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STOCHASTIC INCLUSIONS DRIVEN BY TWO-PARAMETER MARTINGALES

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ABSTRACT. The aim of the paper is the analysis of existence and properties of solutions to stochastic integral inclusions driven by two-parameter martingales. In our investigations we apply set-valued stochastic integral equations and we establish their connections with stochastic integral inclusions. Finally, we show how some particular two-parameter stochastic models are related to stochastic inclusions.

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1. Introduction

There have been many recent papers involving both the theory and applications of stochastic differential inclusions and stochastic set-valued integral equations in one-parameter case. Such studies have been mainly inspired by the theory of stochastic controlled dynamic systems and appear as their generalizations (see [1]–[4], [6], [15], [19]–[24], [27], [30]–[38], [41]–[45], [48], [52] and references therein). Namely, similarly as in the case of deterministic differential inclusions, stochastic inclusions appear as generalizations of a family of stochastic equations which depend on control parameters.

On the other hand as far as we know there are only few papers dealing with stochastic integral inclusions driven by random fields (see [25], [26], [47]). Such inclusions

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have been considered only with respect to two-parameter Wiener process. Therefore, in this paper we investigate stochastic integral inclusions driven by more general integrators i.e. two-parameter continuous increasing processes and two-parameter continuous martingales. We also consider their connections with set-valued stochastic integral equations. Two-parameter set-valued stochastic integral equations driven by martingales and their applications were recently studied in [39] and [40]. Both two-parameter stochastic integral inclusions and set-valued integral equations reduce in a single-valued case to some hyperbolic stochastic partial differential equations. In particular, they include so called stochastic Goursat problem (see e.g. [5] and [14]). Additionally, such stochastic inclusions and set-valued stochastic integral equations can be treated as a generalizations of stochastic differential equations in the plane which have a wide range of financial applications (see e.g. [10], [16], [17]). In our study we apply similar methods that were used in a single-valued case for stochastic equations in [28], [29] and in a multivalued case in [26].

The paper is organized as follows. In Section 2 we recall some basic notions and facts from the theory of stochastic and set-valued analysis needed in the sequel. Throughout Section 3 we analyze existence and main topological properties of solutions to stochastic integral inclusions. Next, in Section 4 we establish main interrelation between solutions to stochastic integral inclusions and set-valued stochastic integral equations driven by two-parameter martingales. Finally we present some concluding remarks on our results.

2. **Preliminaries**

Let $I \times J = [0,S] \times [0,T]$ denote the parameter set together with the partial ordering $(s,t) \preceq (s',t') \Leftrightarrow s \leqslant s',\ t \leqslant t'$. Suppose that $(\Omega, \mathbb{F}, \{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}, P)$ is a complete filtered probability space, where $\{\mathbb{F}_{s,t}\}_{(s,t)\in[0,S]\times[0,T]}$ is a family of sub-σ-fields of $\mathbb{F}$ satisfying the usual axioms (see [8] for details).

By $L^{p,d}$ we will denote the space $L^p\left(\Omega, \mathbb{F}, P; \mathbb{R}^d\right)$, where $p, d \geqslant 1$. In particular, for $d = 1$ we will write $L^p := L^{p,1}$. Let us denote $L^{2,d}_{s,t} := L^2(\Omega, \mathbb{F}_{s,t}, P; \mathbb{R}^d)$ for $(s,t) \in I \times J$.

A stochastic process $x : I \times J \times \Omega \to \mathbb{R}^d$ is said to be $\{\mathbb{F}_{s,t}\}$-adapted, if $x_{s,t} : \Omega \to \mathbb{R}^d$ is an $\mathbb{F}_{s,t}$-measurable random vector for every fixed $(s,t) \in I \times J$. The process x is called right-continuous, if for every $(s,t) \in [0,S) \times [0,T)$ one has $\lim_{\substack{(s',t')\to(s,t)\\(s,t)\preceq(s',t')}} x_{s',t'} = x_{s,t}$ a.s. Moreover, x is said to be continuous, if $\lim_{(s',t')\to(s,t)} x_{s',t'} = x_{s,t}$ a.s. for every $(s,t) \in [0,S) \times [0,T)$ (cf. [8], [11]).

Definition 2.1 ([8])**.** A process $M : I \times J \times \Omega \to \mathbb{R}$ is L^2-martingale if

(i) M is $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted,

(ii) $\mathbb{E}\left|M_{s,t}\right|^2 < \infty$ for all $(s,t) \in I \times J$,
(iii) $\mathbb{E}\left\{M_{s',t'}|\mathbb{F}_{s,t}\right\} = M_{s,t}$ a.s. for all $(s,t),(s',t') \in I \times J$, where $(s,t) \preceq (s',t')$.

We say that the process $x : I \times J \times \Omega \to \mathbb{R}^d$ vanishes on the axes, if $x_{0,t} = x_{s,0} = 0$ for all $s \in I$, $t \in J$. It is denoted as $\partial x = 0$. By $\Delta_{s,t}^{s',t'}(x)$ we denote the increment of x over the rectangle $(s,s'] \times (t,t']$ i.e.

$$\Delta_{s,t}^{s',t'}(x) = x_{s',t'} - x_{s',t} - x_{s,t'} + x_{s,t}.$$

Definition 2.2 ([8])**.** A process $A : I \times J \times \Omega \to \mathbb{R}$ is increasing if

(i) A is right-continuous and $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted,
(ii) $\partial A = 0$,
(iii) $\Delta_{s,t}^{s',t'}(A) \geqslant 0$ for each rectangle $(s,s'] \times (t,t'] \subset I \times J$ P-a.e.

Denote by $\mathcal{P}$ the σ-field of $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-predictable elements in $I \times J \times \Omega$. More precisely, $\mathcal{P}$ is a σ-field generated by the family

$$\begin{aligned}\mathcal{R} = \{&(s,s'] \times (t,t'] \times F : F \in \mathbb{F}_{s,t}, (s,t) \preceq (s',t'), (s,t), (s',t') \in I \times J\}\\ &\cup \{\{0\} \times (t,t'] \times F : F \in \mathbb{F}_{0,t}, t \leqslant t', t, t' \in I\}\\ &\cup \{(s,s'] \times \{0\} \times F : F \in \mathbb{F}_{s,0}, s \leqslant s', s, s' \in J\}\\ &\cup \{\{0\} \times \{0\} \times F : F \in \mathbb{F}_{0,0}\}.\end{aligned}$$

Then $\mathcal{P} \subset \mathcal{B} \otimes \mathbb{F}$, where $\mathcal{B} := \mathcal{B}(I \times J)$ is the Borel σ-field of subsets of $I \times J$.

A stochastic process $x : I \times J \times \Omega \to \mathbb{R}^d$ is said to be predictable, if x is $\mathcal{P}$-measurable.

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a separable Banach space. Denote by $\mathcal{K}_c(\mathbb{X})$ the family of all nonempty closed and convex subsets of $\mathbb{X}$. By $\mathcal{K}_c^b(\mathbb{X})$ we shall denote those elements in $\mathcal{K}_c(\mathbb{X})$, which are also bounded. The Hausdorff metric $H_{\mathbb{X}}$ in $\mathcal{K}_c(\mathbb{X})$ is defined by

$$H_{\mathbb{X}}(B,C) = \max\left\{\overline{H}_{\mathbb{X}}(B,C), \overline{H}_{\mathbb{X}}(C,B)\right\},$$

where $\overline{H}_{\mathbb{X}}(B,C) = \sup_{b\in B} \operatorname{dist}_{\mathbb{X}}(b,C) = \sup_{b\in B} \inf_{c\in C} \|b-c\|_{\mathbb{X}}$ (see e.g. [13] for details).

For $B,C,D,E \in \mathcal{K}_c^b(\mathbb{X})$ it holds (see [13])

$$H_{\mathbb{X}}(B+C, D+E) \leqslant H_{\mathbb{X}}(B,D) + H_{\mathbb{X}}(C,E) \tag{2.1}$$

and

$$H_{\mathbb{X}}(B+D, C+D) = H_{\mathbb{X}}(B,C), \tag{2.2}$$

where $B + C := \{b + c : b \in B, c \in C\}$ denotes the Minkowski sum of B and C. Moreover, by $B \ominus C$ we denote the Hukuhara difference (if it exists) of $B, C \in \mathcal{K}_c^b(\mathbb{X})$, i.e. the set $D \in \mathcal{K}_c^b(\mathbb{X})$ such that $B = C + D$. By $co\,(C)$ we denote the convex hull of the set $C \subset \mathbb{X}$, i.e. an intersection of all convex subsets of $\mathbb{X}$ containing C. Similarly, $\overline{co}\,(C)$ denotes the closed convex hull of the set C.

The space $(\mathcal{K}_c(\mathbb{X}), H_{\mathbb{X}})$ is complete and $\left(\mathcal{K}_c^b(\mathbb{X}), H_{\mathbb{X}}\right)$ is its a closed subspace.

Let $(U,\mathcal{U},\mu)$ be a measure space. A set-valued mapping (multifunction) $F : U \to \mathcal{K}_c^b(\mathbb{X})$ is said to be $\mathcal{U}$-measurable (or measurable), if it satisfies

$$\{u \in U : F(u) \cap C \neq \emptyset\} \in \mathcal{U} \text{ for every closed set } C \subset \mathbb{X}.$$

A measurable multifunction $F : U \to \mathcal{K}_c^b(\mathbb{X})$ is said to be L^p-integrally bounded $(p \geqslant 1)$, if there exists $h \in L^p(U,\mathcal{U},\mu;\mathbb{R}_+)$ such that $||F||_{\mathbb{X}} \leqslant h$ μ-a.e., where

$$||A||_{\mathbb{X}} = H_{\mathbb{X}}(A,\{0\}) = \sup_{a\in A} \|a\|_{\mathbb{X}} \text{ for } A \in \mathcal{K}_c^b(\mathbb{X}).$$

Then F is L^p-integrally bounded if and only if $||F||_{\mathbb{X}} \in L^p(U,\mathcal{U},\mu;\mathbb{R}_+)$ (see [12]).

In order to consider a stochastic integral inclusion driven by a two-parameter increasing process A and a two-parameter L^2-martingale M we recall the main properties of set-valued functional stochastic integrals. Similarly as in [39] for the process A one can define a measure ν_A on the measurable space $(I \times J \times \Omega, \mathcal{P})$ as follows

$$\nu_A(C) := \mathbb{E}\left\{\int_{I\times J} \mathbb{I}_C(\omega,s,t)\, A_{S,T}(\omega)\, dA_{s,t}(\omega)\right\} \text{ for } C \in \mathcal{P}, \tag{2.3}$$

where $dA_{s,t}(\omega)$ is a random measure on the measurable space $(I \times J, \mathcal{B})$ generated by the trajectory of A.

Hence $\nu_A(I \times J \times \Omega) = \mathbb{E}A^2_{S,T}$ and ν_A is finite if and only if $\mathbb{E}A^2_{S,T} < \infty$. We will assume this property throughout the paper. Let

$$L^{2,d}_{\mathcal{P}}(\nu_A) := L^2(I \times J \times \Omega, \mathcal{P}, \nu_A; \mathbb{R}^d).$$

The set-valued mapping $F : I \times J \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ is called a two-parameter predictable set-valued process, if it is $\mathcal{P}$-measurable in the sense of the set-valued analysis. It is called $L^{2,d}_{\mathcal{P}}(\nu_A)$-integrally bounded, if

$$||F||_{\mathbb{R}^d} \in L^2(I \times J \times \Omega, \mathcal{P}, \nu_A; \mathbb{R}_+).$$

For such a mapping F, by Kuratowski and Ryll-Nardzewski Measurable Selection Theorem (cf. [18]), the set of its predictable and square integrable selections

$$\mathcal{S}^2_{\mathcal{P}}(F,\nu_A) := \{f \in L^{2,d}_{\mathcal{P}}(\nu_A) : f \in F \ \nu_A\text{-}a.e.\}$$

is nonempty.

Definition 2.3 ([39])**.** By the two-parameter set-valued functional stochastic integral of F, driven by a two-parameter increasing process A, we mean the set

$$\int_{[s,s']\times[t,t']} F_{u,v}\, dA_{u,v} := \left\{\int_{[s,s']\times[t,t']} f_{u,v}\, dA_{u,v} : f \in \mathcal{S}^2_{\mathcal{P}}(F,\nu_A)\right\}$$

for every $(s,t), (s',t') \in I \times J$, where $(s,t) \preceq (s',t')$.

Due to the Cauchy-Schwarz inequality we obtain

$$\mathbb{E}\|\int_{[s,s']\times[t,t']} f_{u,v}dA_{u,v}\|^2_{\mathbb{R}^d} \leqslant \int_{[s,s']\times[t,t']\times\Omega} \|f\|^2_{\mathbb{R}^d}\, d\nu_A \tag{2.4}$$

for every (s,t), $(s',t')\in I\times J$ such that $(s,t)\preceq(s',t')$ and every $f\in\mathcal{S}^2_{\mathcal{P}}(F,\nu_A)$.

Theorem 2.4 ([39])**.** *Let $F: I\times J\times\Omega\to\mathcal{K}^b_c(\mathbb{R}^d)$ be a predictable and $L^{2,d}_{\mathcal{P}}(\nu_A)$-integrally bounded set-valued stochastic process. Then*

(i) *$\mathcal{S}^2_{\mathcal{P}}(F,\nu_A)$ is a nonempty, closed, bounded, convex, $\mathcal{P}$-decomposable and weakly compact subset of $L^{2,d}_{\mathcal{P}}(\nu_A)$,*
(ii) *$\int_{[s,s']\times[t,t']} F_{u,v}dA_{u,v}$ is a nonempty, closed, bounded, convex and weakly compact subsets of $L^{2,d}_{s',t'}$ for every (s,t), $(s',t')\in I\times J$ and $(s,t)\preceq(s',t')$.*

Theorem 2.5 ([39])**.** *Let $F,G: I\times J\times\Omega\to\mathcal{K}^b_c(\mathbb{R}^d)$ be predictable and $L^{2,d}_{\mathcal{P}}(\nu_A)$-integrally bounded set-valued stochastic processes. Then*

$$H^2_{L^{2,d}}\left(\int_{[s,s']\times[t,t']} F_{u,v}dA_{u,v}, \int_{[s,s']\times[t,t']} G_{u,v}dA_{u,v}\right) \leqslant \int_{[s,s']\times[t,t']\times\Omega} H^2_{\mathbb{R}^d}(F,G)d\nu_A$$

for every $(s,t)\,,(s',t')\in I\times J$, where $(s,t)\preceq(s',t')$.

Theorem 2.6 ([39])**.** *Suppose that A is a continuous increasing process on $I\times J$. Let $F: I\times J\times\Omega\to\mathcal{K}^b_c(\mathbb{R}^d)$ be a predictable and $L^{2,d}_{\mathcal{P}}(\nu_A)$-integrally bounded set-valued stochastic process. Then the correspondence*

$$I\times J\ni(s,t)\mapsto\int_{[0,s]\times[0,t]} F_{u,v}dA_{u,v}\in\mathcal{K}^b_c(L^{2,d})$$

is a continuous set-valued mapping with respect to the metric $H_{L^{2,d}}$.

Below we also recall the notion of set-valued functional stochastic integral with respect to a two-parameter L^2-martingale M with $\partial M=0$. Similarly as in [39] we will consider Doléan's measure on a σ-field $\mathcal{P}$ defined as follows

$$\mu_M(C)=\mathbb{E}\left\{\int_{I\times J}\mathbb{I}_C(u,v,\cdot)\,d\langle M\rangle_{u,v}(\cdot)\right\} \text{ for } C\in\mathcal{P}, \tag{2.5}$$

where $\langle M\rangle$ denotes a quadratic variation process of M (see [8]). Let

$$L^{2,d}_{\mathcal{P}}(\mu_M):=L^2(I\times J\times\Omega,\mathcal{P},\mu_M;\mathbb{R}^d).$$

Then for every $g\in L^{2,d}_{\mathcal{P}}(\mu_M)$ one can define the stochastic integral with respect to M (see [8] for details). This integral has the following isometry property.

Theorem 2.7 (Theorem 2.2 [8])**.** *Let $g\in L^{2,d}_{\mathcal{P}}(\mu_M)$. Then*

$$\begin{aligned}\mathbb{E}\|\int_{[s,s']\times[t,t']} g_{u,v}dM_{u,v}\|^2_{\mathbb{R}^d} &= \mathbb{E}\left\{\int_{[s,s']\times[t,t']}\|g_{u,v}\|^2_{\mathbb{R}^d}\,d\langle M\rangle_{u,v}\right\}\\ &= \int_{[s,s']\times[t,t']\times\Omega}\|g\|^2_{\mathbb{R}^d}\,d\mu_M\end{aligned} \tag{2.6}$$

for every (s,t), $(s',t')\in I\times J$ such that $(s,t)\preceq(s',t')$.

The integral process $\left(\int_{[0,s]\times[0,t]} g_{u,v} dM_{u,v}\right)_{(s,t)\in I\times J}$ is a right-continuous d-dimensional L^2-martingale (cf. [8]). Consequently by (2.6) and Doob's martingale inequality we obtain

$$\mathbb{E}\left(\sup_{(s,t)\in I\times J} \| \int_{[0,s]\times[0,t]} g_{u,v} dM_{u,v}\|^2_{\mathbb{R}^d}\right) \leqslant 16 \int_{I\times J\times\Omega} \|g\|^2_{\mathbb{R}^d}\, d\mu_M. \tag{2.7}$$

For $G : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ being a predictable and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded set-valued stochastic process, let

$$\mathcal{S}^2_{\mathcal{P}}(G, \mu_M) := \{g \in L^{2,d}_{\mathcal{P}}(\mu_M) : g \in G\ \mu_M\text{-}a.e.\}.$$

Similarly as earlier, since $G : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ is $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded, it follows that $\mathcal{S}^2_{\mathcal{P}}(G, \mu_M) \neq \emptyset$.

Definition 2.8 ([39])**.** By the two-parameter set-valued functional stochastic integral of G, driven by a two-parameter martingale M, we mean the set

$$\int_{[s,s']\times[t,t']} G_{u,v} dM_{u,v} := \left\{\int_{[s,s']\times[t,t']} g_{u,v} dM_{u,v} : g \in \mathcal{S}^2_{\mathcal{P}}(G, \mu_M)\right\}$$

for every $(s,t), (s',t') \in I \times J$, where $(s,t) \preceq (s',t')$.

Theorem 2.9 ([39])**.** *Let $G : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be a predictable and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded set-valued stochastic process. Then*

- (i) *$\mathcal{S}^2_{\mathcal{P}}(G, \mu_M)$ is a nonempty, closed, bounded, convex, $\mathcal{P}$-decomposable and weakly compact subset of $L^{2,d}_{\mathcal{P}}(\mu_M)$,*
- (ii) *$\int_{[s,s']\times[t,t']} G_{u,v} dM_{u,v}$ is a nonempty, closed, bounded, convex and weakly compact subsets of $L^{2,d}_{s',t'}$ for every (s,t), $(s',t') \in I \times J$, where $(s,t) \preceq (s',t')$.*

Theorem 2.10 ([39])**.** *Let $F, G : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be predictable and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded set-valued stochastic processes. Then*

$$H^2_{L^{2,d}}\left(\int_{[s,s']\times[t,t']} F_{u,v} dM_{u,v}, \int_{[s,s']\times[t,t']} G_{u,v} dM_{u,v}\right) \leqslant \int_{[s,s']\times[t,t']\times\Omega} H^2_{\mathbb{R}^d}(F, G) d\mu_M$$

for every $(s,t), (s',t') \in I \times J$ and $(s,t) \preceq (s',t')$.

Theorem 2.11 ([39])**.** *Suppose that M is a continuous L^2-martingale on $I \times J$. Let $G : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be a predictable and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded set-valued stochastic process. Then the correspondence*

$$I \times J \ni (s,t) \mapsto \int_{[0,s]\times[0,t]} G_{u,v} dM_{u,v} \in \mathcal{K}^b_c(L^{2,d})$$

is a continuous set-valued mapping with respect to the metric $H_{L^{2,d}}$.

3. **Stochastic inclusions and their properties**

In this part we consider stochastic integral inclusions driven by two-parameter integrators. We establish the existence and main properties of solutions to such inclusions. In our study we apply multivalued counterpart of methods used for stochastic equations studied in [29].

Assume that $(\Omega, \mathbb{F}, \{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}, P)$ is a complete filtered probability space. Let $S^2(I\times J)$ denote the space of all continuous $\{\mathbb{F}_{s,t}\}$-adapted two-parameter stochastic processes on $I\times J$ which satisfy the condition

$$\|x\|_{S^2}^2 = \left\{\mathbb{E}\left\{\sup_{(s',t')\preceq(s,t)} \|x(s',t')\|_{\mathbb{R}^d}^2\right\}\right\}^{\frac{1}{2}} < \infty \text{ for } (s,t)\in I\times J.$$

Then $(S^2(I\times J), \|\cdot\|_{S^2}^2)$ is a Banach space.

Before we formulate the main result of this section we recall the following version of Carathéodory/Lipschitz Selection Theorem needed in the sequel. Let $(\Sigma, \mathbb{M}, \lambda)$ be a measure space and let $\mathbb{X}$ be a linear normed space. We assume that for a set-valued mapping $\tilde{F}:\Sigma\times\mathbb{X}\to\mathcal{K}_c^b\left(\mathbb{R}^d\right)$ and $\mathbb{M}$-measurable functions $L, K:\Sigma\to\mathbb{R}_+$ the following conditions are satisfied:

(i) $\tilde{F}(\cdot, x)$ is $\mathbb{M}$-measurable for every $x\in\mathbb{X}$,
(ii) $H^2_{\mathbb{R}^d}(\tilde{F}(\sigma,x), \tilde{F}(\sigma,y)) \leqslant L_\sigma \|x-y\|^2_{\mathbb{X}}$ for every $x,y\in\mathbb{X}$ and $\sigma\in\Sigma$,
(iii) $H^2_{\mathbb{R}^d}(\tilde{F}(\sigma,x), \{\theta\}) \leqslant K_\sigma(1+\|x\|^2_{\mathbb{X}})$ for every $x\in\mathbb{X}$ and $\sigma\in\Sigma$, where the symbol θ denotes the zero element in $\mathbb{R}^d$.

Similarly as in [26] one can prove the following result.

Proposition 3.1 (Proposition 2 [26]). *Let $\tilde{F}:\Sigma\times\mathbb{X}\to\mathcal{K}_c^b\left(\mathbb{R}^d\right)$ be a set-valued mapping satisfying conditions (i)–(iii). Then there exists a function $\tilde{f}:\Sigma\times\mathbb{X}\to\mathbb{R}^d$ such that*

(a) $\tilde{f}(\sigma,x)\in\tilde{F}(\sigma,x)$ *for all* $(\sigma,x)\in\Sigma\times\mathbb{X}$,
(b) $\tilde{f}(\cdot,x)$ *is $\mathbb{M}$-measurable for each* $x\in\mathbb{X}$,
(c) $\|\tilde{f}(\sigma,x)-\tilde{f}(\sigma,y)\|^2_{\mathbb{R}^d} \leqslant d^2 L_\sigma\|x-y\|^2_{\mathbb{X}}$ *for all $\sigma\in\Sigma$ and $x,y\in\mathbb{X}$,*
(d) $\|\tilde{f}(\sigma,x)\|^2_{\mathbb{R}^d} \leqslant K_\sigma(1+\|x\|^2_{\mathbb{X}})$ *for every $\sigma\in\Sigma$ and $x\in\mathbb{X}$.*

Now we will focus our attention on stochastic integral inclusions. For this reason let set-valued mappings $F, G: I\times J\times\Omega\times\mathbb{R}^d\to\mathcal{K}_c^b(\mathbb{R}^d)$ be given. By a stochastic integral inclusion we mean the relation

(3.1)
$$\begin{cases}\Delta_{s,t}^{s',t'}(x) \in \int_{[s,s']\times[t,t']} F(u,v,x(u,v))\,dA_{u,v} + \int_{[s,s']\times[t,t']} G(u,v,x(u,v))\,dM_{u,v}\\ x(0,t)=\xi(0,t)\\ x(s,0)=\xi(s,0)\end{cases}$$

for every $(s,t), (s',t') \in I \times J$, where $(s,t) \preceq (s',t')$ and $\xi : I \times J \times \Omega \to \mathbb{R}^d$ being a given $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted and continuous stochastic process.

A stochastic process $x \in S^2(I \times J)$ is a strong solution to stochastic inclusion (3.1) if there exist $f \in S^2_{\mathcal{P}}(F \circ x, \nu_A)$ and $g \in S^2_{\mathcal{P}}(G \circ x, \mu_M)$ such that

$$\begin{aligned} &x(s,t) + \xi(0,0) - \xi(s,0) - \xi(0,t) \\ &\quad = \int_{[0,s]\times[0,t]} f(u,v)\, dA_{u,v} + \int_{[0,s]\times[0,t]} g(u,v)\, dM_{u,v} \text{ for } (s,t) \in I \times J, \end{aligned}$$

where $F \circ x$ and $G \circ x$ are set-valued processes such that $(F \circ x)(s,t,\omega) = F(s,t,\omega,x(s,t,\omega))$ and $(G \circ x)(s,t,\omega) = G(s,t,\omega,x(s,t,\omega))$.

Remark 3.2. Note that for a such solution x we have equivalently

$$\Delta_{s,t}^{s',t'}(x) = \int_{[s,s']\times[t,t']} f(u,v)\, dA_{u,v} + \int_{[s,s']\times[t,t']} g(u,v)\, dM_{u,v}$$

with $x(s,0) = \xi(s,0)$, $x(0,t) = \xi(0,t)$ for every $(0,0) \preceq (s,t) \preceq (s',t') \preceq (S,T)$.

Remark 3.3. In a particular case with $F = \{a\}, G = \{b\}$ where $a, b : I \times J \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, inclusion (3.1) reduces to stochastic equation

$$\begin{aligned} &x(s,t) + \xi(0,0) - \xi(s,0) - \xi(0,t) \\ &\quad = \int_{[0,s]\times[0,t]} a(u,v,x(u,v))\, dA_{u,v} + \int_{[0,s]\times[0,t]} b(u,v,x(u,v))\, dM_{u,v} \end{aligned}$$

for $(s,t) \in I \times J$, studied in [29]. It also reduces to stochastic equations considered in [49], [50] and [51]. Moreover, taking $A_{s,t} = st$ and $M_{s,t} = W_{s,t}$ (with a two-parameter Wiener process W) it reduces further to the stochastic form of the Goursat problem

$$\begin{aligned} &x(s,t) = \xi(s,0) + \xi(0,t) - \xi(0,0) \\ &\quad + \int_{[0,s]\times[0,t]} a(u,v,x(u,v))\, du dv + \int_{[0,s]\times[0,t]} b(u,v,x(u,v))\, dW_{u,v} \end{aligned}$$

for $(s,t) \in I \times J$, considered among others in [5] and [53], which can be formally recast as the stochastic partial differential equation

$$\frac{\partial^2 x}{\partial s \partial t} = a(s,t,x(s,t)) + b(s,t,x(s,t)) \frac{\partial^2 W}{\partial s \partial t}.$$

By $SI(F,G,\xi)$ we will denote the set of all solutions to inclusion (3.1). We assume that the multifunctions F and G satisfy the following conditions

(i1) $F(\cdot,\cdot,\cdot,x)$ and $G(\cdot,\cdot,\cdot,x)$ are predictable for every $x \in \mathbb{R}^d$,

(i2) there exist a non-negative predictable process $L = (L_{s,t})_{(s,t)\in I\times J}$ and an increasing function B on $I \times J$ such that for every $\omega \in \Omega$ and

$$\hat{G}(s,t,\omega) \equiv \int_{[0,s]\times[0,t]} [1 + L_{u,v}(\omega)] \left(dA_{u,v}(\omega) + d\langle M\rangle_{u,v}(\omega)\right)$$

the random measure generated by $\hat{G}(\cdot,\cdot,\omega)$ is dominated by the measure generated by B, i.e. $d\hat{G}(\cdot,\cdot,\omega) \leqslant dB(\cdot,\cdot)$, (i3) for every $x, y \in \mathbb{R}^d$ and $(s,t,\omega) \in I \times J \times \Omega$, it holds

$$H^2_{\mathbb{R}^d}(F(s,t,\omega,x), F(s,t,\omega,y)) + H^2_{\mathbb{R}^d}(G(s,t,\omega,x), G(s,t,\omega,y)) \leqslant L_{s,t}(\omega)\, \|x-y\|^2_{\mathbb{R}^d},$$

where L is the same process as in (i2),

(i4) for every $x \in \mathbb{R}^d$ and $(s,t,\omega) \in I \times J \times \Omega$ we have

$$H^2_{\mathbb{R}^d}(F(s,t,\omega,x), \{0\}) + H^2_{\mathbb{R}^d}(G(s,t,\omega,x), \{0\}) \leqslant d^2 L_{s,t}(\omega)(1+\|x\|^2_{\mathbb{R}^d}),$$

with L as above.

Additionally, we assume that the process ξ satisfies the condition

$$\mathbb{E}\left\{\sup_{(s,t)\in I\times J} ||\xi(0,t)+\xi(s,0)-\xi(0,0)||^2_{\mathbb{R}^d}\right\} < \infty. \tag{3.2}$$

Theorem 3.4. *Let ξ be $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted and continuous stochastic process satisfying inequality* (3.2)*. Assume that $F, G : I \times J \times \mathbb{R}^d \to \mathcal{K}^b_c(\mathbb{R}^d)$ satisfy conditions (i1)–(i4). Then $SI(F,G,\xi)$ is nonempty, bounded and closed subset of the space $S^2(I\times J)$.*

Proof. Taking $\Sigma = I \times J \times \Omega$, $\mathbb{M} = \mathcal{P}$ and $\mathbb{X} = \mathbb{R}^d$ by Proposition 3.1 with processes $L_\sigma = L_{s,t}(\omega)$ and $K_\sigma = d^2 L_{s,t}(\omega)$ there exist selections $a(u,v,\omega,x) \in F(u,v,\omega,x)$, $b(u,v,\omega,x) \in G(u,v,\omega,x)$ such that the functions $a(\cdot,\cdot,\cdot,x)$ and $b(\cdot,\cdot,\cdot,x)$ are predictable. Moreover the functions $a(u,v,\omega,\cdot)$ and $b(u,v,\omega,\cdot)$ are such that for every $(s,t,\omega) \in I \times J \times \Omega$ and $x, y \in \mathbb{R}^d$ it holds:

$$\|a\,(s,t,\omega,x) - a\,(s,t,\omega,y)\|^2_{\mathbb{R}^d} \leqslant d^2 L_{s,t}\,(\omega)\,\|x-y\|^2_{\mathbb{R}^d},$$
$$\|b\,(s,t,\omega,x) - b\,(s,t,\omega,y)\|^2_{\mathbb{R}^d} \leqslant d^2 L_{s,t}\,(\omega)\,\|x-y\|^2_{\mathbb{R}^d}$$

and

$$\|a\,(s,t,\omega,x)\|^2_{\mathbb{R}^d} \leqslant d^2 L_{s,t}\,(\omega)\,(1+\|x\|^2_{\mathbb{R}^d}),$$
$$\|b\,(s,t,\omega,x)\|^2_{\mathbb{R}^d} \leqslant d^2 L_{s,t}\,(\omega)\,(1+\|x\|^2_{\mathbb{R}^d}).$$

Let us consider the equation

$$\begin{aligned} &x(s,t) - \xi(0,t) - \xi(s,0) + \xi(0,0) \\ &= \int_{[0,s]\times[0,t]} a(u,v,x(u,v))dA_{u,v} + \int_{[0,s]\times[0,t]} b(u,v,x(u,v))\, dM_{u,v}. \end{aligned} \tag{3.3}$$

Then similarly as in the proof of Theorem 3.1 in [29] one can show that there exists a process $\widehat{x}$ which is an unique strong solution to equation (3.3). Then $\widehat{x}$ satisfies

$$\Delta^{s',t'}_{s,t}\,(x) = \int_{[s,s']\times[t,t']} a\,(u,v,\widehat{x}(u,v))\,dA_{u,v} + \int_{[s,s']\times[t,t']} b\,(u,v,\widehat{x}(u,v))\,dM_{u,v}$$

with $\widehat{x}(s,0) = \xi\,(s,0)$, $\widehat{x}(0,t) = \xi\,(0,t)$ for every $(0,0) \preceq (s,t) \preceq (s',t') \preceq (S,T)$. Hence $\widehat{x}$ is also a solution to stochastic inclusion (3.1). Thus the set $SI(F,G,\xi)$ is nonempty.

In order to prove the boundedness of $SI(F,G,\xi)$, let $x \in SI(F,G,\xi)$. Then there exist $f \in \mathcal{S}^2_{\mathcal{P}}(F\circ x, \nu_A)$ and $g \in \mathcal{S}^2_{\mathcal{P}}(G\circ x, \mu_M)$ such that

$$x(s,t) - \xi(0,t) - \xi(s,0) + \xi(0,0) = \int_{[0,s]\times[0,t]} f(u,v)dA_{u,v} + \int_{[0,s]\times[0,t]} g(u,v)dM_{u,v}$$

for every $(s,t) \in I \times J$. Thus we get

(3.4)

$$\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} \|x(u,v)\|^2_{\mathbb{R}^d}\right\} \leqslant 3\mathbb{E}\{\sup_{(u,v)\in[0,s]\times[0,t]} \|\xi(0,v)+\xi(u,0)-\xi(0,0)\|^2_{\mathbb{R}^d}\}$$
$$+ 3\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} \|\int_{[0,u]\times[0,v]} f(\tau,\eta)dA_{\tau,\eta}\|^2_{\mathbb{R}^d}\right\}$$
$$+ 3\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} \|\int_{[0,u]\times[0,v]} g(\tau,\eta)dM_{\tau,\eta}\|^2_{\mathbb{R}^d}\right\}.$$

By the Cauchy inequality applied to the second term on the right hand side of inequality (3.4) we have

$$\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} \|\int_{[0,u]\times[0,v]} f(\tau,\eta)dA_{\tau,\eta}\|^2_{\mathbb{R}^d}\right\}$$
$$\leqslant \mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} A_{u,v}\int_{[0,u]\times[0,v]} \|f(\tau,\eta)\|^2_{\mathbb{R}^d}dA_{\tau,\eta}\right\}$$
$$\leqslant \mathbb{E}\left\{A_{s,t}\int_{[0,s]\times[0,t]} \|F(u,v,\omega,x(u,v))\|^2_{\mathbb{R}^d}dA_{u,v}\right\}$$
$$\leqslant d^2\mathbb{E}\left\{A_{s,t}\int_{[0,s]\times[0,t]} L_{s,t}(\omega)(1+\|x(u,v)\|^2_{\mathbb{R}^d})dA_{u,v}\right\}$$
$$\leqslant d^2\mathbb{E}\Bigg\{\left[\int_{[0,s]\times[0,t]}(1+L_{u,v}(\omega))(dA_{u,v}+d\left\langle M\right\rangle_{u,v})\right]^2$$
$$+\int_{[0,s]\times[0,t]}(1+L_{u,v}(\omega))(dA_{u,v}+d\left\langle M\right\rangle_{u,v})$$
$$\times\int_{[0,s]\times[0,t]}\sup_{(\tau,\eta)\in[0,u]\times[0,v]}\|x(\tau,\eta)\|^2_{\mathbb{R}^d}(1+L_{u,v}(\omega))(dA_{u,v}+d\left\langle M\right\rangle_{u,v})\Bigg\}$$
$$= d^2B^2_{s,t} + d^2B_{s,t}\int_{[0,s]\times[0,t]}\mathbb{E}\{\sup_{(\tau,\eta)\in[0,u]\times[0,v]}\|x(\tau,\eta)\|^2_{\mathbb{R}^d}\}dB_{u,v}.$$

Next, by Doob's inequality and assumption (i4) we get the following inequalities for the third part of (3.4)

$$\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]} \|\int_{[0,u]\times[0,v]} g(\tau,\eta)dM_{\tau,\eta}\|^2_{\mathbb{R}^d}\right\}$$
$$\leqslant 16\sup_{(u,v)\in[0,s]\times[0,t]}\mathbb{E}\left\{\int_{[0,u]\times[0,v]}\|g(\tau,\eta)\|^2_{\mathbb{R}^d}d\left\langle M\right\rangle_{\tau,\eta}\right\}$$
$$\leqslant 16d^2\mathbb{E}\left\{\int_{[0,s]\times[0,t]}(1+L_{u,v}(\omega))(1+\|x(u,v)\|^2_{\mathbb{R}^d})d\left\langle M\right\rangle_{u,v}\right\}$$

$$
\begin{aligned}
&\leqslant 16d^2\mathbb{E}\left\{\int_{[0,s]\times[0,t]}(1+L_{u,v}(\omega))(1+\|x(u,v)\|^2_{\mathbb{R}^d})(d\left\langle M\right\rangle_{u,v}+dA_{u,v})\right\}\\
&\leqslant 16d^2\Bigg(\mathbb{E}\left\{\int_{[0,s]\times[0,t]}(1+L_{u,v}(\omega))(d\left\langle M\right\rangle_{u,v}+dA_{u,v})\right\}\\
&\quad+\mathbb{E}\left\{\int_{[0,s]\times[0,t]}\sup_{(\tau,\eta)\in[0,u]\times[0,v]}||x(\tau,\eta)||^2_{\mathbb{R}^d}(1+L_{u,v}(\omega))(d\left\langle M\right\rangle_{u,v}+dA_{u,v})\right\}\Bigg)\\
&\leqslant 16d^2B_{s,t}+16d^2\int_{[0,s]\times[0,t]}\mathbb{E}\{\sup_{(\tau,\eta)\in[0,u]\times[0,v]}\|x(\tau,\eta)\|^2_{\mathbb{R}^d}\}dB_{u,v}.
\end{aligned}
$$

Combining the above inequalities with (3.4) we obtain

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]}||x(u,v)||^2_{\mathbb{R}^d}\right\}\\
&\quad\leqslant 3\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]}\|\xi(0,v)+\xi(u,0)-\xi(0,0)\|^2_{\mathbb{R}^d}\right\}+3d^2B^2_{s,t}+48d^2B_{s,t}\\
&\qquad+3d^2(B_{s,t}+16)\int_{[0,s]\times[0,t]}\mathbb{E}\left\{\sup_{(\tau,\eta)\in[0,u]\times[0,v]}\|x(\tau,\eta)\|^2_{\mathbb{R}^d}\right\}dB_{u,v}.
\end{aligned}
$$

Let us note that the above inequality can be written as

$$a(s,t)\leqslant b(s,t)+c(s,t)\int_{[0,s]\times[0,t]}a(u,v)dB_{u,v},$$

where

$$a(s,t):=\mathbb{E}\left\{\sup_{(\tau,\eta)\in[0,s]\times[0,t]}\|x(\tau,\eta)\|^2_{\mathbb{R}^d}\right\},$$

$$b(s,t):=3\mathbb{E}\left\{\sup_{(u,v)\in[0,s]\times[0,t]}\|\xi(0,v)+\xi(u,0)-\xi(0,0)\|^2_{\mathbb{R}^d}\right\}+3d^2B^2_{s,t}+48d^2B_{s,t}$$

and

$$c(s,t):=3d^2(16+B_{s,t}).$$

Hence by Gronwall's inequality (Theorem 2.3 in [29]) we get

$$a(s,t)\leqslant b(s,t)\exp\left\{3c(s,t)B_{s,t}\right\},$$

what proves the boundedness of $SI(F,G,\xi)$.

Now we will show that $SI(F,G,\xi)$ is a closed subset of $S^2(I\times J)$. Let $(x^n)\subset SI(F,G,\xi)$ be such that $x^n\to\widehat{x}$ in $S^2(I\times J)$. Then, there exist sequences $(f^n)\subset\mathcal{S}^2_{\mathcal{P}}(F\circ x^n,\nu_A)$ and $(g^n)\subset\mathcal{S}^2_{\mathcal{P}}(G\circ x^n,\mu_M)$ such that

$$
\begin{aligned}
&x^n(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)\\
&\quad=\int_{[0,s]\times[0,t]}f^n(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]}g^n(u,v)dM_{u,v}
\end{aligned}
\tag{3.5}
$$

for $n=1,2,\ldots$ and $(s,t)\in I\times J$. Since $x^n\to\widehat{x}$ in $S^2(I\times J)$, it holds

$$\mathbb{E}\|x^n(s,t)-\widehat{x}(s,t)\|^2_{\mathbb{R}^d}\to 0,\text{ as }n\to\infty$$

for every $(s,t) \in I \times J$. In view of (3.5) we have

$$\hat{x}(s,t) - \xi(0,t) - \xi(s,0) + \xi(0,0) = \lim_{n\to\infty}\left\{\int_{[0,s]\times[0,t]} f^n(u,v)dA_{u,v} + \int_{[0,s]\times[0,t]} g^n(u,v)dM_{u,v}\right\} \tag{3.6}$$

in $L^{2,d}$-norm for $(s,t) \in I \times J$. On the other hand, by condition (i4) and (2.3) we obtain

$$\begin{aligned}
\int_{I\times J\times\Omega} \|f^n\|^2_{\mathbb{R}^d} d\nu_A &\leqslant \int_{I\times J\times\Omega} \|F(s,t,x^n(s,t))\|^2_{\mathbb{R}^d} d\nu_A \\
&\leqslant d^2 \int_{I\times J\times\Omega} L_{s,t}(1+\|x^n(s,t)\|^2_{\mathbb{R}^d}) d\nu_A \\
&\leqslant d^2\mathbb{E}\left\{\int_{I\times J}(1+L_{s,t})(1+\|x^n(s,t)\|^2_{\mathbb{R}^d})A_{S,T}dA_{s,t}\right\} \\
&\leqslant d^2\mathbb{E}\left\{\int_{I\times J}(1+L_{s,t})(1+\|x^n(s,t)\|^2_{\mathbb{R}^d})(A_{S,T}+\langle M\rangle_{S,T})(dA_{s,t}+d\langle M\rangle_{s,t})\right\} \\
&\leqslant d^2\mathbb{E}\left\{\left[\int_{I\times J}(1+L_{s,t})(dA_{s,t}+d\langle M\rangle_{s,t})\right]^2\right\} \\
&\quad + d^2\mathbb{E}\left\{\sup_{(s,t)\in I\times J}\|x^n(s,t)\|^2_{\mathbb{R}^d}[\int_{I\times J}(1+L_{s,t})(dA_{s,t}+d\langle M\rangle_{s,t})]^2\right\} \\
&\leqslant d^2B^2_{S,T} + d^2B^2_{S,T}\mathbb{E}\left\{\sup_{(s,t)\in I\times J}\|x^n(s,t)\|^2_{\mathbb{R}^d}\right\} < \infty.
\end{aligned}$$

By the boundedness of $SI(F,G,\xi)$ we get

$$\sup_{n\geqslant 1}\int_{I\times J\times\Omega}\|f^n\|^2_{\mathbb{R}^d}d\nu_A < \infty.$$

In a similar way one can show that

$$\sup_{n\geqslant 1}\int_{I\times J\times\Omega}\|g^n\|^2_{\mathbb{R}^d}d\mu_M < \infty.$$

Indeed, by Theorem 2.7 and (i4) we have

$$\begin{aligned}
\int_{I\times J\times\Omega}\|g^n\|^2_{\mathbb{R}^d}d\mu_M &= \mathbb{E}\left\{\int_{I\times J}\|g^n(s,t)\|^2_{\mathbb{R}^d}d\langle M\rangle_{s,t}\right\} \\
&\leqslant \mathbb{E}\left\{\int_{I\times J}\|G(s,t,x^n(s,t))\|^2_{\mathbb{R}^d}d\langle M\rangle_{s,t}\right\} \\
&\leqslant d^2\mathbb{E}\left\{\int_{I\times J}(1+L_{s,t})(1+\|x^n(s,t)\|^2_{\mathbb{R}^d})(dA_{s,t}+d\langle M\rangle_{s,t})\right\} \\
&\leqslant d^2\mathbb{E}\left\{\int_{I\times J}(1+L_{s,t})(dA_{s,t}+d\langle M\rangle_{s,t})\right\} \\
&\quad + d^2\mathbb{E}\left\{\sup_{(s,t)\in I\times J}\|x^n(s,t)\|^2_{\mathbb{R}^d}\int_{I\times J}(1+L_{s,t})(dA_{s,t}+d\langle M\rangle_{s,t})\right\} \\
&\leqslant d^2B_{S,T} + d^2B_{S,T}\mathbb{E}\{\sup_{(s,t)\in I\times J}\|x^n(s,t)\|^2_{\mathbb{R}^d}\} < \infty.
\end{aligned}$$

Hence

$$\sup_{n\geqslant 1}\int_{I\times J\times\Omega}\|g^n\|^2_{\mathbb{R}^d}d\mu_M<\infty.$$

Then the sequences (f^n) and (g^n) are bounded in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$, respectively. Thus there exist subsequences $(f^{n_k})\subset(f^n)$ and $(g^{n_k})\subset(g^n)$ such that $f^{n_k}\rightharpoonup f$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $g^{n_k}\rightharpoonup g$ in $L^{2,d}_{\mathcal{P}}(\mu_M)$ for some $f\in L^{2,d}_{\mathcal{P}}(\nu_A)$ and $g\in L^{2,d}_{\mathcal{P}}(\mu_M)$, where "$\rightharpoonup$" denotes a weak convergence. Moreover $f^{n_k}\in S^2_{\mathcal{P}}(F\circ x^{n_k},\nu_A)$, $g^{n_k}\in S^2_{\mathcal{P}}(G\circ x^{n_k},\mu_M)$, and

$$\begin{aligned}&x^{n_k}(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)\\&\quad=\int_{[0,s]\times[0,t]}f^{n_k}(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]}g^{n_k}(u,v)dM_{u,v}\text{ for every }k\geqslant 1.\end{aligned}$$

By (2.4) and (2.7) the linear operators $I_{s,t}:L^{2,d}_{\mathcal{P}}(\nu_A)\to L^{2,d}$ and $J_{s,t}:L^{2,d}_{\mathcal{P}}(\mu_M)\to L^{2,d}$ such that

$$I_{s,t}(f):=\int_{[0,s]\times[0,t]}f(u,v)\,dA_{u,v}\text{ and }J_{s,t}(g):=\int_{[0,s]\times[0,t]}g(u,v)\,dM_{u,v}$$

are norm-to-norm continuous. Hence by Theorem 3.4.12 in [9] they are also continuous with respect to weak topologies in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}$, and in $L^{2,d}_{\mathcal{P}}(\mu_M)$ and $L^{2,d}$, respectively. Hence

$$I_{s,t}(f^{n_k})=\int_{[0,s]\times[0,t]}f^{n_k}(u,v)\,dA_{u,v}\rightharpoonup I_{s,t}(f)=\int_{[0,s]\times[0,t]}f(u,v)\,dA_{u,v}$$

and

$$J_{s,t}(g^{n_k})=\int_{[0,s]\times[0,t]}g^{n_k}(u,v)\,dM_{u,v}\rightharpoonup J_{s,t}(g)=\int_{[0,s]\times[0,t]}g(u,v)\,dM_{u,v},$$

as $k\to\infty$. By (3.6) we have

$$\int_{[0,s]\times[0,t]}f^{n_k}(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]}g^{n_k}(u,v)dM_{u,v}\rightharpoonup\widehat{x}(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)$$

in $L^{2,d}$. Thus

$$\widehat{x}(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)=\int_{[0,s]\times[0,t]}f(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]}g(u,v)dM_{u,v}.$$

In order to finish the proof it suffices to show that $f\in S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$ and $g\in S^2_{\mathcal{P}}(G\circ\widehat{x},\mu_M)$. For this purpose, let $P^1_C(\cdot)$ and $P^2_D(\cdot)$ denote metric projections from $L^{2,d}_{\mathcal{P}}(\nu_A)$ on $C\subset L^{2,d}_{\mathcal{P}}(\nu_A)$ and from $L^{2,d}_{\mathcal{P}}(\mu_M)$ on $D\subset L^{2,d}_{\mathcal{P}}(\mu_M)$, respectively. Let $\widehat{f}^{n_k}:=P^1_C(f^{n_k})$, where $C=S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$ and $\widehat{g}^{n_k}:=P^2_D(g^{n_k})$, where $D=S^2_{\mathcal{P}}(G\circ\widehat{x},\mu_M)$. Then by Theorem 2.2 in [12] we have

$$\begin{aligned}\|f^{n_k}-\widehat{f}^{n_k}\|^2_{L^{2,d}_{\mathcal{P}}(\nu_A)}&=\mathrm{dist}^2_{L^{2,d}_{\mathcal{P}}(\nu_A)}(f^{n_k},S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A))\\&=\inf_{f\in S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)}\int_{I\times J\times\Omega}\|f^{n_k}-f\|^2_{\mathbb{R}^d}\,d\nu_A\\&=\int_{I\times J\times\Omega}\mathrm{dist}^2_{\mathbb{R}^d}(f^{n_k},F\circ\widehat{x})d\nu_A\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{I\times J\times\Omega} H^2_{\mathbb{R}^d}(F\circ x^{n_k}, F\circ\widehat{x})d\nu_A \leqslant \int_{I\times J\times\Omega} L_{s,t}\left\|x^{n_k}-\widehat{x}\right\|^2_{\mathbb{R}^d} d\nu_A \\
&\leqslant \mathbb{E}\left\{\int_{I\times J}(1+L_{s,t})\left\|x^{n_k}(s,t)-\widehat{x}(s,t)\right\|^2_{\mathbb{R}^d} A_{S,T}dA_{s,t}\right\} \\
&= \mathbb{E}\left\{A_{S,T}\int_{I\times J}\sup_{(s,t)\in I\times J}\left\|x^{n_k}(s,t)-\widehat{x}(s,t)\right\|^2_{\mathbb{R}^d}(1+L_{s,t})(dA_{s,t}+d\left\langle M\right\rangle_{s,t})\right\} \\
&\leqslant \mathbb{E}\left\{\left[\int_{I\times J}(1+L_{s,t})(dA_{s,t}+d\left\langle M\right\rangle_{s,t})\right]^2\sup_{(s,t)\in I\times J}\left\|x^{n_k}(s,t)-\widehat{x}(s,t)\right\|^2_{\mathbb{R}^d}\right\} \\
&\leqslant B^2_{S,T}\mathbb{E}\left\{\sup_{(s,t)\in I\times J}\left\|x^{n_k}(s,t)-\widehat{x}(s,t)\right\|^2_{\mathbb{R}^d}\right\}.
\end{aligned}$$

Since $x^{n_k}\to\widehat{x}$ in $S^2(I\times J)$ it follows that

$$\|f^{n_k}-\widehat{f}^{n_k}\|^2_{L^{2,d}_{\mathcal{P}}(\nu_A)}\to 0,\ \text{ as } k\to\infty.$$

Hence $f^{n_k}-\widehat{f}^{n_k}\rightharpoonup 0$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$. Since we have observed earlier that $f^{n_k}\rightharpoonup f$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$ it follows that $\widehat{f}^{n_k}\rightharpoonup f$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$. Since $\widehat{f}^{n_k}\in S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$ and $S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$ is weakly compact (by Theorem 2.4) we obtain that $f\in S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$. In a similar way one can show that

$$\|g^{n_k}-\widehat{g}^{n_k}\|^2_{L^{2,d}_{\mathcal{P}}(\mu_M)}\to 0,\ \text{ as } k\to\infty$$

and consequently $g^{n_k}-\widehat{g}^{n_k}\rightharpoonup 0$ in $L^2_{\mathcal{P}}(\mu_M)$, as $k\to\infty$. Therefore $\widehat{g}^{n_k}\rightharpoonup g$ in $L^{2,d}_{\mathcal{P}}(\mu_M)$ because $g^{n_k}\rightharpoonup g$ in $L^{2,d}_{\mathcal{P}}(\mu_M)$. Since $\widehat{g}^{n_k}\in S^2_{\mathcal{P}}(G\circ\widehat{x},\mu_M)$ we claim by Theorem 2.9 that $g\in S^2_{\mathcal{P}}(G\circ\widehat{x},\mu_M)$. Hence the limit process $\widehat{x}$ satisfies the equation

$$\widehat{x}(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)=\int_{[0,s]\times[0,t]}f(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]}g(u,v)dM_{u,v},$$

where $f\in S^2_{\mathcal{P}}(F\circ\widehat{x},\nu_A)$, $g\in S^2_{\mathcal{P}}(G\circ\widehat{x},\mu_M)$. Thus $\widehat{x}\in SI(F,G,\xi)$. □

Theorem 3.5. *Let $F,G: I\times J\times\Omega\times\mathbb{R}^d\to\mathcal{K}^b_c(\mathbb{R}^d)$ and $F^{(n)},G^{(n)}: I\times J\times\Omega\times\mathbb{R}^d\to\mathcal{K}^b_c(\mathbb{R}^d)$ for $n\geqslant 1$ be set-valued functions satisfying (i1)–(i4). Assume that*

$$\begin{gathered}F^{(1)}(s,t,\omega,x)\supset F^{(2)}(s,t,\omega,x)\supset\cdots\supset F(s,t,\omega,x)\\ F(s,t,\omega,x)=\bigcap_{n\geqslant 1}F^{(n)}(s,t,\omega,x)\end{gathered}\tag{3.7}$$

$$\begin{gathered}G^{(1)}(s,t,\omega,x)\supset G^{(2)}(s,t,\omega,x)\supset\cdots\supset G(s,t,\omega,x)\\ G(s,t,\omega,x)=\bigcap_{n\geqslant 1}G^{(n)}(s,t,\omega,x)\end{gathered}\tag{3.8}$$

for every $(s,t,\omega,x)\in I\times J\times\Omega\times\mathbb{R}^d$. Then

$$SI(F,G,\xi)=\bigcap_{n\geqslant 1}SI(F^{(n)},G^{(n)},\xi).$$

Proof. By virtue of Theorem 3.4 the sets $SI(F,G,\xi)$ and $SI(F^{(n)},G^{(n)},\xi)$ for $n\geqslant 1$ are nonempty, bounded and closed subsets of $S^2(I\times J)$. Moreover by (3.7) and (3.8)

$$SI(F^{(1)},G^{(1)},\xi)\supset SI(F^{(2)},G^{(2)},\xi)\supset\cdots\supset SI(F,G,\xi).$$

Then $\bigcap_{n\geqslant 1} SI(F^{(n)},G^{(n)},\xi)\supset SI(F,G,\xi)$. Let $x\in\bigcap_{n\geqslant 1} SI(F^{(n)},G^{(n)},\xi)$. Then $x\in S^2(I\times J)$ and there exist $f^{(n)}\in S^2_{\mathcal{P}}(F^{(n)}\circ x,\nu_A)$ and $g^{(n)}\in S^2_{\mathcal{P}}(G^{(n)}\circ x,\mu_M)$ such that

$$x(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)=\int_{[0,s]\times[0,t]} f^{(n)}(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]} g^{(n)}(u,v)dM_{u,v}$$

for every $n\geqslant 1$ and $(s,t)\in I\times J$. By Theorem 2.4 and Theorem 2.9 the sets $S^2_{\mathcal{P}}(F^{(n)}\circ x,\nu_A)$ and $S^2_{\mathcal{P}}(G^{(n)}\circ x,\mu_M)$ for $n\geqslant 1$ are weakly compact in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$, respectively. Moreover by (3.7) and (3.8) the sets $S^2_{\mathcal{P}}(F^{(n)}\circ x,\nu_A)$ and $S^2_{\mathcal{P}}(G^{(n)}\circ x,\mu_M)$ are decreasing with respect to n in the sense of inclusion. Thus, it holds $S^2_{\mathcal{P}}(F\circ x,\nu_A)=\bigcap_{n\geqslant 1} S^2_{\mathcal{P}}(F^{(n)}\circ x,\nu_A)$ and $S^2_{\mathcal{P}}(G\circ x,\mu_M)=\bigcap_{n\geqslant 1} S^2_{\mathcal{P}}(G^{(n)}\circ x,\mu_M)$. Hence we can select subsequences $(f^{(n_k)})$ of $(f^{(n)})$ and $(g^{(n_k)})$ of $(g^{(n)})$ and processes $f\in L^{2,d}_{\mathcal{P}}(\nu_A)$ and $g\in L^{2,d}_{\mathcal{P}}(\mu_M)$ such that $f^{(n_k)}\rightharpoonup f$ and $g^{(n_k)}\rightharpoonup g$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$, respectively. Thus $f\in S^2_{\mathcal{P}}(F\circ x,\nu_A)$ and $g\in S^2_{\mathcal{P}}(G\circ x,\mu_M)$. Using a similar argumentation as in the proof of the closedness of $SI(F,G,\xi)$ in Theorem 3.4 one can show that

$$\begin{aligned}
&x(s,t)-\xi(0,t)-\xi(s,0)+\xi(0,0)\\
&\quad=\int_{[0,s]\times[0,t]} f^{(n_k)}(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]} g^{(n_k)}(u,v)dM_{u,v}\\
&\quad\rightharpoonup\int_{[0,s]\times[0,t]} f(u,v)dA_{u,v}+\int_{[0,s]\times[0,t]} g(u,v)dM_{u,v}
\end{aligned}$$

in $L^{2,d}$ for every $(s,t)\in I\times J$. Thus $x\in SI(F,G,\xi)$. □

Remark 3.6. Let us consider an investment with a horizon of time from 0 to T. Assume that the symbol $P(s,T)$ denotes a price of the zero-coupon bond at the time s with a nominal value 1 and the maturity T. Then

$$f(s,T)=-\frac{\partial\ln P(s,T)}{\partial T}$$

denotes a temporary forward rate of a such investment. Hence the price $P(s,T)$ is given by

$$P(s,T)=\exp\left\{-\int_s^T f(s,u)\,du\right\}.$$

For a fixed maturity T the dynamics of the forward rate is usually modeled by the following stochastic equation of Itô type

$$df(s,T)=a(s,f(s,T))\,ds+b(s,f(s,T))\,dW_s, \tag{3.9}$$

i.e. by diffusion models of interest rates. Such models are well-known and there exists a wide literature devoted to this subject. Taking particular choices of mappings a and b in (3.9), we arrive to popular diffusion models (see [46]):
- the Merton model

$$a\left(s,f\left(s,T\right)\right)=\alpha,\quad b\left(s,f\left(s,T\right)\right)=\gamma,$$

- the Vasiček model

$$a\left(s,f\left(s,T\right)\right)=\alpha-\beta f\left(s,T\right),\quad b\left(s,f\left(s,T\right)\right)=\gamma,$$

- the Dothan model

$$a\left(s,f\left(s,T\right)\right)=\alpha f\left(s,T\right),\quad b\left(s,f\left(s,T\right)\right)=\gamma f\left(s,T\right),$$

- the Cox-Ingersoll-Ross model

$$a\left(s,f\left(s,T\right)\right)=\alpha-\beta f\left(s,T\right),\quad b\left(s,f\left(s,T\right)\right)=\gamma f^{\frac{1}{2}}\left(s,T\right),$$

- the Hull-White model

$$a\left(s,f\left(s,T\right)\right)=\alpha(s)-\beta(s)f\left(s,T\right),\quad b\left(s,f\left(s,T\right)\right)=\gamma(s)f^{\frac{1}{2}}\left(s,T\right),$$

- the Black-Karasiński model

$$a\left(s,f\left(s,T\right)\right)=f\left(s,T\right)\left(\alpha(s)-\beta(s)\ln f\left(s,T\right)\right),\quad b\left(s,f\left(s,T\right)\right)=\gamma(s)f\left(s,T\right).$$

Let us suppose now that the maturity T is not fixed, but it may be varied. More precisely it may be postponed. Let us denote it by t. In this case we have

$$f\left(s,t\right)=-\frac{\partial\ln P\left(s,t\right)}{\partial t}\text{ and }P\left(s,t\right)=\exp\left\{-\int_s^t f\left(s,u\right)du\right\}.$$

Then one can consider the following diffusion model, which models dynamics of forward rate, as a stochastic equation with two-parameter Wiener process $W=\left(W_{s,t}\right)_{s\leqslant t}$, i.e.

$$df\left(s,t\right)=g\left(s,t,f\left(s,t\right)\right)dsdt+h\left(s,t,f\left(s,t\right)\right)dW_{s,t}$$

or

$$\begin{aligned}f\left(s,t\right)-f\left(0,0\right)&=f\left(s,0\right)+f\left(0,t\right)\\&+\int_{[0,s]\times[0,t]}g\left(s,t,f\left(s,t\right)\right)dsdt+\int_{[0,s]\times[0,t]}h\left(s,t,f\left(s,t\right)\right)dW_{s,t}.\end{aligned}\tag{3.10}$$

It is evident that equation (3.10) is a particular case of stochastic inclusion (3.1).

4. **Stochastic inclusions and multivalued equations**

Using the notions of set-valued stochastic integrals discussed in part 2 one can consider connections between stochastic integral inclusions and appropriately defined set-valued stochastic integral equation driven by two-parameter processes.

From now on we assume that the σ-field $\mathbb{F}$ is separable with respect to the probability measure P. Then $L^{2,d} = L^2\left(\Omega, \mathbb{F}, P; \mathbb{R}^d\right)$ is a separable Banach space.

As earlier we consider the stochastic integral inclusion generated by the triple (F, G, ξ)

$$\tag{4.1} \begin{cases} \Delta_{s,t}^{s',t'}(x) \in \int_{[s,s']\times[t,t']} F(u,v,x(u,v))\, dA_{u,v} \\ \quad + \int_{[s,s']\times[t,t']} G(u,v,x(u,v))\, dM_{u,v} \\ x(0,t) = \xi(0,t) \\ x(s,0) = \xi(s,0) \end{cases}$$

for every $(s,t), (s',t') \in I \times J$, where $(s,t) \preceq (s',t')$. Now we assume that set-valued mappings $F, G : I \times J \times \Omega \times L^{2,d} \to \mathcal{K}_c^b\left(\mathbb{R}^d\right)$ satisfy the following conditions:

(A1) for every $\eta \in L^{2,d}$ the mappings

$$F(\cdot,\cdot,\cdot,\eta), G(\cdot,\cdot,\cdot,\eta) : I \times J \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$$

are predictable,

(A2) there exists a constant $L > 0$ such that

$$\max\Big\{ H_{\mathbb{R}^d}(F(s,t,\omega,\eta_1), F(s,t,\omega,\eta_2)), \\ H_{\mathbb{R}^d}(G(s,t,\omega,\eta_1), G(s,t,\omega,\eta_2)) \Big\} \leqslant L \|\eta_1 - \eta_2\|_{L^{2,d}}$$

for every $(s,t,\omega) \in I \times J \times \Omega$, and every $\eta_1, \eta_2 \in L^{2,d}$,

(A3) there exists a constant $K > 0$ such that

$$\max\Big\{ H_{\mathbb{R}^d}(F(s,t,\omega,\eta), \{\theta\}), H_{\mathbb{R}^d}(G(s,t,\omega,\eta), \{\theta\}) \Big\} \\ \leqslant K(1 + \|\eta\|_{L^{2,d}})$$

for every $(s,t,\omega) \in I \times J \times \Omega$, and every $\eta \in L^{2,d}$, while $\xi : I \times J \times \Omega \to \mathbb{R}^d$ is now $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted and square integrable stochastic process such that the mapping $I \times J \ni (s,t) \mapsto \xi(s,t,\cdot) \in L^{2,d}$ is continuous.

By a solution to stochastic integral inclusion (4.1) generated by the triple (F, G, ξ) we mean now $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$-adapted stochastic process $x : I \times J \times \Omega \to \mathbb{R}^d$ such that the mapping $I \times J \ni (s,t) \mapsto x(s,t,\cdot) \in L^{2,d}$ is continuous and which has the following representation

$$x(s,t) + \xi(0,0) - \xi(s,0) - \xi(0,t) = \int_{[0,s]\times[0,t]} f_1(u,v)\, dA_{u,v} + \int_{[0,s]\times[0,t]} g_1(u,v)\, dM_{u,v}$$

for some $f_1 \in \mathcal{S}^2_{\mathcal{P}}(F \circ x, \nu_A)$ and $g_1 \in \mathcal{S}^2_{\mathcal{P}}(G \circ x, \mu_M)$.

In order to introduce a set-valued stochastic integral equation associated with inclusion (4.1), let $\hat{F}, \hat{G} : I \times J \times \Omega \times \mathcal{K}^b_c\left(L^{2,d}\right) \to \mathcal{K}^b_c\left(\mathbb{R}^d\right)$ be set-valued maps defined as follows

$$\hat{F}(s,t,\omega,B) := \overline{co}\left(\bigcup_{b\in B} F(s,t,\omega,b)\right) \tag{4.2}$$

and

$$\hat{G}(s,t,\omega,B) := \overline{co}\left(\bigcup_{b\in B} G(s,t,\omega,b)\right) \tag{4.3}$$

for $(s,t,\omega,B) \in I \times J \times \Omega \times \mathcal{K}^b_c\left(L^{2,d}\right)$.

Moreover we assume that the multifunction $\psi : I \times J \to \mathcal{K}^b_c\left(L^{2,d}\right)$ is continuous and such that

(A4) $\xi(s,0) + \xi(0,t) - \xi(0,0) \in (\psi(s,0) + \psi(0,t)) \ominus \psi(0,0)$ for every $(s,t) \in I \times J$.

By a set-valued stochastic integral equation associated with inclusion (4.1) we mean the following relation in the metric space $\left(\mathcal{K}^b_c\left(L^{2,d}\right), H_{L^{2,d}}\right)$

$$\begin{aligned} X(s,t) + \psi(0,0) = {} & \psi(s,0) + \psi(0,t) \\ & + \int_{[0,s]\times[0,t]} \hat{F}(u,v,X(u,v))\, dA_{u,v} \\ & + \int_{[0,s]\times[0,t]} \hat{G}(u,v,X(u,v))\, dM_{u,v} \end{aligned} \tag{4.4}$$

for $(s,t) \in I \times J$.

By a solution to equation (4.4) we mean $H_{L^{2,d}}$-continuous map $X : I \times J \to \mathcal{K}^b_c\left(L^{2,d}\right)$ that satisfies (4.4). Let $\mathcal{C} = C\left(I \times J, \mathcal{K}^b_c\left(L^{2,d}\right)\right)$ be the space of all continuous mappings from $I \times J$ to $\mathcal{K}^b_c\left(L^{2,d}\right)$ with a metric

$$\rho(X,Y) = \sup_{(s,t)\in I\times J} H_{L^{2,d}}(X(s,t), Y(s,t))$$

for $X, Y \in \mathcal{C}$. Then $(\mathcal{C}, \rho)$ is a complete metric space.

Moreover let us assume that $\psi : I \times J \to \mathcal{K}^b_c\left(L^{2,d}\right)$ satisfies

(A5) the mapping $\psi : I \times J \to \mathcal{K}^b_c\left(L^{2,d}\right)$ is continuous with respect to the Hausdorff metric $H_{L^{2,d}}$ and such that the Hukuhara difference $(\psi(s,0) + \psi(0,t)) \ominus \psi(0,0)$ exists for every $(s,t) \in I \times J$, and

$$\int_{I\times J\times\Omega} H^2_{L^{2,d}}\left((\psi(s,0) + \psi(0,t)) \ominus \psi(0,0), \{\Theta\}\right)(d\nu_A + d\mu_M) < \infty.$$

Let $\mathcal{P}^{\Omega} := \{B \times \Omega : B \in \mathcal{B}\}$. Then $\mathcal{P}^{\Omega}$ is a sub-σ-field of a predictable σ-field $\mathcal{P}$. Let $\tilde{\nu}_A$ and $\tilde{\mu}_M$ be marginals of ν_A and μ_M, respectively, defined on $(I \times J, \mathcal{B})$ as follows

$$\tilde{\nu}_A(B) := \nu_A(B \times \Omega) \text{ and } \tilde{\mu}_M(B) := \mu_M(B \times \Omega) \text{ for } B \in \mathcal{B}.$$

Let $\gamma : I \times J \to \mathbb{R}_+$ be a function given by

$$\gamma(s,t) := \tilde{\nu}_A([0,s]\times[0,t]) + \tilde{\mu}_M([0,s]\times[0,t]) \text{ for } (s,t) \in I \times J.$$

Theorem 4.1. *Let $F, G : I \times J \times \Omega \times L^{2,d} \to \mathcal{K}_c^b\left(\mathbb{R}^d\right)$ satisfy conditions (A1)–(A3) and suppose that $\psi : I \times J \to \mathcal{K}_c^b\left(L^{2,d}\right)$ satisfies assumption* $(A5)$. *Then equation* (4.4) *admits an unique solution.*

Proof. The proof is similar to the proof of Theorem 27 in [39]. Therefore, we only sketch its main parts.
(*Existence*) Firstly, by (A1)–(A3) one can show in a similar way as in Proposition 1 in [26] that the mappings $\hat{F}$ and $\hat{G}$ defined by (4.2) and (4.3) satisfy conditions:
$(B1)$ for every $B \in \mathcal{K}_c^b\left(L^{2,d}\right)$ the mappings

$$\hat{F}(\cdot,\cdot,\cdot,B), \hat{G}(\cdot,\cdot,\cdot,B) : I \times J \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$$

are predictable,
$(B2)$ there exists a constant $L_1 > 0$ such that

$$\max\{H_{\mathbb{R}^d}(\hat{F}(s,t,\omega,B), \hat{F}(s,t,\omega,C)), H_{\mathbb{R}^d}(\hat{G}(s,t,\omega,B), \hat{G}(s,t,\omega,C))\} \leqslant L_1 H_{L^{2,d}}(B,C)$$

for every $(s,t,\omega) \in I \times J \times \Omega$, and every $B, C \in \mathcal{K}_c^b\left(L^{2,d}\right)$,
$(B3)$ there exists a constant $K_1 > 0$ such that

$$\max\{H_{\mathbb{R}^d}(\hat{F}(s,t,\omega,B), \{\theta\}), H_{\mathbb{R}^d}(\hat{G}(s,t,\omega,B), \{\theta\})\} \leqslant K_1(1 + H_{L^{2,d}}(B, \{\Theta\}))$$

for every $(s,t,\omega) \in I \times J \times \Omega$, and every $B \in \mathcal{K}_c^b\left(L^{2,d}\right)$, where the symbol Θ denotes the zero element in $L^{2,d}$.

Let us define set-valued mappings $X_n : I \times J \to \mathcal{K}_c^b\left(L^{2,d}\right)$ for $n \geqslant 0$ as follows

$$X_0(s,t) + \psi(0,0) = \psi(s,0) + \psi(0,t)$$

and for $n \geqslant 1$

$$\begin{aligned} X_n(s,t) + \psi(0,0) &= \psi(s,0) + \psi(0,t) \\ &+ \int_{[0,s]\times[0,t]} \hat{F}(u,v,X_{n-1}(u,v))\, dA_{u,v} \\ &+ \int_{[0,s]\times[0,t]} \hat{G}(u,v,X_{n-1}(u,v))\, dM_{u,v} \text{ for } (s,t) \in I \times J. \end{aligned} \tag{4.5}$$

Due to Theorem 8.2.8 in [7] it follows that for every continuous multifunction $X : I \times J \to \mathcal{K}_c^b\left(L^{2,d}\right)$ the mappings $(s,t,\omega) \mapsto cl_{\mathbb{R}^d}(F(s,t,\omega,X(s,t)))$ and $(s,t,\omega) \mapsto cl_{\mathbb{R}^d}(G(s,t,\omega,X(s,t)))$ are predictable. Then by Proposition 2.26 in [13] we know that the mappings $(s,t,\omega) \mapsto \hat{F}(s,t,\omega,X(s,t))$ and $(s,t,\omega) \mapsto \hat{G}(s,t,\omega,X(s,t))$ are predictable too. On the other hand, by condition $(B3)$ they are $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded, respectively. Hence, due to the definition of the sequence $\{X_n\}$

we deduce that the multifunctions $(s,t,\omega) \mapsto \hat{F}(s,t,\omega,X_n(s,t))$ and $(s,t,\omega) \mapsto \hat{G}(s,t,\omega,X_n(s,t))$ are predictable, and $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$-integrally bounded, respectively, for every $n \geqslant 0$. Therefore, the integrals in (4.5) are correctly defined. Now applying Theorem 2.5 and $(B3)$ we obtain

$$H^2_{L^{2,d}}\left(\int_{[0,s]\times[0,t]} \hat{F}(u,v,X_0(u,v))\,dA_{u,v},\{\Theta\}\right) \leqslant \int_{[0,s]\times[0,t]\times\Omega} 2K_1^2\left(1+H^2_{L^{2,d}}(X_0(u,v),\{\Theta\})\right)d\nu_A.$$

In a similar way, due to Theorem 2.10, we obtain

$$H^2_{L^{2,d}}\left(\int_{[0,s]\times[0,t]} \hat{G}(u,v,X_0(u,v))\,dM_{u,v},\{\Theta\}\right) \leqslant \int_{[0,s]\times[0,t]\times\Omega} 2K_1^2\left(1+H^2_{L^{2,d}}(X_0(u,v),\{\Theta\})\right)d\mu_M.$$

Thus, by properties (2.1) and (2.2) we get

$$H^2_{L^{2,d}}(X_1(s,t),X_0(s,t)) \leqslant 2\int_{[0,s]\times[0,t]\times\Omega} 2K_1^2\left(1+H^2_{L^{2,d}}(X_0(u,v),\{\Theta\})\right)(d\nu_A+d\mu_M) \leqslant 2\eta,$$

where

$$\eta = \int_{[0,S]\times[0,T]\times\Omega} 2K_1^2\left(1+H^2_{L^{2,d}}(X_0(u,v),\{\Theta\})\right)(d\nu_A+d\mu_M).$$

Similarly, by condition $(B2)$ we have

$$\begin{aligned} H^2_{L^{2,d}}&(X_n(s,t),X_{n-1}(s,t)) \\ &\leqslant 2L_1^2\int_{[0,s]\times[0,t]\times\Omega} H^2_{L^{2,d}}(X_{n-1}(u,v),X_{n-2}(u,v))(d\nu_A+d\mu_M) \\ &= 2L_1^2\int_{[0,s]\times[0,t]} H^2_{L^{2,d}}(X_{n-1}(u,v),X_{n-2}(u,v))\,d\gamma(u,v) \ \text{ for } n\in\mathbb{N}, \end{aligned}$$

where $\gamma(s,t) := \tilde{\nu}_A([0,s]\times[0,t]) + \tilde{\mu}_M([0,s]\times[0,t])$ for $(s,t)\in I\times J$. In particular, we get

$$H^2_{L^{2,d}}(X_2(s,t),X_1(s,t)) \leqslant 2L_1^2\cdot 2\eta\cdot\gamma(s,t).$$

Since the processes A and M are continuous, $\partial A = \partial M = 0$, it follows that γ is continuous, increasing (both in the sense of Definition 2.2 and with respect to the order $\preceq$) and $\partial\gamma = 0$. Hence, by the two-parameter version of Ito's formula (see Theorem 2.1 in [29]) it holds

$$\int_{[0,s]\times[0,t]} \gamma^n(u,v)d\gamma(u,v) \leqslant \frac{\gamma^{n+1}(s,t)}{n+1} \text{ for } n\in\mathbb{N}.$$

Therefore, by mathematical induction we get

$$H^2_{L^{2,d}}(X_n(s,t),X_{n-1}(s,t)) \leqslant 2^n\left(L_1^2\right)^{n-1}\eta\frac{\gamma^{n-1}(S,T)}{(n-1)!}.$$

Consequently, for $m < n$ we have

$$\rho\left(X_n, X_m\right) \leqslant \sqrt{2\eta} \sum_{k=m+1}^{n} \sqrt{\left(2L_1^2\right)^{k-1} \frac{\gamma^{k-1}\left(S,T\right)}{(k-1)!}}.$$

Thus $\{X_n\}$ is a Cauchy sequence in a complete metric space $(\mathcal{C}, \rho)$. Hence, there exists $X \in \mathcal{C}$ such that $\rho\left(X_n, X\right) \to 0$, as $n \to \infty$.

In order to show that X is a solution to equation (4.4), let us fix $(s,t) \in I \times J$. Then by properties of Hausdorff distance we have

$$\begin{aligned}
&H^2_{L^{2,d}}\Big(X\left(s,t\right) + \psi\left(0,0\right), \psi\left(s,0\right) + \psi\left(0,t\right) \\
&\quad + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X\left(u,v\right)\right) dA_{u,v} + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X_{u,v}\right) dM_{u,v}\Big) \\
&\quad \leqslant 3H^2_{L^{2,d}}\left(X\left(s,t\right) + \psi\left(0,0\right), X_n\left(s,t\right) + \psi\left(0,0\right)\right) \\
&\quad + 3H^2_{L^{2,d}}\Big(X_n\left(s,t\right) + \psi\left(0,0\right), \psi\left(s,0\right) + \psi\left(0,t\right) \\
&\quad + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X_{n-1}\left(u,v\right)\right) dA_{u,v} + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X_{n-1}\left(u,v\right)\right) dM_{u,v}\Big) \\
&\quad + 3H^2_{L^{2,d}}\Big(\psi\left(s,0\right) + \psi\left(0,t\right) + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X_{n-1}\left(u,v\right)\right) dA_{u,v} \\
&\quad + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X_{n-1}\left(u,v\right)\right) dM_{u,v}, \psi\left(s,0\right) + \psi\left(0,t\right) \\
&\quad + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X\left(u,v\right)\right) dA_{u,v} + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X\left(u,v\right)\right) dM_{u,v}\Big).
\end{aligned}$$

The first and the second term on the right-hand side of the above inequality converges to zero as $n \to \infty$. It is also true for the last term above. Indeed, again by Theorem 2.5, 2.10 and $(B2)$ we have

$$\begin{aligned}
&H^2_{L^{2,d}}\Big(\psi\left(s,0\right) + \psi\left(0,t\right) + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X_{n-1}\left(u,v\right)\right) dA_{u,v} \\
&\qquad + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X_{n-1}\left(u,v\right)\right) dM_{u,v}, \psi\left(s,0\right) + \psi\left(0,t\right) \\
&\qquad + \int_{[0,s]\times[0,t]} \hat{F}\left(u,v,X\left(u,v\right)\right) dA_{u,v} + \int_{[0,s]\times[0,t]} \hat{G}\left(u,v,X\left(u,v\right)\right) dM_{u,v}\Big) \\
&\quad \leqslant 2L_1^2 \int_{[0,s]\times[0,t]} H^2_{L^{2,d}}\left(X_{n-1}\left(u,v\right), X\left(u,v\right)\right) d\gamma(u,v) \\
&\quad \leqslant 2L_1^2 \rho\left(X_{n-1}, X\right) \gamma\left(S,T\right).
\end{aligned}$$

Hence we infer that X is a solution to equation (4.4).
(*Uniqueness*) Let us suppose that X and Y are any solutions to (4.4). Then similarly as above we get

$$H^2_{L^{2,d}}\left(X\left(s,t\right), Y\left(s,t\right)\right) \leqslant 2L_1^2 \int_{[0,s]\times[0,t]} H^2_{L^{2,d}}\left(X\left(u,v\right), Y\left(u,v\right)\right) d\gamma\left(u,v\right).$$

Hence we have

$$H_{L^{2,d}}^{2}\left(X\left(u,v\right),Y\left(u,v\right)\right)\leqslant 2L_{1}^{2}\int_{[0,s]\times[0,t]}\rho_{a,b}^{2}\left(X,Y\right)d\gamma\left(a,b\right)$$

for every $(u,v)\in[0,s]\times[0,t]$, where

$$\rho_{s,t}\left(X,Y\right)=\sup_{(u,v)\in[0,s]\times[0,t]}H_{L^{2,d}}\left(X\left(u,v\right),Y\left(u,v\right)\right).$$

Thus

$$\rho_{s,t}^{2}\left(X,Y\right)\leqslant 2L_{1}^{2}\int_{[0,s]\times[0,t]}\rho_{u,v}^{2}\left(X,Y\right)d\gamma\left(u,v\right)$$

for every $(s,t)\in I\times J$. Using again Gronwall's inequality (Theorem 2.3 in [29]) it follows that $\rho_{s,t}\left(X,Y\right)=0$ for $(s,t)\in I\times J$, what shows the uniqueness of solutions to equation (4.4). □

In a similar way one can prove the following property.

Theorem 4.2. *Under assumptions of Theorem* 4.1 *the solution X to equation* (4.4) *satisfies:*

$$\begin{aligned}\sup_{(u,v)\in[0,s]\times[0,t]}&H_{L^{2,d}}^{2}\left(X\left(u,v\right),\{\Theta\}\right)\\&\leqslant 3[\sup_{(s,t)\in I\times J}H_{L^{2,d}}^{2}\left(\psi\left(s,0\right)+\psi\left(0,t\right),\psi\left(0,0\right)\right)+2K_{1}^{2}\gamma\left(s,t\right)]\\&\cdot\exp\left\{18K_{1}^{2}\gamma\left(s,t\right)\right\}\end{aligned}$$

for every $(s,t)\in I\times J$.

Now we proceed with a further analysis of inclusion (4.1). Again by $SI\left(F,G,\xi\right)$ we denote the set of its solutions.

By Theorem 2.1 in [29], i.e. by the two-parameter version of Itô's formula one can prove the following Lemma.

Lemma 4.3. *Let B be a continuous increasing process. Then*

$$\int_{[0,s]\times[0,t]}e^{hB_{u,v}}dB_{u,v}\leqslant\frac{1}{h}\left(e^{hB_{s,t}}-1\right)$$

for every $h>0$ and $(s,t)\in I\times J$.

Now we formulate the main result of this section.

Theorem 4.4. *Assume that F and G satisfy conditions (A1)–(A3) , and ψ satisfies assumption (A5). Moreover, let ψ and ξ satisfy condition (A4). Then there exists a solution $X:I\times J\to\mathcal{K}_{c}^{b}\left(L^{2,d}\right)$ of equation* (4.4) *and a solution $x:I\times J\times\Omega\to\mathbb{R}^{d}$ of* (4.1) *such that $x\left(s,t\right)\in X\left(s,t\right)$ for every $(s,t)\in I\times J$.*

Proof. By Theorem 4.1 there exists a unique solution X to set-valued stochastic integral equation (4.4). Now, let us consider the set

$$K(\xi,X) := \Big\{x \in C\left(I\times J, L^{2,d}\right) : x(s,t)+\xi(0,0)-\xi(s,0)-\xi(0,t) = \int_{[0,s]\times[0,t]} f(u,v)\,dA_{u,v} + \int_{[0,s]\times[0,t]} g(u,v)\,dM_{u,v} P\text{-a.e. for every } (s,t)\in I\times J, \text{ and some } f\in\mathcal{S}^2_{\mathcal{P}}(\hat{F}\circ X,\nu_A),\ g\in\mathcal{S}^2_{\mathcal{P}}(\hat{G}\circ X,\mu_M)\Big\}.$$

Notice that $K(\xi,X)$ is a nonempty subset of $C\left(I\times J, L^{2,d}\right)$. Indeed, similarly as in the proof of Theorem 4.1 by the continuity of X it follows that the set-valued mappings $\hat{F}\circ X : I\times J\times\Omega \to \mathcal{K}^b_c\left(\mathbb{R}^d\right)$ and $\hat{G}\circ X : I\times J\times\Omega\to\mathcal{K}^b_c\left(\mathbb{R}^d\right)$ are predictable. Hence by Kuratowski and Ryll-Nardzewski Measurable Selection Theorem (cf. [18]), there exist predictable selections for $\hat{F}\circ X$ and for $\hat{G}\circ X$. Since by $(A3)$ we have $(B3)$ (at it was mentioned in the beginning of the proof of Theorem 4.1), i.e.

$$\begin{aligned}\|(\hat{F}\circ X)(s,t,\omega)\|^2_{\mathbb{R}^d} &= \|\hat{F}(s,t,\omega,X(s,t))\|^2_{\mathbb{R}^d}\\ &\leqslant 2K_1^2(1+H^2_{L^{2,d}}(X(s,t),\{\Theta\})),\end{aligned}$$

it follows due to Theorem 4.2 that

$$\sup_{(s,t)\in I\times J}\|(\hat{F}\circ X)(s,t,\omega)\|^2_{\mathbb{R}^d}<\infty.$$

Thus the set $\mathcal{S}^2_{\mathcal{P}}(\hat{F}\circ X,\nu_A)$ is nonempty. In a similar way one can show the nonemptiness of the set $\mathcal{S}^2_{\mathcal{P}}(\hat{G}\circ X,\mu_M)$. It shows the nonemptiness of the set $K(\xi,X)$.

Next, let us note that if $x\in K(\xi,X)$, then for every $(s,t)\in I\times J$ it holds

$$\operatorname{dist}_{L^{2,d}}(x(s,t),X(s,t))=0. \tag{4.6}$$

Indeed, by the definition of $K(\xi,X)$ and assumption $(A4)$ we have

$$\begin{aligned}x(s,t) &= \xi(s,0)+\xi(0,t)-\xi(0,0)\\ &\quad+\int_{[0,s]\times[0,t]} f(u,v)\,dA_{u,v}+\int_{[0,s]\times[0,t]} g(u,v)\,dM_{u,v}\\ &\in \psi(s,0)+\psi(0,t)\ominus\psi(0,0)+\int_{[0,s]\times[0,t]}\hat{F}(u,v,X(u,v))\,dA_{u,v}\\ &\quad+\int_{[0,s]\times[0,t]}\hat{G}(u,v,X(u,v))\,dM_{u,v}=X(s,t)\ \text{ for }\ (s,t)\in I\times J.\end{aligned}$$

Now we will show that the set $K(\xi,X)$ is bounded. Let $x\in K(\xi,X)$. Then there exist $f\in\mathcal{S}^2_{\mathcal{P}}(\hat{F}\circ X,\nu_A)$ and $g\in\mathcal{S}^2_{\mathcal{P}}(\hat{G}\circ X,\mu_M)$ such that

$$\begin{aligned}&x(s,t)+\xi(0,0)-\xi(s,0)-\xi(0,t)\\ &\qquad=\int_{[0,s]\times[0,t]} f(u,v)\,dA_{u,v}+\int_{[0,s]\times[0,t]} g(u,v)\,dM_{u,v}\end{aligned}$$

for $(s,t) \in I \times J$. Hence by $(A4)$, (2.4) and two-parameter Itô's isometry (2.6) we have

$$\begin{aligned}\sup_{(s,t)\in I\times J} \mathbb{E}\left\|x\left(s,t\right)\right\|_{\mathbb{R}^d}^2 &\leqslant 3 \sup_{(s,t)\in I\times J} \mathbb{E}\left\|\xi\left(s,0\right)+\xi\left(0,t\right)-\xi\left(0,0\right)\right\|_{\mathbb{R}^d}^2 \\ &\quad + 3 \sup_{(s,t)\in I\times J} \mathbb{E}\| \int_{[0,s]\times[0,t]} f\left(u,v\right)dA_{u,v}\|_{\mathbb{R}^d}^2 \\ &\quad + 3 \sup_{(s,t)\in I\times J} \mathbb{E}\| \int_{[0,s]\times[0,t]} g\left(u,v\right)dM_{u,v}\|_{\mathbb{R}^d}^2 \\ &\leqslant 3 \sup_{(s,t)\in I\times J} H_{L^{2,d}}^2\left(\left(\psi\left(s,0\right)+\psi\left(0,t\right)\right)\ominus\psi\left(0,0\right),\{\Theta\}\right) \\ &\quad + 3\int_{I\times J\times\Omega}\left\|f\left(u,v\right)\right\|_{\mathbb{R}^d}^2 d\nu_A + 3\int_{I\times J\times\Omega}\left\|g\left(u,v\right)\right\|_{\mathbb{R}^d}^2 d\mu_M.\end{aligned}$$

On the other hand by $(B3)$ related to $(A3)$, Theorem 4.1 and Theorem 4.2 for every $(s,t) \in I \times J$ we infer

$$\begin{aligned}\left\|f\left(s,t,\omega\right)\right\|_{\mathbb{R}^d}^2 &\leqslant \|\hat{F}\left(s,t,\omega,X\left(s,t\right)\right)\|_{\mathbb{R}^d}^2 = H_{\mathbb{R}^d}^2(\hat{F}\left(s,t,\omega,X\left(s,t\right)\right),\{\Theta\}) \\ &\leqslant 2K_1^2(1+H_{L^{2,d}}^2\left(X\left(s,t\right),\{\Theta\}\right)) < \infty.\end{aligned}$$

A similar argumentation gives

$$\left\|g\left(s,t,\omega\right)\right\|_{\mathbb{R}^d}^2 < \infty.$$

Therefore we have

$$\sup_{(s,t)\in I\times J} \mathbb{E}\left\|x\left(s,t\right)\right\|_{\mathbb{R}^d}^2 < m,$$

where m is a positive constant, which does not depend on x. Hence the boundedness of $K(\xi, X)$ in $C\left(I \times J, L^{2,d}\right)$ follows.

In the next step we will show that $K(\xi, X)$ is a closed subset of $C\left(I \times J, L^{2,d}\right)$. Let $\{x^n\} \subset K(\xi, X)$ and $x^n \to x$ in the space $C\left(I \times J, L^{2,d}\right)$, as $n \to \infty$, with $x \in C\left(I \times J, L^{2,d}\right)$. Since $x^n \in K(\xi, X)$ for every $n \in \mathbb{N}$ it follows

$$\begin{aligned}x^n\left(s,t\right) &= \xi\left(s,0\right)+\xi\left(0,t\right)-\xi\left(0,0\right) \\ &\quad + \int_{[0,s]\times[0,t]} f^n\left(u,v\right)dA_{u,v} + \int_{[0,s]\times[0,t]} g^n\left(u,v\right)dM_{u,v},\end{aligned}$$

for every $(s,t) \in I \times J$ and for some $f^n \in \mathcal{S}_{\mathcal{P}}^2(\hat{F} \circ X, \nu_A)$, $g^n \in \mathcal{S}_{\mathcal{P}}^2(\hat{G} \circ X, \mu_M)$. Thus for every $(s,t) \in I \times J$

$$\begin{aligned}&\xi\left(s,0\right)+\xi\left(0,t\right)-\xi\left(0,0\right) \\ &\quad + \int_{[0,s]\times[0,t]} f^n\left(u,v\right)dA_{u,v} + \int_{[0,s]\times[0,t]} g^n\left(u,v\right)dM_{u,v} \to x\left(s,t\right)\end{aligned} \tag{4.7}$$

in $L^{2,d}$ for $n \to \infty$. By Theorem 2.4(i) and Theorem 2.9(i) there exist subsequences (f^{n_k}) and (g^{n_k}) of (f^n) and (g^n), respectively, and $f \in \mathcal{S}_{\mathcal{P}}^2(\hat{F} \circ X, \nu_A)$, $g \in \mathcal{S}_{\mathcal{P}}^2(\hat{G} \circ$

$X,\mu_M)$ such that $f^{n_k} \rightharpoonup f$ in $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $g^{n_k} \rightharpoonup g$ in $L^{2,d}_{\mathcal{P}}(\mu_M)$. Therefore by (4.7) we get

$$\xi(s,0)+\xi(0,t)-\xi(0,0)+\int_{[0,s]\times[0,t]} f^{n_k}(u,v)\,dA_{u,v} + \int_{[0,s]\times[0,t]} g^{n_k}(u,v)\,dM_{u,v} \to x(s,t) \tag{4.8}$$

in $L^{2,d}$, as $k\to\infty$, and $(s,t)\in I\times J$. Similarly as in the proof of Theorem 3.4, by the weak convergence of the sequences (f^{n_k}) and (g^{n_k}), we have

$$\int_{[0,s]\times[0,t]} f^{n_k}(u,v)dA_{u,v} \rightharpoonup \int_{[0,s]\times[0,t]} f(u,v)dA_{u,v}$$

and

$$\int_{[0,s]\times[0,t]} g^{n_k}(u,v)dM_{u,v} \rightharpoonup \int_{[0,s]\times[0,t]} g(u,v)dM_{u,v}$$

in $L^{2,d}$, as $k\to\infty$. Thus for every $(s,t)\in I\times J$ we have

$$\begin{aligned}&\xi(s,0)+\xi(0,t)-\xi(0,0)\\ &\quad+\int_{[0,s]\times[0,t]} f^{n_k}(u,v)\,dA_{u,v}+\int_{[0,s]\times[0,t]} g^{n_k}(u,v)\,dM_{u,v}\\ &\quad\rightharpoonup \xi(s,0)+\xi(0,t)-\xi(0,0)\\ &\quad+\int_{[0,s]\times[0,t]} f(u,v)\,dA_{u,v}+\int_{[0,s]\times[0,t]} g(u,v)\,dM_{u,v}\end{aligned}$$

in $L^{2,d}$, as $k\to\infty$. This convergence and (4.8) allow us to claim that

$$\begin{aligned}x(s,t)&=\xi(s,0)+\xi(0,t)-\xi(0,0)\\ &\quad+\int_{[0,s]\times[0,t]} f(u,v)\,dA_{u,v}+\int_{[0,s]\times[0,t]} g(u,v)\,dM_{u,v} P\text{-a.e.}\end{aligned}$$

Thus $x\in K(\xi,X)$ which proves the closedness of $K(\xi,X)$ in $C\left(I\times J,L^{2,d}\right)$.

In order to finish the proof, we will show that there exists $\hat{x}\in SI(F,G,\xi)$ such that $\hat{x}\in K(\xi,X)$. Because F and G satisfy conditions (A1)–(A3), then by Proposition 3.1 there exist functions $\overline{f},\overline{g}: I\times J\times\Omega\times L^{2,d}\to\mathbb{R}^d$ such that:

(i) $\overline{f}(s,t,\omega,\eta)\in F(s,t,\omega,\eta)$, $\overline{g}(s,t,\omega,\eta)\in G(s,t,\omega,\eta)$ for every $(s,t,\omega,\eta)\in I\times J\times\Omega\times L^{2,d}$,

(ii) for every $\eta\in L^{2,d}$ the mappings $\overline{f}(\cdot,\cdot,\cdot,\eta),\overline{g}(\cdot,\cdot,\cdot,\eta): I\times J\times\Omega\to\mathbb{R}^d$ are predictable,

(iii) for every $(s,t,\omega)\in I\times J\times\Omega$, $\eta_1,\eta_2\in L^{2,d}$ it holds

$$\begin{aligned}&\max\{\left\|\overline{f}(s,t,\omega,\eta_1)-\overline{f}(s,t,\omega,\eta_2)\right\|^2_{\mathbb{R}^d},\left\|\overline{g}(s,t,\omega,\eta_1)-\overline{g}(s,t,\omega,\eta_2)\right\|^2_{\mathbb{R}^d}\}\\ &\qquad\leqslant d^2L^2\left\|\eta_1-\eta_2\right\|^2_{L^{2,d}},\end{aligned}$$

(iv) for every $(s,t,\omega)\in I\times J\times\Omega$, and every $\eta\in L^{2,d}$

$$\max\left\{\left\|\overline{f}(s,t,\omega,\eta)\right\|_{\mathbb{R}^d},\left\|\overline{g}(s,t,\omega,\eta)\right\|_{\mathbb{R}^d}\right\}\leqslant K\left(1+\|\eta\|_{L^{2,d}}\right).$$

Then for every $x \in K(\xi, X)$ the mappings

$$I \times J \times \Omega \ni (s,t,\omega) \mapsto \overline{f}(s,t,\omega,x(s,t)) \in \mathbb{R}^d,$$

$$I \times J \times \Omega \ni (s,t,\omega) \mapsto \overline{g}(s,t,\omega,x(s,t)) \in \mathbb{R}^d$$

are elements of $L^{2,d}_{\mathcal{P}}(\nu_A)$ and $L^{2,d}_{\mathcal{P}}(\mu_M)$, respectively. Moreover $\overline{f}(s,t,\omega,x(s,t)) \in F(s,t,\omega,x(s,t))$ and $\overline{g}(s,t,\omega,x(s,t)) \in G(s,t,\omega,x(s,t))$. Let us define the operator $V : K(\xi, X) \to C(I \times J, L^{2,d})$ as follows

$$\begin{aligned} V(x)(s,t) = \xi(s,0) + \xi(0,t) - \xi(0,0) \\ + \int_{[0,s]\times[0,t]} \overline{f}(u,v,x(u,v))\, dA_{u,v} \\ + \int_{[0,s]\times[0,t]} \overline{g}(u,v,x(u,v))\, dM_{u,v} \end{aligned}$$

for every $x \in K(\xi, X)$ and $(s,t) \in I \times J$. By (4.6), properties of $\overline{f}$ and (4.2) we claim that for every $x \in K(\xi, X)$

$$\begin{aligned} \overline{f}(s,t,\omega,x(s,t)) \in F(s,t,\omega,x(s,t)) \\ \subset \bigcup_{\eta \in X(s,t)} F(s,t,\omega,\eta) \subset \hat{F}(s,t,\omega,X(s,t)). \end{aligned}$$

In a similar way we conclude the same relations for $\overline{g}, G$ and $\hat{G}$. Thus we get $V(x) \in K(\xi, X)$ for every $x \in K(\xi, X)$. Hence it is sufficient to show that the mapping V has a fixed point. Such a fixed point will be also a solution to stochastic integral inclusion (4.1) generated by the triple (F, G, ξ). We will show that V is a contraction under the metric

$$\rho(x,y) := \sup_{(s,t)\in I\times J} e^{-dL\gamma(s,t)} \left[\mathbb{E}\|x(s,t) - y(s,t)\|^2_{\mathbb{R}^d}\right]^{\frac{1}{2}}$$

in $C\left(I \times J, L^{2,d}\right)$, where $\gamma(s,t) := \tilde{\nu}_A([0,s]\times[0,t]) + \tilde{\mu}_M([0,s]\times[0,t])$ for $(s,t) \in I \times J$. Let $x, y \in K(\xi, X)$. Then by properties of the mappings $\overline{f}$ and $\overline{g}$, inequality (2.4) and Theorem 2.7 we have

$$\begin{aligned} \rho^2(V(x), V(y)) &\leqslant 2 \sup_{(s,t)\in I\times J} e^{-2dL\gamma(s,t)} \cdot \\ &\left[\mathbb{E}\| \int_{[0,s]\times[0,t]} \left(\overline{f}(u,v,x(u,v)) - \overline{f}(u,v,y(u,v))\right) dA_{u,v}\|^2_{\mathbb{R}^d} \right. \\ &\left. + \mathbb{E}\| \int_{[0,s]\times[0,t]} \left(\overline{g}(u,v,x(u,v)) - \overline{g}(u,v,y(u,v))\right) dM_{u,v}\|^2_{\mathbb{R}^d}\right] \\ &\leqslant 2 \sup_{(s,t)\in I\times J} e^{-2dL\gamma(s,t)} \left[\int_{[0,s]\times[0,t]\times\Omega} \left\|\overline{f}(u,v,x(u,v)) - \overline{f}(u,v,y(u,v))\right\|^2_{\mathbb{R}^d} d\nu_A \right. \\ &\left. + \int_{[0,s]\times[0,t]\times\Omega} \|\overline{g}(u,v,x(u,v)) - \overline{g}(u,v,y(u,v))\|^2_{\mathbb{R}^d}\, d\mu_M\right] \\ &\leqslant 2dL \sup_{(s,t)\in I\times J} e^{-2dL\gamma(s,t)} \int_{[0,s]\times[0,t]\times\Omega} \|x(u,v) - y(u,v)\|^2_{L^{2,d}} (d\nu_A + d\mu_M) \end{aligned}$$

$$= 2dL \sup_{(s,t)\in I\times J} e^{-2dL\gamma(s,t)} \int_{[0,s]\times[0,t]} \|x(u,v) - y(u,v)\|_{L^{2,d}}^2 \, d\gamma(u,v)$$
$$\leqslant 2dL \sup_{(s,t)\in I\times J} e^{-2dL\gamma(s,t)} \rho^2(x,y) \int_{[0,s]\times[0,t]} e^{2dL\gamma(u,v)} d\gamma(u,v).$$

Therefore by Lemma 4.3 we get

$$\rho^2(V(x), V(y)) \leqslant \sup_{(s,t)\in I\times J} \rho^2(x,y)\left(1 - e^{-2dL\gamma(s,t)}\right)$$
$$\leqslant \rho^2(x,y)\left(1 - e^{-2dL\gamma(S,T)}\right).$$

Thus by Banach's Contraction Principle we infer that there exists a unique element $\hat{x} \in K(\xi, X)$ such that

$$\hat{x}(s,t) = \xi(s,0) + \xi(0,t) - \xi(0,0) + \int_{[0,s]\times[0,t]} \overline{f}(u,v,\hat{x}(u,v))\, dA_{u,v} + \int_{[0,s]\times[0,t]} \overline{g}(u,v,\hat{x}(u,v))\, dM_{u,v}.$$

Thus the proof is completed. □

We finish our considerations with the following remarks. Let $CS(X)$ denote the set of all continuous selections for $X : I \times J \to \mathcal{K}_c^b\left(L^{2,d}\right)$ being a solution to set-valued stochastic integral equation (4.4). Since X is a continuous multifunction, it follows by Michael's Continuous Selection Theorem (see [7], [9]) that $CS(X) \neq \emptyset$. Hence by Theorem 4.4 we have the following result.

Corollary 4.5. *Under assumptions of Theorem* 4.4 *it holds*

$$CS(X) \cap SI(F,G,\xi) \neq \emptyset.$$

Moreover Theorem 4.4 can be also expressed in the spirit of reachable sets of solutions to stochastic inclusion (4.1) generated by (F, G, ξ). Namely, for $(s,t) \in I \times J$ let

$$\mathcal{A}((s,t), \xi, F, G) := \left\{x(s,t) \in L^{2,d} : x \in SI(F,G,\xi)\right\},$$

i.e. $\mathcal{A}((s,t), \xi, F, G)$ is the set of all possible values that are attained by trajectories of solutions to stochastic integral inclusion (4.1) at the point (s,t). Then the following result can be stated.

Corollary 4.6. *Let assumptions of Theorem* 4.4 *be satisfied and* $X : I \times J \to \mathcal{K}_c^b\left(L^{2,d}\right)$ *be a unique solution to the equation* (4.4). *Then* $\mathcal{A}((s,t), \xi, F, G) \cap X(s,t) \neq \emptyset$ *for every* $(s,t) \in I \times J$.

Let us also note that all of the above considerations can be applied to stochastic inclusions with expectations in the coefficients:

$$(4.9)\qquad \begin{cases} \Delta_{s,t}^{s',t'}(x) \in \int_{[s,s']\times[t,t']} F_1\left(u,v,\mathbb{E}\left(x\left(u,v\right)\right), \left\|x\left(u,v\right)\right\|_{L^{2,d}}\right) dA_{u,v} \\ \qquad + \int_{[s,s']\times[t,t']} G_1\left(u,v,\mathbb{E}\left(x\left(u,v\right)\right), \left\|x\left(u,v\right)\right\|_{L^{2,d}}\right) dM_{u,v} \\ x\left(0,t\right) = \xi\left(0,t\right) \\ x\left(s,0\right) = \xi\left(s,0\right) \end{cases},$$

where $F_1, G_1 : I \times J \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathcal{K}_c^b\left(\mathbb{R}^d\right)$ are appropriately regular set-valued mappings. Indeed, the stochastic inclusion (4.9) can be transformed to inclusion (4.1) if one takes set-valued mappings $F, G : I \times J \times \Omega \times L^{2,d} \to \mathcal{K}_c^b\left(\mathbb{R}^d\right)$ such that

$$F\left(s,t,\omega,\eta\right) := F_1\left(s,t,\omega,\mathbb{E}\left(\eta\right), \left\|\eta\right\|_{L^{2,d}}\right)$$

and

$$G\left(s,t,\omega,\eta\right) := G_1\left(s,t,\omega,\mathbb{E}\left(\eta\right), \left\|\eta\right\|_{L^{2,d}}\right)$$

for $(s,t,\omega,\eta) \in I \times J \times \Omega \times L^{2,d}$. Next, one can state an associated set-valued stochastic integral equation with coefficients defined by (4.2) and (4.3) or by mappings

$$\hat{F}\left(s,t,\omega,B\right) := \overline{co}\left(\bigcup_{\eta\in B} F_1\left(s,t,\omega,\mathbb{E}\left(\eta\right), \left\|\eta\right\|_{L^{2,d}}\right)\right)$$

and

$$\hat{G}\left(s,t,\omega,B\right) := \overline{co}\left(\bigcup_{\eta\in B} G_1\left(s,t,\omega,\mathbb{E}\left(\eta\right), \left\|\eta\right\|_{L^{2,d}}\right)\right)$$

for $(s,t,\omega,B) \in I \times J \times \Omega \times \mathcal{K}_c^b\left(L^{2,d}\right)$. In a single-valued case the above stochastic inclusion reduces to the stochastic integral equation

$$\begin{aligned} x\left(s,t\right) + \xi\left(0,0\right) - \xi\left(s,0\right) - \xi\left(0,t\right) \\ = \int_{[0,s]\times[0,t]} f\left(u,v,\mathbb{E}\left(x\left(u,v\right)\right), \left\|x\left(u,v\right)\right\|_{L^{2,d}}\right) dA_{u,v} \\ + \int_{[0,s]\times[0,t]} g\left(u,v,\mathbb{E}\left(x\left(u,v\right)\right), \left\|x\left(u,v\right)\right\|_{L^{2,d}}\right) dM_{u,v}. \end{aligned}$$

Such equations driven by a two-parameter Wiener process were used in the theory of term structure of interest rates (see e.g. [10], [16], [17] and references therein).

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