L^p(R^n;E_0)} + \|u\|_{H^{s,p}(R^n;E)} < \infty\Big\}.$$

Let $B^s_{p,q}(R^n; E)$ denote E-valued Besov space (see e.g. [34, § 15]). Let $B^s_{p,q}(R^n; E_0, E)$ denote the space $L^p(R^n; E_0) \cap B^s_{p,q}(R^n; E)$ with the norm

$$\|u\|_{B^s_{p,q}(R^n;E_0,E)} = \|u\|_{L^p(R^n;E_0)} + \|u\|_{B^s_{p,q}(R^n;E)} < \infty.$$

For estimating lower order derivatives we use following embedding theorem that is obtained from [22, Theorem 1]:

Theorem 1.1. *Suppose the following conditions are satisfied:*

(1) E is a UMD space and A is an R-positive operator in E (see e.g. [29] *for definitions);*

(2) $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-tuples of nonnegative integer number and s is a positive number such that

$$\varkappa = \frac{1}{s}\left[|\alpha| + n\left(\frac{1}{p} - \frac{1}{q}\right)\right] \le 1, \quad 0 \le \mu \le 1 - \varkappa, \quad 1 < p \le q < \infty; \quad 0 < h \le h_0,$$

where h_0 is a fixed positive number.

Then the embedding $D^{\alpha}H^{s,p}(R^n;E(A),E)\subset L^q(R^n;E(A^{1-\varkappa-\mu}))$ is continuous and for $u\in H^{s,p}(R^n;E(A),E)$ the following uniform estimate holds

$$\|D^{\alpha}u\|_{L^q(R^n;E(A^{1-\varkappa-\mu}))}\leq h^{\mu}\|u\|_{H^{s,p}(R^n;E(A),E)}+h^{-(1-\mu)}\|u\|_{L^p(R^n;E)}.$$

In a similar way as [19, Theorem A_0] we obtain:

Proposition $\mathbf{A}_1$. Let $1<p\leq q\leq\infty$ and E be UMD space. Suppose $\Psi_h\in C^n(R^n\backslash\{0\};B(E))$ and there is a positive constant K such that

$$\sup_{h\in Q}R\left(\left\{|\xi|^{|\beta|+n\left(\frac{1}{p}-\frac{1}{q}\right)}D^{\beta}\Psi_h(\xi):\ \xi\in R^n\backslash\{0\},\ \beta_k\in\{0,1\}\right\}\right)\leq K.$$

Then Ψ_h is a uniformly bounded collection of Fourier multiplier from $L^p(R^n;E)$ to $L^{\infty}(R^n;E)$.

2. Estimates for linearized equation

Here, we make the necessary estimates for solutions of Cauchy problem for the linearized abstract Boussinesq equation

$$u_{tt}-\Delta u_{tt}+Au=\Delta g(x,t),\quad x\in R^n,\quad t\in(0,T), \tag{2.1}$$

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x). \tag{2.2}$$

Let

$$X_p=L^p(R^n;E),\quad Y^{s,p}=H^{s,p}(R^n;E),\quad Y_1^{s,p}=H^{s,p}(R^n;E)\cap L^1(R^n;E),$$

$$Y_{\infty}^{s,p}=H^{s,p}(R^n;E)\cap L^{\infty}(R^n;E).$$

Condition 2.1. Assume E is an UMD space and the operator A is positive in E. Moreover suppose φ, $\psi\in Y_{\infty}^{s,p}$, $g(.,t)\in Y_{\infty}^{s,p}$ for $t\in(0,T)$ and $s>\frac{n}{p}$ for $1<p<\infty$.

First we need the following lemmas

Lemma 2.2. *Suppose the Condition* 2.1 *hold. Then problem* (2.1)–(2.2) *has a unique generalized solution.*

Proof. By using of the Fourier transform we get from (2.1)–(2.2)

$$\hat{u}_{tt}(\xi,t)+A_{\xi}\hat{u}(\xi,t)=|\xi|^2\left(1+|\xi|^2\right)^{-1}\hat{g}(\xi,t), \tag{2.3}$$

$$\hat{u}(\xi,0)=\hat{\varphi}(\xi),\quad \hat{u}_t(\xi,0)=\hat{\psi}(\xi),\quad \xi\in R^n,\ t\in(0,T),$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ with respect to x and

$$A_{\xi}=\left(1+|\xi|^2\right)^{-1}A,\quad \xi\in R^n.$$

By virtue of [10, 11] we obtain that A_ξ is a generator of a strongly continuous cosine operator function and problem (2.3) has a unique solution for all $\xi \in R^n$, moreover, the solution can be written as

$$\hat{u}(\xi,t) = C(t)\hat{\varphi}(\xi) + S(t)\hat{\psi}(\xi) + \int_0^t S(t-\tau,\xi,A)|\xi|^2 \hat{g}(\xi,\tau)\,d\tau,\ t \in (0,T), \tag{2.4}$$

where $C(t) = C(t,\xi,A)$ is a cosine and $S(t) = S(t,\xi,A)$ is a sine operator-functions (see e.g. [11]) generated by parameter dependent operator A_ξ. From (2.4) we get that, the solution of the problem (2.1)–(2.2) can be expressed as

$$\begin{aligned} u(x,t) &= S_1(t,A)\varphi(x) + S_2(t,A)\psi(x) \\ &+ (2\pi)^{-\frac{1}{n}} \int_{R^n} \int_0^t e^{ix\xi} S(t-\tau,\xi,A)|\xi|^2 \hat{g}(\xi,\tau)\,d\tau d\xi, \quad t \in (0,T), \end{aligned} \tag{2.5}$$

where $S_1(t,A)$ and $S_2(t,A)$ are linear operators in E defined by

$$\begin{aligned} S_1(t,A)\varphi &= (2\pi)^{-\frac{1}{n}} \int_{R^n} e^{ix\xi} C(t)\hat{\varphi}(\xi)\,d\xi, \\ S_2(t,A)\psi &= (2\pi)^{-\frac{1}{n}} \int_{R^n} e^{ix\xi} S(t)\hat{\psi}(\xi)\,d\xi. \end{aligned} \tag{2.6}$$

□

Lemma 2.3. *Suppose the Condition* 2.1 *hold. Then the solution* (2.1)–(2.2) *satisfies the following uniform estimate*

$$\begin{aligned} \left(\|u\|_{X_\infty} + \|u_t\|_{X_\infty}\right) &\le C\left(\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1}\right. \\ &\left. + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|\Delta g(.,\tau)\|_{Y^{s,p}} + \|\Delta g(.,\tau)\|_{X_1}\right) d\tau. \right. \end{aligned} \tag{2.7}$$

Proof. Let $N \in \mathbb{N}$ and

$$\Pi_N = \{\xi : \xi \in R^n, \quad |\xi| \le N\},\ \Pi_N' = \{\xi : \xi \in R^n,\ |\xi| \ge N\}.$$

It is clear to see that

$$\|u(.,t)\|_{L^\infty(R^n;E)} \le \left\|F^{-1}C(t)\hat{\varphi}(\xi)\right\|_{L^\infty\left(\Pi_N';E\right)} + \left\|F^{-1}S(t)\hat{\psi}(\xi)\right\|_{L^\infty\left(\Pi_N';E\right)}. \tag{2.8}$$

Using Hölder inequality we have

$$\left\|F^{-1}C(t)\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi_N;E)} + \left\|F^{-1}S(t)\hat{\psi}(\xi)\right\|_{L^\infty(\Pi_N;E)} \le C\left[\|\varphi\|_{X_1} + \|\psi\|_{X_1}\right]. \tag{2.9}$$

By using the resolvent properties of operator A, representation of

$$|\xi|^{|\alpha|+\frac{n}{p}} \|D^\alpha C_1(t,\xi)\|_{B(E)} \le C_1, \quad |\xi|^{|\alpha|+\frac{n}{p}} \|D^\alpha S_1(t,\xi)\|_{B(E)} \le C_2, \tag{2.10}$$

for $s > \frac{n}{p}$ and all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_k \in \{0,1\}$, $\xi \in R^n$, $\xi \ne 0$, $t \in [0,T]$, here

$$C_1(t,\xi) = \left(1+|\xi|^2\right)^{-\frac{s}{2}} C(t), \quad S_1(t,\xi) = \left(1+|\xi|^2\right)^{-\frac{s}{2}} S(t).$$

By Proposition A$_1$ from (2.9)–(2.10) we get that the operator-valued functions $C_1(t,\xi)$ and $S_1(t,\xi)$ are $L^p(R^n;E) \to L^\infty(R^n;E)$ Fourier multipliers uniformly in $t \in [0,T]$. Then by Minkowski's inequality for integrals, the semigroups estimates (see e.g. [10, 11]) and (2.9) we obtain

(2.11)
$$\left\|F^{-1}C(t)\hat{\varphi}(\xi)\right\|_{L^\infty(\Pi'_N;E)} + \left\|F^{-1}S(t)\hat{\psi}(\xi)\right\|_{L^\infty(\Pi'_N;E)} \leq C\left[\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}}\right].$$

By reasoning as the above we get

(2.12)
$$\left\|F^{-1}\int_0^t S(t-\tau,\xi,A)\hat{g}(\xi,\tau)\,d\tau\right\|_{X_\infty} \leq C\int_0^t \left(\|\Delta g(.,\tau)\|_{Y^s} + \|\Delta g(.,\tau)\|_{X_1}\right)d\tau.$$

By differentiating, in view of (2.6) we obtain from (2.5) the estimate of type (2.9), (2.11), (2.12) for u_t. Then by using (2.9) , (2.11), (2.12) we get the estimate (2.7). □

Lemma 2.4. *Assume the Condition* 2.1 *hold. Then the solution of* (2.1)–(2.2) *satisfies the following uniform estimate*

$$\left(\|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}}\right) \leq C\left(\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|\Delta g(.,\tau)\|_{Y^{s,p}}\,d\tau\right). \tag{2.13}$$

Proof. From (2.4) we have the following estimate

$$\begin{aligned} &\left(\left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{X_p} + \left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}_t\right\|_{X_p}\right) \\ &\leq C\left\{\left\|F^{-1}C_1(t,\xi)\hat{\varphi}\right\|_{X_p} + \left\|F^{-1}S_1(t,\xi)\hat{\psi}\right\|_{X_p}\right. \\ &\left.+\int_0^t \left\|F^{-1}|\xi|^2\left(1+|\xi|^2\right)^{\frac{s}{2}}S(t-\tau,\xi,A)\hat{g}(.,\tau)\right\|_{X_p}d\tau\right\}. \end{aligned} \tag{2.14}$$

By construction and in view of Proposition A$_1$ we get that $C(t)$ and $S(t)$ are $L^p(R^n;E)$ Fourier multipliers uniformly in $t \in [0,T]$. So, the estimate (2.14) by using the Minkowski's inequality for integrals implies (2.13). □

From Lemmas 2.2–2.4 we obtain

Theorem 2.5. *Let the Condition* 2.1 *hold. Then problem* (2.1)–(2.2) *has a unique solution* $u \in C^{(2)}\left([0,T];Y_1^{s,p}\right)$ *and the following uniform estimates hold*

(2.15)
$$\begin{aligned} \|u\|_{X_\infty} + \|u_t\|_{X_\infty} &\leq C\left(\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1}\right. \\ &+ \|\psi\|_{Y^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|\Delta g(.,\tau)\|_{Y^{s,p}} + \|\Delta g(.,\tau)\|_{X_1}\right)d\tau, \end{aligned}$$

$$\|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}} \leq C\left(\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|\Delta g(.,\tau)\|_{Y^{s,p}}\,d\tau\right). \tag{2.16}$$

Proof. From Lemma 2.2 we obtain that, problem (2.1)–(2.2) has a unique generalized solution. From the representation of solution (2.5) and Lemmas 2.3, 2.4 we get that there is a solution $u \in C^{(2)}\left([0,T]\,; Y_1^s\right)$ and estimates (2.15), (2.16) hold. □

3. **Initial value problem for nonlinear equation**

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1)–(1.2). For the study of the nonlinear problem (1.1)–(1.2) we need the following lemmas

Lemma 3.1 (Abstract Nirenberg's inequality)**.** *Let E be an UMD space. Assume that $u \in L^p\left(R^n; E\right)$, $D^m u \in L^q\left(R^n; E\right)$, $p,q \in (1,\infty)$. Then for i with $0 \leq i \leq m$, $m > \frac{n}{q}$ we have*

$$\left\|D^i u\right\|_r \leq C \left\|u\right\|_p^{1-\mu} \sum_{k=1}^{n} \left\|D_k^m u\right\|_q^{\mu}, \tag{3.1}$$

where

$$\frac{1}{r} = \frac{i}{m} + \mu\left(\frac{1}{q} - \frac{m}{n}\right) + (1-\mu)\frac{1}{p}, \quad \frac{i}{m} \leq \mu \leq 1.$$

Proof. By virtue of interpolation of Banach spaces [33, § 1.3.2], in order to prove (3.1) for any given i, one has only to prove it for the extreme values $\mu = \frac{i}{m}$ and $\mu = 1$. For the case of $\mu = 1$, i.e., $\frac{1}{r} = \frac{i}{m} + \frac{1}{q} - \frac{m}{n}$ the above estimate is obtained from Theorem 1.1. The case $\mu = \frac{i}{m}$ is derived by reasoning as in [36, § 2] and in replacing absolute value of complex valued function u by the E-norm of E-valued function. Note that, for $E = \mathbb{C}$ the lemma considered by L. Nirenberg [35]. □

Using the chain rule of the composite function, from Lemma 3.1 we can prove the following result

Lemma 3.2. *Let E be an UMD space. Assume that $u \in W^{m,p}\left(R^n; E\right) \cap L^{\infty}\left(R^n; E\right)$, and $f(u)$ possesses continuous derivatives up to order $m \geq 1$. Then $f(u) - f(0) \in W^{m,p}(\Omega; E)$ and there is a constant $C_0 > 0$ such that*

$$\left\|f(u) - f(0)\right\|_p \leq \left\|f^{(1)}(u)\right\|_{\infty} \left\|u\right\|_p,$$

$$\left\|D^k f(u)\right\|_p \leq C_0 \sum_{j=1}^{k} \left\|f^{(j)}(u)\right\|_{\infty} \left\|u\right\|_{\infty}^{j-1} \left\|D^k u\right\|_p, \quad 1 \leq k \leq m.$$

For $E = \mathbb{C}$ the lemma coincide with the corresponding inequality in [35]*. Let*

$$X = L^p\left(R^n; E\right), \quad Y = W^{2,p}\left(R^n; E(A), E\right), \quad E_0 = (X,Y)_{\frac{1}{2p},p},$$

where $(X,Y)_{\theta,p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ denotes the real interpolation [33].

Remark 3.3. By using J. Lions-I. Petree result [18, § 1.8.] we obtain that the map $u \to u(t_0)$, $t_0 \in [0,T]$ is continuous from $W^{2,p}(0,T;X,Y)$ onto E_0 and there is a constant C_1 such that

$$\|u(t_0)\|_{E_0} \leq C_1 \|u\|_{W^{2,p}(0,T;X,Y)}, \quad 1 \leq p \leq \infty.$$

Let $Y(T) = C([0,T];Y_\infty^{2,p})$ be the space equipped with the norm defined by

$$\|u\|_{Y(T)} = \max_{t\in[0,T]} \|u\|_{Y^{2,p}} + \max_{t\in[0,T]} \|u\|_{X_\infty}, \quad u \in Y(T).$$

It is easy to see that $Y(T)$ is a Banach space. For φ, $\psi \in Y^{2,p}$, let

$$M = \|\varphi\|_{Y^{2,p}} + \|\varphi\|_{X_\infty} + \|\psi\|_{Y^{2,p}} + \|\psi\|_{X_\infty}.$$

Definition 3.4. For any $T > 0$ if v, $\psi \in Y_\infty^{2,p}$ and $u \in C([0,T];Y_\infty^{2,p})$ satisfies the equation (1.1)–(1.2) then $u(x,t)$ is called the continuous solution or the strong solution of the problem (1.1)–(1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem (1.1)–(1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of the problem (1.1)–(1.2).

Condition 3.5. Assume the operator A generates continuous cosine operator function in UMD space E, φ, $\psi \in Y_\infty^{2,p}$ and $1 < p < \infty$ for $\frac{n}{p} < 2$. Moreover, suppose the function $u \to f(u)$: $R^n \times [0,T] \times E_0 \to E$ is a measurable in $(x,t) \in R^n \times [0,T]$ for $u \in E_0$, $f(x,t,.,.)$ is continuous in $u \in E_0$ for $x \in R^n$, $t \in [0,T]$ and $f(u) \in C^{(3)}(E_0;E)$.

Main aim of this section is to prove the following result:

Theorem 3.6. *Let the Condition* 3.5 *hold. Then problem* (1.1)–(2.2) *has a unique local strange solution* $u \in C^{(2)}([0,\ T_0);Y_\infty^{2,p})$, *where* T_0 *is a maximal time interval that is appropriately small relative to* M. *Moreover, if*

$$\sup_{t\in[0,\ T_0)} \left(\|u\|_{Y^{2,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{2,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

Proof. First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1)–(1.2) by contraction mapping principle. Suppose that $u \in C^{(2)}([0,T];Y_\infty^{2,p})$ is a strong solution of the problem (1.1)–(1.2). Consider a map G on $Y(T)$ such that $G(u)$ is the solution of the Cauchy problem

$$G_{tt}(u) - \Delta G_{tt}(u) + AG(u) = \Delta f(G(u)), \quad x \in R^n, \quad t \in (0,T), \tag{3.2}$$

$$G(u)(x,0) = \varphi(x), \quad G_t(u)(x,0) = \psi(x).$$

From Lemma 3.2 we know that $f(u) \in L^p(0,T;Y_\infty^{2,p})$ for any $T > 0$. Thus, by Theorem 2.5, problem (3.2) has a unique solution which can be written as

(3.3)
$$G(u)(t,x) = S_1(t,A)\varphi(x) + S_2(t,A)\psi(x) + \int_0^t \int_{R^n} S_0(t,\tau,\xi)\hat{f}(u)(\xi,\tau)\,d\xi d\tau,$$

$$S_0(t,\tau,\xi) = (2\pi)^{-\frac{1}{n}} F^{-1}|\xi|^2 S(t-\tau,\xi,A).$$

For the sake of convenience, we assume that $f(0) = 0$. Otherwise, we can replace $f(u)$ with $f(u) - f(0)$. Hence, from Lemma 3.2 we have $f(u) \in Y^{2,p}$ iff $f \in C^2(R;E)$. Consider the operator in $Y(T)$ defined as

$$Gu = S_1(t,A)\varphi(x) + S_2(t,A)\psi(x) + \int_0^t \int_{R^n} S_0(t,\tau,\xi)\hat{f}(u)(\xi,\tau)\,d\xi d\tau. \tag{3.4}$$

From Lemma 3.2 we get that the operator G is well defined for $f \in C^{(2)}(R;E)$. Moreover, from Lemma 3.2 it is easy to see that the map G is well defined for $f \in C^{(2)}(X_0;E)$. We put

$$Q(M;T) = \left\{u \mid u \in Y(T),\ \|u\|_{Y(T)} \leq M+1\right\}.$$

By reasoning as in [9] let us prove that the map G has a unique fixed point in $Q(M;T)$. For this aim, it is sufficient to show that the operator G maps $Q(M;T)$ into $Q(M;T)$ and $G : Q(M;T) \to Q(M;T)$ is strictly contractive if T is appropriately small relative to M. Consider the function $\bar{f}(\xi) : [0,\infty) \to [0,\infty)$ defined by

$$\bar{f}(\xi) = \max_{|x|\leq\xi}\left\{\left\|f^{(1)}(x)\right\|_E, \left\|f^{(2)}(x)\right\|_E\right\}, \quad \xi \geq 0.$$

It is clear to see that the function $\bar{f}(\xi)$ is continuous and nondecreasing on $[0,\infty)$. From Lemma 3.2 we have

$$\begin{aligned}\|f(u)\|_{Y^{2,p}} &\leq \left\|f^{(1)}(u)\right\|_{X_\infty}\|u\|_{X_p} + \left\|f^{(1)}(u)\right\|_{X_\infty}\|Du\|_{X_p} \\ &\leq 2C_0\bar{f}(M+1)(M+1)\|u\|_{Y^{2,p}}.\end{aligned} \tag{3.5}$$

By using the Theorem 2.5 we obtain from (3.4)

$$\|G(u)\|_{X_\infty} \leq \|\varphi\|_{X_\infty} + \|\psi\|_{X_\infty} + \int_0^t \|\Delta f(u(\tau))\|_{X_\infty}, \tag{3.6}$$

$$\|G(u)\|_{Y^{2,p}} \leq \|\varphi\|_{Y^{2,p}} + \|\psi\|_{Y^{2,p}} + \int_0^t \|\Delta f(u(\tau))\|_{Y^{2,p}}\,d\tau. \tag{3.7}$$

Thus, from (3.5)–(3.7) and Lemma 3.2 we get

$$\|G(u)\|_{Y(T)} \leq M + T(M+1)\left[1 + 2C_0(M+1)\bar{f}(M+1)\right].$$

If T satisfies

$$T \leq \left\{(M+1)\left[1 + 2C_0(M+1)\bar{f}(M+1)\right]\right\}^{-1} \tag{3.8}$$

then $\|Gu\|_{Y(T)} \leq M+1$. Therefore, if (3.8) holds, then G maps $Q(M;T)$ into $Q(M;T)$. Now, we are going to prove that the map G is strictly contractive. Assume $T>0$ and u_1, $u_2 \in Q(M;T)$ given. We get

$$G(u_1) - G(u_2) = \int_0^t \int_{R^n} S_0(t,\tau,\xi) \left[\hat{f}(u_1)(\xi,\tau) - \hat{f}(u_2)(\xi,\tau)\right] d\xi d\tau.$$

By using the mean value theorem and using Hölder's and Nirenberg's, Minkowski's inequalities for integrals, Fourier multiplier theorems for operator-valued functions in X_p spaces and Young's inequality, we obtain

$$\|Gu_1 - Gu_2\|_{Y(T)} \leq \frac{1}{2}\|u_1 - u_2\|_{Y(T)}.$$

That is, G is a constructive map. By contraction mapping principle, we know that $G(u)$ has a fixed point $u(x,t) \in Q(M;T)$ that is a solution of the problem (1.1)–(1.2). From (2.5) we get that u is a solution of the following integral equation

$$u(t,x) = S_1(t,A)\varphi(x) + S_2(t,A)\psi(x) + \int_0^t \int_{R^n} S_0(t,\tau,\xi)\hat{f}(u)(\xi,\tau)\, d\xi d\tau.$$

Let us show that this solution is a unique in $Y(T)$. Let $u_1, u_2 \in Y(T)$ be two solution of the problem (1.1)–(1.2). Then, by Lemmas 2.4, Minkowski's inequality for integrals and Theorem 2.5 we obtain from (3.8)

$$\|u_1 - u_2\|_{Y^{2,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{2,p}}\, d\tau. \tag{3.9}$$

From (3.9) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{2,p}} = 0$, i.e. problem (1.1)–(1.2) has a unique solution which belongs to $Y(T)$. That is, we obtain the first part of the assertion. Now, let $[0, T_0)$ be the maximal time interval of existence for $u \in Y(T_0)$. It remains only to show that if (3.4) is satisfied, then $T_0 = \infty$. Assume contrary that, (3.4) holds and $T_0 < \infty$. For $T \in [0, T_0)$, we consider the following integral equation

$$v(x,t) = S_1(t,A)u(x,T) + S_2(t,A)u_t(x,T) + \int_0^t \int_{R^n} S_0(t,\tau,\xi)\hat{f}(v)(\xi,\tau)\, d\xi d\tau. \tag{3.10}$$

By virtue of (3.4), for $T' > T$ we have

$$\sup_{t\in[0,T)} \left(\|u\|_{Y^{2,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{2,p}} + \|u_t\|_{X_\infty}\right) < \infty.$$

By reasoning as a first part of theorem and by contraction mapping principle, there is a $T^* \in (0, T_0)$ such that for each $T \in [0, T_0)$, the equation (3.10) has a unique solution $v \in Y(T^*)$. The estimates (3.8) and (3.9) imply that T^* can be selected independently of $T \in [0, T_0)$. Set $T = T_0 - \frac{T^*}{2}$ and define

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & t \in [0,T] \\ v(x,t-T), & t \in \left[T, T_0 + \frac{T^*}{2}\right] \end{cases}.$$

By construction $\tilde{u}(x,t)$ is a solution of the problem (1.1)–(1.2) on $\left[T, T_0 + \frac{T^*}{2}\right]$ and in view of local uniqueness, $\tilde{u}(x,t)$ extends u. This is against to the maximality of $[0, T_0)$, i.e we obtain $T_0 = \infty$. □

4. The Cauchy problem for the system of Boussinesq equation of infinite order

Consider the Cauchy problem for the following nonlinear system

$$(4.1) \qquad (u_m)_{tt} - \Delta (u_m)_{tt} + \sum_{j=1}^{N} a_{mj} u_j (x,t) = \Delta f_m (u), \quad x \in R^n, \quad t \in (0,T),$$

$$(4.2) \qquad u_m (x,0) = \varphi_m (x), \quad (u_m)_t (x,0) = \psi_m (x), \quad m = 1, 2, \ldots, N, \quad N \in \mathbb{N},$$

where $u = (u_1, u_2, \ldots, u_N)$, a_{mj} are complex numbers, $\varphi_m (x)$ and $\psi_m (x)$ are data functions. Let $l_q = l_q (N)$ (see [33, § 1.18]) and A be the operator in $l_q (N)$ defined by

$$D(A) = \left\{ u = \{u_j\}, \ \|u\|_{l_q^s(N)} = \left(\sum_{j=1}^{N} 2^{sj} u_j^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$A = [a_{mj}], \quad a_{mj} = g_m 2^{sj}, \quad m, j = 1, 2, \ldots, N.$$

Let $X_{pq} = L^p (R^n; l_q), Y^{s,p,q} = H^{s,p} (R^n; l_q)$ and

$$E_{0q} = B_{p,q}^{2\left(1-\frac{1}{2p}\right)} \left(R^n; l_q^{s\left(1-\frac{1}{2p}\right)}, l_q \right), \quad Y_\infty^{s,p,q} = H^{s,p} (R^n; l_q) \cap L^\infty (R^n; l_q).$$

From Theorem 2.5 we obtain the following result

Theorem 4.1. *Assume φ_m, $\psi_m \in Y_\infty^{2,p,q}$ and $1 < p < \infty$ for $\frac{n}{p} < 2$. Suppose the function $u \to f(u)$: $R^n \times [0,T] \times E_{0q} \to l_q$ is a measurable function in $(x,t) \in R^n \times [0,T]$ for $u \in E_{0q}$; $f(x,t.,.)$ and this function is continuous in $u \in E_{0q}$ for $x, t \in R^n \times [0,T]$; moreover $\Delta f(u) \in C^{(3)} (E_{0q}; l_q)$. Then problem (4.1)–(4.2) has a unique local strange solution $u \in C^{(2)} ([0\ , T_0); Y_\infty^{2,p,q})$, where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if*

$$(4.3) \qquad \sup_{t \in [0\,, T_0)} \left(\|u\|_{Y^{2,p,q}} + \|u\|_{X_{\infty,q}} + \|u_t\|_{Y^{2,p,q}} + \|u_t\|_{X_{\infty,q}} \right) < \infty$$

then $T_0 = \infty$.

Proof. It is known that $l_q (N)$ is a UMD space. It is easy to see that the operator A is R-positive in $l_q (N)$. Moreover, by interpolation theory of Banach spaces [33, § 1.3], we have

$$E_{0q} = \left(H^{2,p} \left(R^n; l_q^s, l_q \right), L^p (R^n; l_q) \right)_{\frac{1}{2p}, q} = B_{p,q}^{2\left(1-\frac{1}{2p}\right)} \left(R^n; l_q^{s\left(1-\frac{1}{2p}\right)}, l_q \right).$$

By using the properties of spaces $Y^{s,p,q}$, $Y_\infty^{s,p,q}$, E_{0q} we get that all conditions of Theorem 2.5 are hold, i,e., we obtain the conclusion. □

5. **The Wentzell-Robin type mixed problem for Boussinesq equations**

Consider the problem (1.6)–(1.8). Let

$$E = L^{\mathbf{p}}\left(\tilde{\Omega}\right), \quad E_0 = \left(L^{p_1}\left(\tilde{\Omega}\right), H^{2,\mathbf{p}}\left(\tilde{\Omega}\right)\right)_{\frac{1}{2p},p} = B^{\frac{1}{p}}_{p_1,p}\left(\tilde{\Omega}\right),$$

$$Y_1^{s,p} = H^{s,p}\left(\tilde{\Omega}\right) \cap L^1\left(\tilde{\Omega}\right), \quad Y_\infty^{s,p} = H^{s,p}\left(\tilde{\Omega}\right) \cap L^\infty\left(\tilde{\Omega}\right).$$

Suppose $\nu = (\nu_1, \nu_2, ..., \nu_n)$ are nonnegative real numbers. In this section, we present the following result:

Condition 5.1. Assume $\varphi, \psi \in Y_\infty^{2,p}$ and $1 < p < \infty$ for $\frac{n}{p} < 2$. Moreover, the function $u \to f(u) : \tilde{\Omega} \times [0, T] \times B^{\frac{1}{p}}_{2,p}\left(\tilde{\Omega}\right) \to L^{p_1}(0,1)$ is a measurable for $u \in B^{\frac{1}{p}}_{2,p}\left(\tilde{\Omega}\right)$, $f(x, y, t, ., .)$ is continuous with respect to $u \in B^{\frac{1}{p}}_{2,p}\left(\tilde{\Omega}\right)$ for $x \in R^n$, $y \in (0,1)$, $t \in [0, T]$ and $f(u) \in C^{(3)}\left(B^{\frac{1}{p}}_{2,p}\left(\tilde{\Omega}\right); L^{p_1}(0,1)\right)$ for a.e. $x \in R^n$, $y \in (0,1)$. The main aim of this section is to prove the following result:

Theorem 5.2. *Suppose the Condition* 5.1 *hold and* $a \in W^{1,\infty}(0,1)$, $a(x) \geq \delta > 0$, *b,* $c \in L^\infty(0,1)$. *Then problem* (1.5)–(1.7) *has a unique local strange solution* $u \in C^{(2)}([0, T_0); Y_\infty^{2,p})$, *where* T_0 *is a maximal time interval that is appropriately small relative to* M. *Moreover, if*

$$\sup_{t \in [0, T_0)} \left(\|u\|_{Y^{2,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{2,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

Proof. Let $E = L^2(0,1)$. It is known [29] that $L^2(0,1)$ is an UMD space. Consider the operator A defined by (1.5). Then, the problem (1.6)–(1.8) can be rewritten in the form of (1.1), where $u(x) = u(x, .)$, $f(x) = f(x, .)$ are functions with values in $E = L^2(0,1)$. By virtue of [31, 32] the operator A generates analytic semigroup in $L^2(0,1)$. Then in view of Hill- Yosida theorem (see e.g. [33, § 1.13]) this operator is positive in $L^2(0,1)$. Since all uniform bounded set in Hilbert space is an R-bounded, we get that the operator A is R-positive in $L^2(0,1)$. Then from Theorem 2.5 we obtain the assertion. □

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