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GROUP ACTIONS WITH TOPOLOGICALLY STABLE MEASURES

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ABSTRACT: We prove that if an action T of a finitely generated group G on a compact metric space X is measure expansive and has the measure shadowing property then it is measure topologically stable. This represents a measurable version of the main result in [4]. Moreover we prove that if G is a finitely generated virtually nilpotent group and there exists $g \in G$ such that T_g is expansive and has the invariant measure shadowing property then T is invariant measure topologically stable. Finally we show that minimal actions approximated by periodic ones have no topologically stable measures.

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1. INTRODUCTION

Walters [12] introduced the notion of topological stability, a kind of stability for homeomorphisms for which continuous pertubations are allowed, and proved that every expansive homeomorphism with the shadowing property on a compact metric space is topologically stable. Recently, Lee and Morales [6] obtained a measurable version of this result that any expansive measure with the shadowing property is topologically stable. This represents a further contribution to the study of expansive measures developed elsewhere in the recent literature [2, 3, 8, 11].

Very recently, Chung and Lee [4] introduced the notion of topological stability for finitely generated group actions, and Pilyugin et. al [7, 9, 10] introduced the notions of shadowing and invese shadowing for finitely generated group actions which are generalizations of those of topological stability, shadowing and inverse shadowing, respectively, for homeomorphisms on compact metric spaces.

In this paper, we consider a measure version of the dynamics of finitely generated group actions on compact metric spaces which were developed in [4, 7, 9, 10]. More precisely, we introduce the notions of measure shadowing property and measure topological stability of group actions, and prove that if an action T of a finitely generated group G is measure expansive and has the measure shadowing property then it is measure topologically stable. Moreover we show that if G is a finitely generated virtually nilpotent group and there exists $g \in G$ such that T_g is expansive and has the invariant measure shadowing property then T is invariant measure topologically stable. Finally we claim that minimal actions approximated by periodic ones have no topologically stable measures.

2. PRELIMINARIES

We round out the introduction with some notations that we will use in the paper. Let G be a finitely generated group and X be a compact metric space with a metric d. Let $Homeo(X)$ be the space of all homeomorphisms of X. We denote by $Act(G,X)$ the set of all continuous actions T of G on X, *i.e.*, $T : G \times X \to X$ is a continuous map such that $T(e,x) = x$ and $T(g,T(h,x)) = T(gh,x)$ for $x \in X$ and $g,h \in G$, where e is the identity element of G. For briefness, $T(g,x)$ will be denoted by $T_g(x)$. Let $Homeo(X)^G = \prod_G Homeo(X)$ be the set of homeomorphisms from G to $Homeo(X)$ with the product topology. Then $Act(G,X)$ can be considered as a subset of $Homeo(X)^G$. Let A be a symmetric finitely generating set of G, i.e., for any $a \in A$, $a^{-1} \in A$. If A is a finintely generating set of G, then there always exists a symmetric finitely generating set containing A. Throughout the paper, a finitely generating set A of G implies a symmetric finitely generating set. We define a metric d_A on $Act(G,X)$ by

$$d_A(T,S) = \sup\{d(T_a x, S_a x) \mid x \in X, a \in A\}$$

for $T,S \in Act(G,X)$. Then the topology on $Act(G,X)$ induced by d_A coincides with the product topology on $Act(G,X)$ inherited from $Homeo(X)^G$. Hence the space $Act(G,X)$ is a separable complete metrizable topological space, and so a Baire space.

Recall that the *Borel σ-algebra* of X is the smallest σ-algebra generated by all open sets. Its elements will be referred to as *Borelians* of X. A *Borel measure* on X is a σ-additive measure defined on the Borelians of X. All Borel measures μ will be assumed to be *nontrivial* (i.e., $\mu(X) > 0$), and the set of all Borel measues on X will be denoted by $\mathcal{M}(X)$. A point $x \in X$ is an *atom* of $\mu \in \mathcal{M}$ if $\mu(\{x\}) > 0$ for every $x \in X$. We say that $\mu \in \mathcal{M}(X)$ is *nonatomic* if it has no atoms, and the set of all nonatomic measures on X will be deonted by $\mathcal{M}^*(X)$. In the sequel we proceed to extend the notions of expansivity, shadowing property and topological stability for group actions to Borel measures.

We say that $T \in A(G, X)$ is *expansive* if there is $c > 0$ called an *expansive constant* of T such that $d(T_g x, T_g y) \leq c$ for all $g \in G$ implies $x = y$. Equivalently, $\Gamma_c^T(x)) = \{x\}$ for all $x \in X$, where $\Gamma_c^T(x) = \{y \in X|\ d(T_g x, T_g y) \leq c \text{ for every } g \in G\}$. A Borel measure $\mu \in \mathcal{M}(X)$ is said to be *expansive* with respect to a homeomorphism f of X if there is $\varepsilon > 0$ such that $\mu(\Gamma_\varepsilon^f(x)) = 0$ for every $x \in X$ (see [6]). By this motivation, we introduce a notion of expansive measure with respect to an action $T \in Act(G, X)$.

Definition 2.1. For any Borel measure $\mu \in \mathcal{M}(X)$, an action $T \in Act(G, X)$ is said to be μ-expansive (or μ is expansive with respect to T) if there is $c > 0$ such that $\mu(\Gamma_c^T(x)) = 0$ for every $x \in X$. Moreover, we say that T is measure expansive if T is μ-expansive for all $\mu \in \mathcal{M}^*(X)$.

It is clear that every expansive measure for a group action is nonatomic, and any nonatomic measure with repect to an expansive action $T \in Act(G, X)$ is expansive.

If T and S are two continuous actions of G on X with $d_A(T, S) < \delta$, then the S-orbit $\{S_g x\}_{g \in G}$ of $x \in X$ is nearly a T-orbit in the sense that $d(T_a S_g x, S_{ag} x) < \delta$ for all $a \in A$ and $g \in G$. This observation motivates the following definition. Let A be a finitely generating set of G and $\delta > 0$. A *δ-pseudo orbit* of $T \in Act(G, X)$ with respect to A is a sequence $\{x_g\}_{g \in G}$ in X such that $d(T_a x_g, x_{ag}) < \delta$ for all $a \in A, g \in G$. An action $T \in Act(G, X)$ is said to have the *shadowing property* with respect to A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_g\}_{g \in G}$ for T with respect to A is ε-traced by some point x of X, that is, $d(T_g x, x_g) < \varepsilon$ for all $g \in G$.

Now we extend the notion of shadowing property of group actions to Borel measures. Given a subset B of X, we say that a sequence $\{x_g\}_{g \in G}$ is through B if $x_e \in B$.

Definition 2.2. Let A be a finitely generating set of G and $\mu \in \mathcal{M}(X)$. We say that an action $T \in Act(G, X)$ has the μ-shadowing property with respect to A (or μ has the shadowing property with respect to T and A) if for every $\varepsilon > 0$ there are $\delta > 0$ and a Borelian $B \subset X$ with $\mu(X \setminus B) = 0$ such that every δ-pseudo orbit $\{x_g\}_{g \in G}$ of T with respect to A through B is ε-traced by some point $x \in X$, i.e., $d(T_g x, x_g) < \varepsilon$

for all $g \in G$. Moreover T has the measure shadowing property with respect to A if T has the μ-shadowing property with respect to A for all $\mu \in \mathcal{M}^*(X)$.

It is clear that the definition of shadowing property of T does not depend on the choice of a compatible metric d on X. Furthermore, we can see that measure shadowing property of T does not depend on the choice of a finitely generating set A of G as we can see in the following lemma.

Lemma 2.3. *Let A_1 and A_2 be two finitely generating sets of G. An action $T \in Act(G,X)$ has the measure shadowing property with respect to A_1 if and only if it has the measure shadowing property with respect to A_2.*

Proof. Assume that $T \in Act(G,X)$ has the measure shadowing property with respect to A_1, and let $\mu \in \mathcal{M}(X)$. Then for any $\varepsilon > 0$, by definition there exist $\delta > 0$ and a Borelian $B \subset X$ with $\mu(X \setminus B) = 0$ such that every δ-pseudo orbit for T with respect to A_1 through B is ε-traced by some point of X. Put $m = \max_{a \in A_1} l_{A_2}(a)$, where l_{A_2} is the word length metric on G induced by A_2. Choose $\delta' > 0$ such that $m\delta' < \delta$. Since X is compact, A_1 and A_2 are finite and T is continuous, there exists $\delta'' > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta''$, one has $d(T_g x, T_g y) < \delta'$ for every $g \in G$ with $l_{A_2}(g) \le m$. Choose $0 < \delta_1 < \min\{\delta', \delta''\}$ such that for all $x, y \in X$ with $d(x,y) < \delta_1$, one has $d(T_{b^{-1}}x, T_{b^{-1}}y) < \delta''$ for every $b \in A_2$. It suffices to show that every δ_1-pseudo orbit of T with respect to A_2 through B is a δ-pseudo orbit of T with respect to A_1 through B. Let $\{x_g\}_{g \in G}$ be a δ_1-pseudo orbit for T with respect to A_2 i.e., $d(T_b x_g, x_{bg}) < \delta_1$ for every $b \in A_2, g \in G$. For every $b \in A_2, g \in G$, since $d(x_g, T_b x_{b^{-1}g}) = d(x_{bb^{-1}g}, T_b x_{b^{-1}g}) < \delta_1$, one has

$$d(T_{b^{-1}} x_g, T_{b^{-1}} T_b x_{b^{-1}g}) = d(T_{b^{-1}} x_g, x_{b^{-1}g}) < \delta''.$$

Thus $d(T_h T_{b^{-1}} x_g, T_h x_{b^{-1}g}) = d(T_h T_b x_g, T_h x_{bg}) < \delta'$ for every $b \in A_2, g \in G$ and $h \in G$ with $l_{A_2}(h) \le m$. For any $a \in A_1$, we write a as $b_1 \cdots b_{l(a)}$ where $l(a) = l_{A_2}(a), b_i \in A_2$. Then for any $a \in A_1, g \in G$, we have

$$\begin{aligned} d(T_a x_g, x_{ag}) &= d(T_{b_1 \cdots b_{l(a)}} x_g, x_{b_1 \cdots b_{l(a)} g}) \\ &\le d(T_{b_1 \cdots b_{l(a)}} x_g, T_{b_1 \cdots b_{l(a)-1}} x_{b_{l(a)} g}) \\ &\quad + d(T_{b_1 \cdots b_{l(a)-1}} x_{b_{l(a)} g}, T_{b_1 b_2 \cdots b_{l(a)-2}} x_{b_{l(a)-1} b_{l(a)} g}) \\ &\quad + \cdots + d(T_{b_1 b_2} x_{b_3 \cdots b_{l(a)} g}, T_{b_1} x_{b_2 \cdots b_{l(a)} g}) \\ &\quad + d(T_{b_1} x_{b_2 \cdots b_{l(a)} g}, x_{b_1 \cdots b_{l(a)} g}) \\ &< (m-1)\delta' + \delta_1 < (m-1)\delta' + \delta' = m\delta' < \delta. \end{aligned}$$

This means that $\{x_g\}_{g \in G}$ is a δ-pseudo orbit of T with respect to A_1 through B, and so completes the proof. ∎

Definition 2.4. We say that an action $T \in Act(G, X)$ has the measure shadowing property if T has the measure shadowing property with respect to A for some finitely generating set of G.

It is clear that if an action $T \in Act(G, X)$ has the shadowing property then it has the measure shadowing property, but the converse is not true in general.

Recently, Chung and lee [4] introduced the notion of topological stability for group actions as follows. Let A be a finitely generating set of G. An action $T \in Act(G, X)$ is said to be *topologically stable with respect to* A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$ then there exists a continuous map $f : X \to X$ satisfying $T_g \circ f = f \circ S_g$ for every $g \in G$ and $d(f, Id_X) \leq \varepsilon$. T is said to be *topologically stable* if it is topologically stable with respect to A for some finitely generating set A of G.

To define topologically stable measure we will use some basic notations. Denote by 2^X the set formed by the subsets of X. A map $H : X \to 2^X$ will be referred to as a *set-valued map* of X. We define the domain of H by $Dom(H) = \{x \in X : H(x) \neq \emptyset\}$. We say that H is *compact-valued* if $H(x)$ is compact for every $x \in X$. We write $d(H, Id_X) < \varepsilon$ for some $\varepsilon > 0$ if $H(x) \subset B[x, \varepsilon]$ where $B[.,.]$ denotes the closed ball operation (note that such an inclusion is obvious for points $x \notin Dom(H)$). A set-valued map H of X is *upper semicontinuous* if for every $x \in Dom(H)$ and every neighborhood O of $H(x)$ there is $\eta > 0$ such that $H(y) \subset O$ whenever $y \in X$ with $d(x, y) < \eta$. With these terminologies, we introduce the notion of measure topological stability of group actions.

Definition 2.5. Let A be a finitely generating set of G and $T \in Act(G, X)$. For any $\mu \in \mathcal{M}(X)$, we say that an action T is μ-topologically stable with respect to A (or μ is topologically stable with respect to T and A) if for every $\varepsilon > 0$, there is $\delta > 0$ such that for each continuous action S of G on X with $d_A(T, S) < \delta$ there is an upper semicontinuous compact-valued map H of X with measurable domain such that $\mu(X \setminus Dom(H)) = 0$, $\mu \circ H = 0$, $d(H, Id_X) \leq \varepsilon$, and $T_g \circ H = H \circ S_g$ for $g \in G$. Moreover we say that T is measure topologically stable if T is μ-topologically stable with respect to A for all $\mu \in \mathcal{M}^*(X)$.

The definition of measure topological stability of $T \in Act(G, X)$ is independent of the choice of generating sets of G as we can see in the following lemma.

Lemma 2.6. *Let A_1 and A_2 be finitely generating sets of G. An action $T \in Act(G, X)$ is measure topologically stable with respect to A_1 if any only if it is measure topologically stable with respect to A_2.*

Proof. Suppose T is measure topologically stable with respect to A_1. Then for any

$\varepsilon > 0$, there exists $\delta' > 0$ such that if S is another continuous action of G on X with $d_{A_1}(T, S) < \delta'$ then there is an upper semicontinuous compact-valued map H of X with measurable domain such that $\mu(X \setminus Dom(H)) = 0$, $\mu \circ H = 0$, $d(H, Id_X) \leq \varepsilon$, and $T_g \circ H = H \circ S_g$ for $g \in G$. It suffices to show that there exists $\delta > 0$ such that for every $S \in Act(G, X)$ if $d_{A_2}(T, S) < \delta$ then $d_{A_1}(T, S) < \delta'$. Put $m = \max_{a \in A_1} l_{A_2}(a)$, where l_{A_2} is the word length metric on G induced by A_2. Choose $\delta_1 > 0$ such that $m\delta_1 < \delta'$. Since X is compact and T is continuous, there exists $\delta > 0$ such that $d(T_h x, T_h y) < \delta_1$ for all $x, y \in X$ with $d(x, y) < \delta$ and for every $h \in G$ with $l_{A_2}(h) \leq m$. For any $a \in A_1$, we write a as $b_1 \cdots b_{l(a)}$, where $l(a) = l_{A_2}(a) \leq m, b_i \in A_2, i = 1, 2, \cdots, l(a)$. Then for every $S \in Act(G, X)$ with $d_{A_2}(T, S) < \delta$, one has

$$\begin{aligned}
d(T_a x, S_a x) \quad &= d(T_{b_1 \cdots b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\
&\leq d(T_{b_1 \cdots b_{l(a)-1}} T_{b_{l(a)}} x, T_{b_1 \cdots b_{l(a)-1}} S_{b_{l(a)}} x) \\
&+ d(T_{b_1 \cdots b_{l(a)-2}} T_{b_{l(a)-1}} S_{b_{l(a)}} x, T_{b_1 \cdots b_{l(a)-2}} S_{b_{l(a)-1}} S_{b_{l(a)}} x) \\
&+ \cdots + d(T_{b_1} T b_2 S_{b_3 \cdots \cdots b_{l(a)}} x, T_{b_1} S_{b_2} S_{b_3} \cdots \cdots S_{l(a)} x) \\
&+ d(T_{b_1} S_{b_2 \cdots b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\
&< m\delta_1 < \delta'.
\end{aligned}$$

This means that $d_A(T, S) < \delta'$, and so completes the proof. □

Finally we introduce the notion of measure topological stability for group actions as follows.

Definition 2.7. An action $T \in Act(G, X)$ is *measure topologically stable* if it is measure topologically stable with respect to some finitely generating set A of G.

3. ACTIONS WITH TOPOLOGICALLY STABLE MEASURES

In the previous section, we extended the concept of topological stability for group actions to Borel measures. First, we see that if an action $T \in Act(G, X)$ is topologically stable, then it is measure topologically stable.

Theorem 3.1. *If an action $T \in Act(G, X)$ is topologically stable, then it is measure topologically stable.*

Proof. Suppose an action $T \in Act(G, X)$ is topologically stable and $\mu \in \mathcal{M}^*(X)$. Let A be a finitely generating set of G. Fix $\varepsilon > 0$ and $\delta > 0$ as in the definition of topological stability for T. Take $S \in Act(G, X)$ with $d_A(T, S) < \delta$. Then there is a continuous map $h : X \to X$ with $T_g \circ h = h \circ S_g$ for all $g \in G$, and $d(h, Id_X) \leq \varepsilon$. Define a map H of X by $H(x) = \{h(x) | x \in X\}$. Then we have $Dom(H) = X$, and H is compact-valued. Since h is continuous, H is upper semicontinuous and $\mu(X \setminus Dom(H)) = 0$. Since μ is nonatomic, $\mu(H(x)) = \mu(\{h(x)\}) = 0$ for $x \in X$. Moreover, we have $d(H(x), x) = d(h(x), x) \le \varepsilon$ for all $x \in X$. Since $T_g \circ h = h \circ S_g$, we get $T_g \circ H = H \circ S_g$, and so completes the proof. □

As we have mentioned before, every expansive measure for group action is nonatomic, but topologically stable measure need not be nonatomic in general. In fact, let f_0 be a homeomorphism on $[0, 1]$ given by

$$f_0(t) = \begin{cases} \frac{1}{2}t, & 0 \le t \le \frac{1}{4}, \\ \frac{3}{2}t - \frac{1}{4}, & \frac{1}{4} \le t \le \frac{3}{4}, \\ \frac{1}{2}t + \frac{1}{2}, & \frac{3}{4} \le t \le 1. \end{cases}$$

Put $p_n = 1/2^n$, $p_{-n} = 1 - 1/2^n$ for each $n \in \mathbb{N}$. Consider a homeomorphism f on $S^1 = [0, 1]/\sim$ which has the following properties: $f(0) = 0$ and

$$f(x) = \begin{cases} p_{n+1} + (1/2^{n+1})f_0(2^{n+1}(x - p_{n+1})), p_{n+1} \le x \le p_n, \\ p_{-n} + (1/2^{n+1})f_0(2^{n+1}(x - p_{-n})), p_{-n} \le x \le p_{-n-1}, \end{cases}$$

where $n \in \mathbb{N}$. Then Yano [13] proved that f is not topologically stable, but it has the shadowing property. Moreover we have

$$\Omega(f) = \{p_n | n \in \mathbb{Z}\} \cup \left\{ \frac{p_n + p_{n+1}}{2} | n \in \mathbb{Z} \right\} \cup \{0\}.$$

By Colloary 4.5 in [6], we see that the Dirac measure μ_p ($p \in S^1 \setminus \Omega(f)$) is topologically stable with respect to f.

Now we prove that if an action of a finitely generated group on a compact metric space is measure expansive and has the measure shadowing property then it is measure topologically stable. This extends the main result of Theorem 2.8 in [4] to Borel measures. The techiques of the proof resembles that of Theorem 3.1 in [6], but we will give a detailed proof here for safety.

Theorem 3.2. *Any measure expansive action $T \in Act(G, X)$ with the measure shadowing property is measure topologically stable. More precisely, if T is μ-expansive and has the μ-shadowing property then it is μ-topologically stable for all $\mu \in \mathcal{M}^*(X)$.*

Proof. Let A be a finitely generating set of G and $\mu \in \mathcal{M}^*(X)$. Suppose that $T \in Act(G, X)$ is μ-expansive and has the μ-shadowing property. Let c be a μ-expansive constant of T. Take $\varepsilon > 0$ and $0 < \varepsilon' < \min\{\frac{c}{2}, \varepsilon\}$. For this ε', we let δ and B be given by the definition of the μ-shadowing property of T. Let S be another continuous group action of G on X with $d_A(S, T) \le \delta$. Define a set-valued map H of X by

$$H(x) = \bigcap_{g \in G} {T_g}^{-1}(B[S_g(x), \varepsilon']), \quad \forall x \in X.$$

Clearly, H is a compact-valued map.

Let us prove that $Dom(H)$ is measurable. Take a sequence $x_n \in Dom(H)$ conversing to some $x \in X$. Since $x_n \in Dom(H)$, we can choose a sequence $y_n \in X$ such that

$$d(T_g y_n, S_g x_n) \leq \varepsilon', \ \ \forall g \in G, \forall n \in \mathbb{Z}.$$

Since X is compact, we can assume that $y_n \to y$ for some $y \in X$. By fixing $g \in G$ and letting $n \to \infty$, we have $d(T_g y, S_g x) \leq \varepsilon'$ for every $g \in G$. This means that $y \in H(x)$, and so $x \in Dom(H)$. Consequently $Dom(H)$ is measurable.

Next we show that $\mu(X \setminus Dom(H)) = 0$. Since $d_A(T, S) \leq \delta$, the S-orbit $(S_g(x))_{g \in G}$ of every point $x \in X$ is a δ-pseudo orbit for T with respect to A. By taking $S_e(x) = x \in B$ we have that such an S-orbit is through B, hence it can be ε'-traced by some point in X. It follows that there is $y \in X$ such that $d(T_g y, S_g x) \leq \varepsilon'$ for every $g \in G$ and $x \in X$. From this we have $H(x) \neq \emptyset$ for every $x \in B$. Then $B \subset Dom(H)$ and so $\mu(X \setminus Dom(H)) \leq \mu(X \setminus B) = 0$.

To show that H is upper semicontinuous, fix $x \in Dom(H)$ and an open neighborhood O of $H(x)$. Define a map $H : X \to 2^X$ by

$$H(y) = \bigcap_{m=0}^{\infty} H_m(y),$$

where $H_m(y) = \bigcap_{g \in B(m)} T_{g^{-1}}(B[S_g(y), \varepsilon'])$, $B(m) = \{g \in G \mid l_A(g) \leq m\}$, and $l_A(g)$ is the word length metric on G induced by A. Note that each $H_m(y)$ is compact and $H_{m+1}(y) \subset H_m(y)$ for m and $y \in X$. By taking $y = x$, we have $H_m(x) \subset O$. We claim that there is $\eta > 0$ such that $H_m(y) \subset O$ whenever $d(x, y) < \eta$. Otherwise, there are sequences $y_{g_k} \to x$ and $z_{g_k} \in H_m(y_{g_k}) \setminus O$ for all g_k with $l_A(g_k) = k$. Since X is compact, we can assume that $z_{g_k} \to z$ for some $z \in X$. Clearly, $z \notin O$. However $z_{g_k} \in H_m(y_{g_k})$ so $d(T_g(z_{g_k}), S_g(y_{g_k}) \leq \varepsilon'$ for $g \in B(m)$ and all g_k with $l_A(g_k) = k$. Then we get,

$$\begin{aligned} d(T_g z, S_g x) &\leq d(T_g(z), T_g(z_{g_k})) + d(T_g(z_{g_k}), S_g(y_{g_k})) + d(S_g(y_{g_k}), S_g(x)) \\ &\leq d(T_g(z), T_g(z_{g_k})) + d(S_g(y_{g_k}), S_g(x)) + \varepsilon', \end{aligned}$$

for $g \in B(m)$. Letting $k \to \infty$ we get $d(T_g(z), S_g(x)) \leq \varepsilon'$ for any $g \in B(m)$, so $z \in H_m(x)$. Since $z \notin O$ and $H_m(x) \subset O$. We arrive at a contradiction. Consequently, if $d(x, y) < \eta$ then $H(y) \subset H_m(y) \subset O$, and so H is uppersemicontinuous.

Now we prove $\mu \circ H = 0$ and $d(H, Id_X) \leq \varepsilon$. Take $x \in X$ and $y \in H(x)$. If $z \in H(x)$, we have $d(T_g z, S_g x) \leq \varepsilon'$ for $g \in G$. Since $y \in H(x)$ we have $d(T_g y, S_g x) \leq \varepsilon'$. This implies $d(T_g y, T_g z) \leq 2\varepsilon'$ for $g \in G$. Hence $z \in \Gamma_c^T(y)$ and $H(x) \subset \Gamma_c^T(y)$.

Therefore, $\mu(H(x)) \le \mu(\Gamma_c^T(y)) = 0$. Since c is an expansive constant of μ, we have $\mu \circ H = 0$. It follows from the definition of H that $H(x) \subset B[x, \varepsilon']$. Since $\varepsilon' < \varepsilon$, we have $d(H, Id_X) \le \varepsilon$.

Finally, we prove $T_g \circ H = H \circ S_g$ for $g \in G$. If $x \in Dom(H)$, then we have

$$\begin{aligned} T_g(H(x)) &= T_g\left(\bigcap_{h\in G} T_h^{-1}(B[S_h(x), \varepsilon']\right) = \bigcap_{h\in G} T_{gh^{-1}}(B[S_h(x), \varepsilon']) \\ &= \bigcap_{g_1\in G} T_{g_1}^{-1}(B[S_{g_1 g}(x), \varepsilon']) = \bigcap_{g_1\in G} T_{g_1}^{-1}(B[S_{g_1}(S_g x), \varepsilon']) = H(S_g(x)), \end{aligned}$$

where $g_1 = hg^{-1}$. So $Dom(H)$ is S-invariant and $T_g \circ H = H \circ S_g$ in $Dom(H)$. It follows that $S_g(x) \notin Dom(H)$ if $x \notin Dom(H)$. Hence we obtain $T_g(H(x)) = \emptyset = H(S_g(x))$, and so $T_g \circ H = H \circ S_g$ on $X \setminus Dom(H)$. This completes the proof. □

Next we will provide a class of measure topologically stable actions, and study the dynamical properties of those actions. First, we recall the definition of nilpotent group. Let G be a countable group. The lower central series of G is the sequence $\{G_i\}_{i\ge 0}$ of subgroups of G defined by $G_0 = G$ and $G_{i+1} = [G_i, G]$, where $[G_i, G]$ is the subgroup of G generated by all commutators $[a, b] := aba^{-1}b^{-1}$, $a \in G_i, b \in G$. The group G is said to be *nilpotent* if there exists $n \ge 0$ such that $G_n = \{e_G\}$. The such smallest n is called the *nilpotent degree* of G.

We say that $\mu \in \mathcal{M}(X)$ is *invariant* for group action T if $\mu(B) = \mu(T_g(B))$ for any Borel set B and $g \in G$. We denote by $\mathcal{M}_T(X)$ and $\mathcal{M}_T^*(X)$ the set of all invariant Borel probability measures and the set of all invariant nonatomic Borel probability measures on X, respectively. We say that a group action T has *invariant measure shadowing property* if and only if T has μ-shadowing for all $\mu \in \mathcal{M}_T^*(X)$.

Theorem 3.3. *Let G be a finitely generated virtually nilpotent group, i.e., there exists a nilpotent normal subgroup H of G with finite index. Let T be a continuous group action of G on a compact metric space X. Suppose T_g is expansive and has the invariant measure shadowing property for some $g \in G$, then the action T is invariant measure topologically stable.*

To prove the above theorem, we need some following lemmas.

Lemma 3.4. *Let $T \in Act(G, X)$ and $\mu \in \mathcal{M}(X)$. If T is expansive with expansive constant η and has the μ-shadowing property. Let $\varepsilon < \frac{\eta}{2}$, then every δ-pseudo orbit of T as in the definition of μ-shadowing property with respect to ε has a unique shadowing point in X.*

Proof. For any $\varepsilon > 0$ choose $\delta > 0$ and a Borel set B with $\mu(B) = 1$ as in the definition of μ-shadowing property. Let $\{x_g\}_{g\in G}$ be a δ-pseudo orbit of T through B

and let η be an expansive constant with $\eta > 2\varepsilon$. Assume that x, y be two points that ε-traced $\{x_g\}_{g\in G}$. Then one has

$$d(T_g x, T_g y) \leq d(T_g x, x_g) + d(x_g, T_g y) < \varepsilon + \varepsilon < \eta$$

for every $g \in G$. Since T is expansive, we have $x = y$. □

Lemma 3.5. *Let G be a finitely generated group and H be a finitely generated normal subgroup of G. Let $T \in Act(G, X)$ and $\mu \in \mathcal{M}_T(X)$. If the restriction map T_H on $H \times X$ is expansive and has the μ-shadowing property, then T has the μ-shadowing property.*

Proof. Let $\mu \in \mathcal{M}_T(X)$ and A_H be a symmetric finitely generating set of H. We can add more elements to A_H to get a finitely generating set A of G. Let c be the expansive constant of T_H. Since X is compact and A is finite, there exists $0 < \eta < \frac{c}{3}$ such that $d(T_b x, T_b y) < \frac{c}{3}$ for every $b \in A$ and all $x, y \in X$ with $d(x, y) < \eta$. Let $\varepsilon > 0$ be a constant with $\varepsilon < \eta$, since T_H has the μ-shadowing property, we can choose $0 < \delta < \varepsilon$ and Borel set B_H with $\mu(X \setminus B_H) = 0$ such that every δ-pseudo orbit of T_H with respect to A_H through B_H is $\frac{\varepsilon}{2}$-traced by some point of X. Let

$$B = B_H \cap supp(\mu).$$

First, we claim that $\mu(B) = 1$. Indeed, by contradiction, we suppose that $0 \leq \mu(B) < 1$. Let $B' = B_H \bigcap (supp(\mu))^c$, then $B_H = B' \bigcup B$ and hence $\mu(B') > 0$. For any $x \in B'$, $x \notin supp(\mu)$, there exists a neighborhood U_x such that $\mu(U_x) = 0$. Then

$$B' \subset \bigcup_{x\in B'} U_x, \text{ so } \mu(\bigcup_{x\in B'} U_x) > \mu(B') > 0.$$

However, $(\bigcup_{x\in B'} U_x) \bigcap supp(\mu) = \emptyset$ and then $\mu(\bigcup_{x\in B'} U_x) = 0$. The contradiction shows that $\mu(B) = 1$. Let $\{x_g\}_{g\in G}$ be a $\frac{\delta}{2}$-pseudo orbit with respect to A through B. For every $g \in G$, the sequence $\{x_{hg}\}_{h\in H}$ is a $\frac{\delta}{2}$-pseudo orbit of T_H with respect to A_H. Since μ is invariant for T and B is a subset of $supp(\mu)$ with $\mu(B) = 1$, B is dense in $supp(\mu)$. For $\frac{\delta}{2} > 0$, we choose $0 < \eta' < \frac{\delta}{2}$ and $y_g \in B$ such that $d(y_g, x_g) < \eta' < \frac{\delta}{2}$ and $d(T_a x_g, T_a y_g) < \frac{\delta}{2}$ for every $g \in G$, $a \in A_H$. We define a new sequence $\{y_{hg}\}_{h\in H}$ by

$$y_{hg} = \begin{cases} y_g & h = e, \\ x_{hg} & h \neq e. \end{cases}$$

for every $g \in G$ such that

$$d(T_a y_{hg}, y_{ahg}) = \begin{cases} d(T_a x_{a^{-1}g}, y_g) & h = a^{-1}, \\ d(T_a y_g, x_{ag}) & h = e, \\ d(T_a x_{hg}, x_{ahg}) & h \neq e, a^{-1}. \end{cases}$$

Then,

$$
\begin{aligned}
&d(T_a x_{a^{-1}g}, y_g) \le d(T_a x_{a^{-1}g}, x_{aa^{-1}g}) + d(x_g, y_g) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,\\
&d(T_a y_g, x_{ag}) \le d(T_a y_g, T_a x_g) + d(T_a x_g, x_{ag}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,\\
&d(T_a x_{hg}, x_{ahg}) < \delta.
\end{aligned}
$$

So, $\{y_{hg}\}_{h\in H}$ is a δ-pseudo orbit of T_H with respect to A_H through B hence through B_H. Since T_H is expansive, by Lemma 3.4 there exists a unique point $z_g \in X$ such that

$$
d(y_{hg}, T_h(z_g)) < \frac{\varepsilon}{2} \text{ for every } h \in H. \tag{1}
$$

Now we prove that $z_g = T_g z_e$ for every $g \in G$. Fix $g \in G$ and $b \in A$. For each $h \in H$, there exists $h' \in H$ such that $hb = bh'$ by H is a normal subgroup of G. Then we have

$$
d(y_{bh'g}, T_h z_{bg}) = d(y_{hbg}, T_h z_{bg}) < \frac{\varepsilon}{2}
$$

and

$$
d(T_b y_{h'g}, T_h T_b z_g) = d(T_b y_{h'g}, T_b T_{h'} z_g) < \frac{c}{3}.
$$

Hence we get

$$
\begin{aligned}
d(T_h z_{bg}, T_h T_b z_g) &\le d(T_h z_{bg}, y_{bh'g}) + d(y_{bh'g}, T_b y_{h'g}) + d(T_b y_{h'g}, T_h T_b z_g)\\
&\le \frac{\varepsilon}{2} + \delta + \frac{c}{3} < c.
\end{aligned}
$$

Since T_H is expansive, we have $T_b z_g = z_{bg}$ for every $b \in A$. As A is symmetric generating set of G, we get $z_g = T_g z_e$ for every $g \in G$. On the other hand, by applying $h = e$ for $d(y_{hg}, T_h(z_g)) < \frac{\varepsilon}{2}$, we have $d(y_g, z_g) < \frac{\varepsilon}{2}$ and then

$$
d(x_g, z_g) = d(x_g, y_g) + d(y_g, z_g) < \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon
$$

for every $g \in G$. So $d(T_g z_e, x_g) = d(z_g, x_g) < \varepsilon$ for all $g \in G$ thus T has the μ-shadowing property. □

Recall that for an action $T \in Act(G, X)$, T_g is a homeomorphism for every $g \in G$. We call that T_g has the *μ-shadowing property* if for any $\varepsilon > 0$ there are $\delta > 0$ and a Borel set B with $\mu(X \setminus B) = 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\in\mathbb{N}}$ of T_g through B is ε-traced by some point in X.

Lemma 3.6. *Let T be a continuous action on a compact metric space X. Then for any $g \in G$, T_g has the μ-shadowing property then T_{g^n} has the μ-shadowing property for any $n \in \mathbb{Z}$.*

Proof. For any $\varepsilon > 0$, choose $\delta_1 > 0$ and a Borel set B corresponding to the definition of μ-shadowing property with respect to T_g. For $\varepsilon > 0$ choose $0 < \delta_n < \delta_1$ such that $\{x_n\}_{n\in\mathbb{N}}$ is a δ_n-pseudo orbit of T_{g_n} through B. Now we show that the δ_n-pseudo orbit $\{x_n\}_{n\in\mathbb{N}}$ of T_{g_n} through B is also a δ_1-pseudo orbit of T_g through B. Define a sequence $\{y_m\}_{m\in\mathbb{N}}$ as $y_{ln} = x_l$ for all $l \in \mathbb{Z}$, $y_{ln+i} = T_{g^i}x_l$ for $1 \le i \le n-1$ and for all $l \in \mathbb{Z}$. Since $d(T_{g^n}x_0, x_1) < \delta_n$, $d(T_g y_{n-1}, y_n) < \delta_n$, it means that $\{y_n\}_{n\in\mathbb{N}}$ is a δ_1-pseudo orbit of T_g through B. Hence there is $y \in X$ such that $d((T_g)^k, y_k) < \varepsilon$, so for $k = ln$ where $l \in \mathbb{N}$, we have $d((T_g)^{ln}y, y_{ln}) = d((T_g^n)^l y, x_l) < \varepsilon$ so complete the proof. □

The following lemma whose proof is similar to that of Lemmas 2.13 in [4]. For satety, we give detailed proof.

Lemma 3.7. *Let G be a finitely generated virtually nilpotent group and T be a continuous action of G on a compact metric space X. If there exists $g \in G$ such that T_g is expansive and has the μ-shadowing property for some Borel measure $\mu \in \mathcal{M}_T^*(X)$ on X, then T has the μ-shadowing property.*

Proof. We prove by induction on the nilpotent degree n of G. If $n = 1$, then the group G is abelian and hence $H =< g >$ is a normal subgroup of G. Thus, by the previous two lemmas T has the μ-shadowing property. Let $n > 1$ and assume that this lemma holds for each nilpotent group with nilpotent degree less than or equal to $n-1$. Put $G_1 = [G, G]$ and $K =< G_1, g >$, then K has the nilpotent degree at most $n-1$ by Proposition 2(N2) in [7]. It is known that G_1 is finitely generated by Lemma 6.8.4 in [5] and hence K is finitely generated. Thus from the assumption, T_K has the μ-shadowing property. Since T_g is expansive and $g \in K$, T_K is expansive. As K is a normal subgroup of G by Proposition 2(N1) in [7], applying Lemma 3.5, we get the result. □

Proof of Theorem 3.3. Let H be a nilpotent normal subgroup of G with finite index. Then H is finitely generated by Proposition 6.6.2 in [5]. Since H has finite index in G, there exists $n \in \mathbb{N}$ such that $g^n \in H$ for any $g \in G$. Since T_g is an expansive homeomorphism and $g^n \in H$, then $T_{g^n} = {T_g}^n$ is also expansive by Theorem 2.2.4 in [1] and also has the μ-shadowing property by the Lemma 3.6. Thus from Lemma 3.7, T_H has the μ-shadowing property. Since T_{g^n} is expansive, T_H is expansive so is T, hence T is μ-expansive for every $\mu \in \mathcal{M}^*(X)$. Therefore, applying Theorem 3.2 and Lemma 3.5, T is μ-topologically stable. □

In the following theorem, we provide a class of group actions which do not have any topologically stable measures. For this, we first recall that an action $T \in Act(G, X)$ is *minimal* if every T-orbit $\{T_g x | g \in G\}$ $(x \in X)$ is dense in X, and T is called *periodic* if $\{T_g x | g \in G\}$ $(x \in X)$ is finite.

Theorem 3.8. *A minimal action $T \in Act(G, X)$ approximated by periodic actions is not μ-topologically stable for any $\mu \in \mathcal{M}(X)$.*

Proof. Let $T \in Act(G, X)$ be a minimal action which can be approximated by periodic actions, and suppose there exists a Borel measure $\mu \in \mathcal{M}(X)$ which is topologically stable with respect to T. For $\varepsilon = 1$, choose a constant $\delta > 0$ corresponding to ε by the definiton of μ-topological stability of T. Choose a periodic action $S \in Act(G, X)$ with $d_A(T, S) \leq \delta$ for some fintely generating set A. Then we have that $\mu(Per(S)) = 0$. Suppose not. Then, by the topological stability of μ with respect to T, there is an upper semicontinuous compact-valued map H with measurable domain X satisfying $\mu(X \setminus Dom(H)) = 0$ and $\mu \circ H = 0$ and $T_g \circ H = H \circ S_g$ for every $g \in G$. Since $\mu(Per(S)) > 0$ and $\mu(X \setminus Dom(H)) = 0$, there exists a point $x \in Per(S) \bigcap Dom(H)$. Since x is a periodic point of S, we can easily show that $\Lambda = \bigcup_{g \in G} H(S_g x)$ is both compact and invariant. As $\mu \circ H = 0$, we also have $\mu(\Lambda) = 0$. As $H(x) \neq \emptyset$ we can choose $y \in H(x)$. Since $H(x) \subset \Lambda$ we have $y \in \Lambda$. But T is minimal so the T-orbit of y is dense in X. As Λ is compact and invariant we obtain that the closure of such an T-orbit is contained in Λ. Hence we reach that $X = \Lambda$, and so $\mu(X) = \mu(\Lambda) = 0$. This shows that $\mu(Per(S)) = 0$. On the other hand, since S is periodic, we have $\mu(Per(S)) = 1$. Consequently the contradiction completes the proof. □

Finally we show that any action $T \in Act(G, X)$ with a topologically stable measure, and which can be approximated by minimal actions is recurrsive, i.e., $\Omega(T) = X$. Recall that a point $x \in X$ is nonwandering for T if, for any neighborhood U of x there is $g \in G \setminus \{e\}$ such that $T_g(U) \bigcap U \neq \emptyset$. The set of all such points is denoted by $\Omega(T)$.

Lemma 3.9. *Let A be a finitely generating set of G, and let $T \in Act(G, X)$ be and action with a topologically stable measure μ. Then for any $\varepsilon > 0$, there is $\delta > 0$ and a subset B of X with $\mu(X \setminus B) = 0$ such that if $d_A(T, S) < \delta$ then every S-orbit of x can be ε-traced in T for all $x \in B$.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by the μ-topological stability of T. Take a continuous action S with $d_A(T, S) \leq \delta$. For this S let H be as in the definition of μ-topological stability of T with $\mu(X \setminus Dom(H)) = 0$. We choose $B = Dom(H)$. Then for any $x \in B$ we have $y \in H(x)$ and hence $T_g y \in T_g(H(x)) = H(S_g x) \subset B[S_g x, \varepsilon]$ and so $d(T_g y, S_g x) \leq \varepsilon$ for all $g \in G$ completing the proof. □

Theorem 3.10. *Let $T \in Act(G, X)$ be an action which can be approximated by* *minimal actions. If T has a topologically stable measure, then $\Omega(T) = X$.*

Proof. Let μ be a topologically stable measure of T. Fix $\varepsilon > 0$ and for any $z \in X$ we show that $z \in \Omega(T)$. For the given ε and μ, by the previous lemma there are $\delta > 0$ and B with $\mu(B) = 1$ such that every S-orbit of any $x \in B$ with $d_A(T, S) < \delta$ is $\frac{\varepsilon}{2}$-traced in T. By hypothesis there is a minimal action $S \in Act(G, X)$ with $d_A(T, S) \leq \delta$ and for any $x \in B$, the S-orbit $\{S_g x | g \in G\}$ of x is dense in X. Since $x \in B$, there is $y \in X$ such that $d(T_g y, S_g x) < \frac{\varepsilon}{2}$ for any $g \in G$. Since $\overline{\{S_g x | g \in G\}} = X$, there is a sequence $(g_n)_{n\in\mathbb{N}} \subset G$ with $l_A(g_n) = n$ such that $\lim_{n\to\infty} S_{g_n}(x) = z$ for some $z \in X$. Then there is $m \in \mathbb{N}$ such that $d(S_{g_n}(x), z) < \frac{\varepsilon}{2}$ for all g_n with $l_a(g_n) \geq m$. Since X is compact we can assume that $T_{g_n} y \to w$ for some $w \in X$. Let U be a neighborhood of w then there exists $g_n, g_{n+1} \in G$ such that $T_{g_n} y, T_{g_{n+1}} y \in U$ for n large enough. Choose $h = g_n g_{n+1}^{-1}$. Then $T_h T_{g_{n+1}} y = T_{g_n} y \in U$ hence $T_{g_{n+1}} y \in T_{h^{-1}} U$, and we have $w \in \Omega(T)$. Since

$$\begin{aligned} d(z, T_{g_n} y) &\leq d(z, S_{g_n} x) + d(S_{g_n} x, T_{g_n} y) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $g_n \in G$ with $l_a(g_n) \geq m$. By letting $n \to \infty$ we have $d(z, w) \leq \varepsilon$. Since $w \in \Omega(T)$ we conclude that

$$d(z, \Omega(T)) = \inf_{w\in\Omega(T)} d(z, w) \leq \varepsilon.$$

As z, ε are arbitrary, we conclude that $\Omega(T)$ is dense in X. As $\Omega(T)$ is closed, we obtain $\Omega(T) = X$ and complete the proof. □

REFERENCES

[1] N. Aoki and K. Hiraide, *Topologically Theory of Dynamical Systems*, North-Holland Mathematical Library, **52**, North-Holland Publishing Co., Amsterdam, 1994.

[2] A. Arbieto and C.A Morales, Some properties of positive entropy maps, *Ergodic Theory Dynam. Systems*, **34** (2014), no. 3, 765-776.

[3] A. Artigue and D. Carrasco-Olivera, A note on measure-expansive diffeomorphisms, *J. Math. Anal. Appl.*, **428** (2015), no. 1, 713-716.

[4] N.P Chung and K. Lee, Topological stability and pseudo-orbit tracing property of group actions, To appear in *Proc. Amer. Math. Soc.*.

[5] M. Coornaert and T.C. Silberstein, *Cellular Automata and Groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[6] K. Lee and C.A Morales, Topological stability and pseudo-orbit tracing property for expansive measures, *J. Differential Equations*, **262** (2017), 3467-3487.

[7] A.V. Osipov and S.B. Tikhomirov, Shadowing for actions of some finitely generated groups, *Dyn. Syst.*, **29** (2014), 337-351.

[8] M.J. Pacifico and J.L. Vietez, On measure expansive diffeomorphisms, *Proc. Amer. Math. Soc.*, **143** (2015), no. 2, 811-819.

[9] S.Y Pilyugin, Inverse shadowing in group action, *Dyn. Syst.*, **32** (2017), 198-210.

[10] S.Y. Pilyugin and S.B. Tikhomirov, Shadowing in actions of some Abelian groups, *Fund. Math.*, **179** (2003), 83-96.

[11] K. Sakai, N. Sumi, and K. Yamamoto, Measure-expansive diffeomorphisms, *J. Math. Anal. Appl.*, **414** (2014), no. 2, 546-552.

[12] P. Walters, On the pseudo orbit tracing property and its relationship to stability, *Lecture Notes in Math.*, **668** (1978), 231-244.

[13] K. Yano, Topologically stable homeomorphisms of the circle, *Nagoya Math. J.*, **79** (1980), 145- 149.