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GLOBAL EXPONENTIAL STABILITY OF POSITIVE PERIODIC SOLUTIONS FOR AN IMPULSIVE LASOTA-WAZEWSKA MODEL WITH DELAYS

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ABSTRACT: In this paper, we consider a class of non-autonomous impulsive Lasota-Wazewska model with delays. First, the equivalent relation between the solution (or positive periodic solution) of the Lasota-Wazewska delayed model with impulsive effects and that of a corresponding delayed Lasota-Wazewska model without impulsive effects is established. Then, by applying algebraic inequalities and Laypunov functional method, some sufficient conditions for the existence and global exponential stability of positive periodic solution of addressed model are given. Finally, an example and its numerical simulation are provided to illustrate the effectiveness of the theoretical results.

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1. INTRODUCTION

In 1988, Wazewska-Czyzewska and Lasota [1] proposed the following nonlinear autonomous delay differential equation as a model to describe the survival of red blood cells in an animal

$$N'(t) = -\alpha N(t) + \beta e^{-\gamma N(t-\tau)}, t \geq 0, \tag{1.1}$$

where $\alpha, \beta, \gamma \in (0, +\infty)$, and $N(t)$ denotes the number of red blood cells at time t, α is the probability of death of a red blood cell, β and γ are positive constants related to the production of red blood cells per unit of time and τ is the time needed to produce a red blood cell. The Lasota-Wazewska model (1.1) and its modifications have also been later to describe population growth of other species, and thus, have gained much research attention. A large amount of results related to these problems have been published (see, e.g.[2-7] and the reference therein).

It is well known that impulsive effective effect are also likely to exist in the ecological models besides delay effects. Many evolution processes in nature are characterized by the fact that at certain moments of time experience an abrupt change of state. That was the reason for the development of the theory of the impulsive delay differential equations [8-10]. Recently, some important results about positive periodic solution for impulsive biology mathematical model with delays have been obtained [11-16].

To the best of our knowledge, however, few authors have considered the global exponential stability of positive periodic solutions of impulsive Lasota-Wazewska model with delays. Motivated by the above discussions, we will study the existence and global exponential stability of positive periodic solutions of impulsive Lasota-Wazewska model with delays.

Firstly, the equivalent relation between the solution (or positive periodic solution)of impulsive Lasota-Wazewska model with delays and that of a corresponding nonimpulsive Lasota-Wazewska model with delays is established. Then, by applying algebraic inequalities and Lyapunov functional method, some sufficient conditions for the existence and global exponential stability of positive periodic solutions of impulsive Lasota-Wazewska model with delays are given. Finally, an example and its numerical simulation are provided to show the effectiveness of the theoretical results.

2. MODEL DESCRIPTION AND PRELIMINARIES

For convenience, we introduce several notations and recall some basic definitions.

Considering the following generalized impulsive Lasota-Wazewska model with delays

$$\begin{cases} y'(t) = -\alpha(t)y(t) + \sum\limits_{i=1}^{m} \beta_i(t)e^{-\gamma_i(t)y(t-\tau_i)}, \quad t \geq 0, \\ y(t_k^+) = (1+b_k)y(t_k^+), \quad k = 1, 2, \cdots \end{cases} \tag{2.1}$$

In equation (2.1), we shall use the following hypotheses:

(H_1) $0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $\lim\limits_{k\to\infty} t_k = \infty$;

(H_2) $\alpha, \beta_i, \gamma_i \in C([0,\infty),(0,\infty))$, where $C([0,\infty),(0,\infty))$ denotes the space of continuous functions from $[0,\infty)$ to $(0,\infty)$;

(H_3) $\{b_k\}$ is a real sequence and $b_k > -1,\ k = 1, 2, \cdots$;

(H_4) $\alpha(t), \beta_i(t), \gamma_i(t)$ and $\prod\limits_{0<t_k<t}(1+b_k)$ are periodic functions with common periodic $\omega > 0,\ i = 1, 2, \cdots, m,\ k = 1, 2, \cdots$.

Here and in the sequel we assume that a product equals unit if the number of factors is equal to zero. We will only consider the solutions of equation (2.1) with initial condition

$$y(s) = \phi(s),\ s \in [-\tau, 0], \tag{2.2}$$

where $\phi \in C([-\tau,0],(0,\infty)),\ \tau = \max\limits_{1\leq i\leq m}\{\tau_i\}$, and $\tau_i > 0$ is time delay, $i = 1, 2, \cdots, m$.

Definition 2.1. A function $y \in C([-\tau,\infty),(0,\infty))$ is said to be a solution of equation (2.1) on $[-\tau,\infty)$ if

(i) $y(t)$ is absolutely continuous on each interval $(0,t_1]$ and $(t_k,t_{k+1}],\ k = 1, 2, \cdots$;

(ii) for any $t_k,\ k = 1, 2, \cdots,\ y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$;

(iii) $y(t)$ satisfies the differential equation of equation (2.1) in $[0,\infty)\backslash\{t_k\}$, and the impulsive conditions for every $t_k, k = 1, 2, \cdots$;

(iv) $y(s) = \phi(s),\ s \in [-\tau, 0]$.

Under the hypotheses (H_1)-(H_4), we consider the following delay differential equation without impulses:

$$x'(t) = -\alpha(t)x(t) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)x(t-\tau_i)}, \tag{2.3}$$

with initial condition

$$x(s) = \phi(s), \text{ for } s \in [-\tau, 0], \quad \phi \in C([-\tau, 0], (0, \infty)), \tag{2.4}$$

where

$$p_i(t) = \prod_{t-\tau_i \le t_k < t} (1 + b_k)^{-1}\beta_i(t) \text{ and } q_i(t) = \prod_{0<t_k<t-\tau_i} (1 + b_k)\gamma_i(t), \quad t \ge 0.$$

$i = 1, 2, \cdots, m.$

Remark 2.2. By a solution $x(t)$ of equation (2.3), we mean an absolutely continuous function $x(t)$ defined on $[-\tau, \infty)$ satisfying equation (2.3) for $t \ge 0$ and initial condition (2.4) on $[-\tau, 0]$.

Similar to the method of [17], we have

Lemma 2.3. *Assume that* (H_1) – (H_4) *hold. Then*
(i) if $x(t)$ is a solution (or positive $\omega-$periodic solution) of equation (2.3) on $[-\tau, \infty)$, then $y(t) = \prod_{0<t_k<t} (1 + b_k)x(t)$ is a solution (or positive $\omega-$periodic solution) of equation (2.1) on $[-\tau, \infty)$;
(ii) if $y(t)$ is a solution (or positive $\omega-$periodic solution) of equation (2.1) on $[-\tau, \infty)$, then $x(t) = \prod_{0<t_k<t} (1 + b_k)^{-1}y(t)$ is a solution (or positive $\omega-$periodic solution) of equation (2.3) on $[-\tau, \infty)$.

Lemma 2.4. *Let* (H_1) – (H_4) *hold. Then every solution $x(t)$ of equation (2.3) and every solution $y(t)$ of equation (2.1) are defined on $[-\tau, \infty)$ and positive on $[0, \infty)$.*

Proof. Clearly, by Lemma 2.3, we only need to prove that every solution $x(t)$ of equation (2.3) defined on $[-\tau, \infty)$ and positive on $[0, \infty)$. From (2.3) and (2.4), we have

$$x(t) = \phi(0)e^{-\int_0^t \alpha(s)ds} + \int_0^t \sum_{i=1}^{m} p_i(t)e^{\int_s^t \alpha(v)dv}e^{-q_i(s)x(s-\tau_i)}ds.$$

Hence, $x(t)$ is defined on $[-\tau,\infty)$ and positive on $[0,\infty)$. the proof is complete. $\square$

For the sake of simplicity of notation, for a bounded continuous function f defined on R, we denote

$$f^+ = \sup_{t\in R} f(t) \text{ and } f^- = \inf_{t\in R} f(t).$$

As usual, $C = C([-\tau,0),(0,\infty))$ is the Banach space of the set of all continuous functions from $[-\tau,0)$ to $(0,\infty)$ equipped with supremum norm $||\cdot||$. Furthermore, for a continuous function x defined on $[t_0-\tau,\sigma)$ with $0 < t_0 < \sigma$ and $t \in [t_0,\sigma)$, we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau,0]$.

Define a continuous map $f : \mathbb{R}\times C \to \mathbb{R}$ by

$$f(t,\phi) = -\alpha(t)\phi(0) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)\phi(-\tau_i)},$$

then f is a locally Lipschitz map with respect to $\phi \in C$, which ensures the existence and uniqueness of the solution of equation (2.3) with initial condition (2.4).

We denote $x_t(t_0,\phi)$ for a solution of (2.3). Also, let $[t_0,\eta(\phi))$ be the maximal right-interval of existence of $x_t(t_0,\phi)$.

In the following, we present two lemmas which lays the foundation for the existence and global exponential stability of positive periodic solution to equation (2.3).

Lemma 2.5. *Let* $(\mathrm{H}_1)-(\mathrm{H}_4)$ *hold. Suppose that there exists a positive number* $\xi > 0$, *such that*

$$0 < \xi < \frac{\sum_{i=1}^{m} p_i(t)}{e\alpha(t)} < \frac{\sum_{i=1}^{m} p_i(t)}{\alpha(t)} < \frac{1}{q^+}, \tag{2.5}$$

where $q^+ = \max_{1\le i\le m}\{q_i^+\}$. *Then, for a solution of equation (2.3)* $x_t(t_0,\phi)$ *with initial condition* $\phi \in C : \xi < \phi(\theta) < \frac{1}{q^+}$ *for all* $\theta \in [-\tau,0]$, $\xi < x_t(t_0,\phi) < \frac{1}{q^+}$ *for all* $t \ge t_0$.

Proof. For a given $\phi \in C : \xi < \phi(\theta) < \frac{1}{q^+}$, we denote $x_t(t_0,\phi)$ by $x(t)$, then we have $x(t) > 0$ for all $t \in [t_0,\eta(\phi))$ by Lemma 2.4.

Firstly, we will show that

$$x(t) < \tfrac{1}{q^+}, \text{ for all } \ t \in [t_0, \eta(\phi)). \tag{2.6}$$

By way of contradiction, there exists $t_1 \in (t_0, \eta(\phi))$ such that $x(t_1) = \frac{1}{q^+}$ and $x(t) < \frac{1}{q^+}$ for all $t \in [t_0 - \tau, t_1)$. Then $x'(t_1) = x'_-(t_1) = \lim\limits_{t \to t_1^-} \frac{x(t)-x(t_1)}{t-t_1} \geq 0$. On the other hand, by the fact that $\sup\limits_{x>0}\{e^{-x}\} = 1$, we obtain from Lemma 2.4, (2.3) and (2.5)

$$x'(t_1) = -\alpha(t_1)x(t_1) + \sum_{i=1}^{m} p_i(t_1)e^{-q_i(t_1)x(t_1-\tau_i)} \leq -\alpha(t_1)\tfrac{1}{q^+} + \sum_{i=1}^{m} p_i(t_1) < 0,$$

which is a contradiction. This proves (2.6).

Secondly, we will show that

$$x(t) > \xi, \text{ for all } \ t \in [t_0, \eta(\phi)). \tag{2.7}$$

By way of contradiction again, there exists $t_2 \in (t_0, \eta(\phi))$ such that $x(t_2) = \xi$ and $x(t) > \xi$ for all $t \in [t_0 - \tau, t_1)$. Then $x'(t_2) = x'_-(t_2) = \lim\limits_{t \to t_2^-} \frac{x(t)-x(t_2)}{t-t_2} \leq 0$. On the other hand, we obtain from (2.3), (2.5) and (2.6) that

$$\begin{aligned} x'(t_2) &= -\alpha(t_2)x(t_2) + \sum_{i=1}^{m} p_i(t_2)e^{-q_i(t_2)x(t_2-\tau_i)} \\ &\geq -\alpha(t_2)\xi + \sum_{i=1}^{m} p_i(t_2)e^{-q_i(t_2)\frac{1}{q^+}} \geq -\alpha(t_2)\xi + \sum_{i=1}^{m} p_i(t_2)e^{-1} > 0. \end{aligned}$$

Which is a contradiction. This proves (2.7).

By the above proof, we have shown that $x(t)$ is bounded on $[t_0 - \tau, \eta(\phi))$. By Theorem 2.3.1 in [18], we easily obtain $\eta(\phi) = \infty$. This completes the proof. □

Lemma 2.6. *Under the assumptions of Lemma 2.5, there exists a positive $\lambda > 0$ such that*

$$|x_t(t_0, \phi) - x_t(t_0, \varphi)| \leq K_{\phi\varphi}e^{-\lambda t} \quad \textit{for} \ \ \phi, \varphi \in C, \quad \xi \leq \phi, \varphi \leq \tfrac{1}{q^+} \ \textit{and} \ \ t \geq t_0, \tag{2.8}$$

where $K_{\phi\varphi} = e^{\lambda t_0}(\max\limits_{\theta \in [-\tau, 0]} |\phi(\theta) - \varphi(\theta)| + 1)$.

Proof. Considering $\Gamma : [0,1] \to R$ defined by

$$\Gamma(u) = -[\alpha(t) - u] + q^+ \sum_{i=1}^{m} p_i(t) e^{u\tau_i} \text{ for } u \in [0,1].$$

Obviously, Γ is continuous. Note that $\Gamma(0) = -\alpha(t) + q^+ \sum_{i=1}^{m} p_i(t) < 0$, which follows from (2.5) and the continuity and periodicity of $\alpha(t)$ and $p_i(t), i = 1, 2, \cdots$. Then there exists $\eta > 0$ and $1 > \lambda > 0$ such that

$$\Gamma(\lambda) = -[\alpha(t) - \lambda] + q^+ \sum_{i=1}^{m} p_i(t) e^{\lambda\tau_i} < -\eta < 0 \text{ for } t \in \mathbb{R}. \tag{2.9}$$

Denote $x_t(t_0, \phi)$ and $x_t(t_0, \varphi)$ by $x(t)$ and $\overline{x}(t)$ respectively for simplicity. By lemma 2.5, we have

$$\xi < x(t), \overline{x}(t) < \frac{1}{q^+} \text{ for all } t \in [t_0 - \tau, \infty).$$

Let $y(t) = x(t) - \overline{x}(t)$ for $t \in [t_0 - \tau, \infty)$. Then, for $t \geq t_0$,

$$y'(t) = -\alpha(t) y(t) + \sum_{i=1}^{m} p_i(t) [e^{-q_i(t) x(t-\tau_i)} - e^{-q_i(t)\overline{x}(t-\tau_i)}]. \tag{2.10}$$

Considering the Lyapunov functional

$$V(t) = |y(t)| e^{\lambda t} \text{ for } t \geq t_0.$$

We claim that $V(t) \leq K_{\phi\varphi}$ for $t \geq t_0$. By way of contradiction, there exists $t_1 > t_0$ such that

$$V(t_1) = K_{\phi\varphi} \text{ and } V(t) < K_{\phi\varphi} \text{ for all } t \in [t_0 - \tau, t_1). \tag{2.11}$$

Then calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of equation (2.10), and in view of (2.9) and (2.11), we have

$$\begin{aligned} 0 \leq & D^+(V(t_1)) \\ \leq & -\alpha(t_1)|y(t_1)| e^{\lambda t_1} + \sum_{i=1}^{m} p_i(t_1) |e^{-q_i(t_1) x(t_1-\tau_i)} - e^{-q_i(t_1)\overline{x}(t_1-\tau_i)}| e^{\lambda t_1} \\ & + \lambda |y(t_1)| e^{\lambda t_1} \\ \leq & [\lambda - \alpha(t_1)] K_{\phi\varphi} + \sum_{i=1}^{m} p_i(t_1) q^+ |y(t_1 - \tau_i)| e^{\lambda(t_1-\tau_i)} e^{\lambda\tau_i} \end{aligned}$$

$$\leq [\lambda - \alpha(t_1) + \sum_{i=1}^{m} p_i(t_1) q^+ e^{\lambda \tau_i}] K_{\phi\varphi} \leq -\eta K_{\phi\varphi} < 0.$$

which is a contradiction. Here we have used the equality $|e^{-rs} - e^{-rt}| \leq re^{-r\zeta}|s-t| \leq r|s-t|$ for $r, s, t, \zeta \in (0, \infty)$, which can be obtained by the mean value theorem. This proves $V(t) \leq K_{\phi\varphi}$ for $t \geq t_0$, which immediately implies (2.8). The proof of Lemma 2.6 is complete. □

3. GLOBAL EXPONENTIAL STABILITY OF POSITIVE PERIODIC SOLUTIONS

In the following, we discuss the existence, uniqueness and exponential stability of positive $\omega-$periodic solutions of equation (2.3).

Theorem 3.1. *Under the assumptions of Lemma 2.5, equation (2.3) has exactly one $\omega-$periodic solution $x^*(t)$.*

Proof. Let $x(t) = x_t(t_0, \phi)$ be a solution of equation (2.3) with initial conditions $\phi : \xi < \phi(\theta) < \frac{1}{q^+}$ for all $\theta \in [-\tau, 0]$. By Lemma 2.5, the solution $x(t)$ satisfies that $\xi < x(t) < \frac{1}{q^+}$ for all $t \in [t_0 - \tau, \infty]$. By the periodicity of coefficients, one can easily see that, for any $h \in \mathbb{N}$, $x(\cdot + h\omega; t_0, \phi)$ is also a solution of equation (2.3) on $[t_0 - \tau - h\omega, \infty)$. Denote $\psi = x_{t_0+\omega}(t_0, \phi)$. From (2.8), for $h \in \mathbb{N}$ and $t + h\omega \geq t_0$, we have

$$\begin{aligned} & |x(t + (h+1)\omega; t_0, \phi) - x(t + h\omega; t_0, \phi)| \\ = \; & |x(t + h\omega; t_0, \psi) - x(t + h\omega; t_0, \phi)| \leq K_{\phi\psi} e^{-\lambda(t+h\omega)}. \end{aligned}$$

where $K_{\phi\psi} = e^{\lambda t_0}(\max_{\theta \in [-\tau, 0]} |\phi(\theta) - \psi(\theta)| + 1)$.

Now, we show that $x(t + h\omega; t_0, \phi)$ is convergent on any compact interval as $h \to \infty$. Let $[a, b] \subset \mathbb{R}$ be an arbitrary subset of $\mathbb{R}$. Choose a $h_0 \in \mathbb{N}$ such that $t_0 + h\omega \geq t_0$ for $t \in [a, b]$. Then for $t \in [a, b]$ and $h > h_0$, we have

$$x(t + h\omega) = x(t + h_0\omega) + \sum_{j=h_0}^{h-1} [x(t + (j+1)\omega) - x(t + j\omega)].$$

It follows Weierstrass principle that $x(t + h\omega)$ converges uniformly to a continuous function, say $x^*(t)$ on $[a, b]$. Because of arbitrariness of $[a, b]$, we have

$x(t+h\omega) \to x^*(t)$ as $h \to \infty$ for $t \in \mathbb{R}$. Moreover, $0 < \xi < x^*(t) < \frac{1}{q^+}$ for all $t \in \mathbb{R}$.

Now we are in the position of prove that $x^*(t)$ is a positive $\omega-$periodic solutions of equation (2.3). It is easily known that $x^*(t)$ is periodic since

$$x^*(t+\omega) = \lim_{h\to\infty} x((t+\omega)+h\omega) = \lim_{h+1\to\infty} x(t+(h+1)\omega) = x^*(t), \text{ for all } t \in \mathbb{R}.$$

In addition, we will prove that $x^*(t)$ is the solution of equation (2.3). Note that $x(t+h\omega)$ is a solution of equation (2.3), that is

$$x(t+h\omega) - x(t_0+h\omega) = \int_{t_0}^{t} -[\alpha(s)x(s+h\omega) + \sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(s-\tau_i)}]ds, \quad (3.1)$$

for $t \geq t_0$. Letting $h \to \infty$ on both side of (3.1), we have

$$x^*(t) - x^*(t_0) = \int_{t_0}^{t} -[\alpha(s)x^*(s) + \sum_{i=1}^{m} p_i(s)e^{-q_i(s)x^*(s-\tau_i)}]ds \quad (3.2)$$

for $t \geq t_0$. From (3.2), it is easy to know that $x^*(t)$ is a solution of equation (2.3). The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Under the assumptions of Lemma 2.5, equation (2.3) has exactly one $\omega-$periodic solution which is globally exponentially stable.*

Proof. By Theorem 3.1, equation (2.3) has a positive $\omega-$periodic solution say $x^*(t)$. It suffices to show that $x^*(t)$ is global exponential stable. Choose $\phi \in C^+([-\tau, 0), (0, \infty)$. For simplicity, denote $x(t, t_0, \phi)$ by $x(t)$. It follows from Theorem 5.2.1 in [19] that $x_t \in C+$ for all $t \in [t_0, \eta(\phi))$. In fact, since $x(t_0) = \phi(0) > 0$, one can easily deduce that

$$x(t) = x(t_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + \int_{t_0}^{t}\sum_{i=1}^{m} p_i(t)e^{-\int_{s}^{t}\alpha(v)dv}e^{-q_i(s)x(s-\tau_i)}ds > 0$$

for all $t \in [t_0, \eta(\phi))$.

We first show that there exists a $t_1 \in [t_0, \eta(\phi))$ such that $x(t_1) < \frac{1}{q^+}$. By way of contradiction, assume that $x(t) \geq \frac{1}{q^+}$ for all $t \in [t_0, \eta(\phi))$. This , together with (2.5), implies that

$$x'(t) = -\alpha(t)x(t) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)x(t-\tau_i)} \leq -\alpha(t)\frac{1}{q^+} + \sum_{i=1}^{m} p_i(t) < 0 \quad \text{for all} \quad t \in [t_0, \eta(\phi)).$$

Therefore, $x(t)$ is bounded and monotone decreasing on $[t_0, \eta(\phi))$. By Theorem 2.3.1 in [18], we easily obtain $\eta(\phi) = \infty$. Then

$$\begin{aligned} x(t) &\leq x(t_0) + \int_{t_0}^{t} [-\alpha(s)\frac{1}{q^+} + \sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(s-\tau_i)}]ds \\ &\leq x(t_0) + \max_{t\in R}\{-\alpha(s)\frac{1}{q^+} + \sum_{i=1}^{m} p_i(t)\}(t - t_0) \\ &\to -\infty \quad as \quad t \to \infty, \end{aligned}$$

which is a contradiction with $x(t) > 0$ for all $t \in [t_0, \eta(\phi))$. This proves that $x(t_1) < \frac{1}{q^+}$ for some $t_1 \in [t_0, \eta(\phi))$. By applying similar proof in Lemma 2.5, we get $x(t) < \frac{1}{q^+}$ for all $t \in [t_1, \eta(\phi))$ and $\eta(\phi)) = \infty$.

We next show that $\liminf_{t\to\infty} x(t) = l > 0$. By way of contradiction again, we assume that $l = 0$. For each $t \geq t_0$, we define $\delta(t) = max\{\zeta | \zeta \leq t, x(\zeta) = \min_{t_0\leq s\leq t} x(t)\}$. It follows from $l = 0$ that $\delta(t) \to \infty$ as $t \to \infty$ and $\lim_{t\to\infty} x(\delta(t)) = 0$. From the definition of $\delta(t)$, we know that $x'(\delta(t)) \leq 0$ or

$$\alpha(\delta(t))x(\delta(t)) \geq \sum_{i=1}^{m} p_i(\delta(t))e^{-q_i(\delta(t))x(\delta(t)-\tau_i)}. \tag{3.3}$$

Choosing a sequence t_n such that $t_n \to \infty$. In view of (3.3), we obtain

$$x(\delta(t_n)) \geq \sum_{i=1}^{m} \frac{p_i(\delta(t_n))}{\alpha(\delta(t_n))} e^{-q_i(\delta(t_n))x(\delta(t_n)-\tau_i)}. \tag{3.4}$$

Taking limits $n \to \infty$ on both side of (3.4), we obtain $\sum_{i=1}^{m} \frac{p_i(t)}{\alpha(t)} \leq 0$, which obviously contradictions with the condition $p_i, \alpha \in (0, \infty)$. Therefore, we have proved that $l > 0$.

In addition, we will prove that $l > \xi$. By way of contradiction, we assume that $l \leq \xi$. By the fluctuation Lemma A.1.[20], there exits a sequence t_k such that $t_k \to \infty, x(t_k) \to l, x'(t_k) \to 0$ as $k \to \infty$. Since $x_{t_k}(t_0, \phi)$ is bounded and equicontinuous, by the Ascoli-Arzelá theorem, there exists a subsequence, still denoted by itself, such that

$$x(t_k) \to \phi^*, \quad \phi^* \in C, \quad \phi(\theta) \geq 0 \text{ for } \theta \in [-\tau, 0],$$

Moreover,

$$\phi^*(0) = l \leq \phi^*(\theta) \leq \frac{1}{q^+} \quad \text{for } \theta \in [-\tau, 0],$$

without loss generality, we assume that all $\alpha(t_k), p_i(t_k)$ and $q_i(t_k)$ are converge to α^*, p_i^* and q_i^* respectively, which can be obtained by periodicity. Note that

$$x'(t_k) = -\alpha(t_k)x(t_k) + \sum_{i=1}^{m} p_i(t_k)e^{-q_i(t_k)x(t_k-\tau_i)},$$

taking limits, in view of the assumption $l \leq \xi$ and (2.5), we have

$$\begin{aligned} 0 &= -\alpha^* l + \sum_{i=1}^{m} p_i^* e^{-q_i^* l} \geq -\alpha^* \xi + \sum_{i=1}^{m} p_i^* e^{-q_i^* \xi} \geq -\alpha^* \xi + \sum_{i=1}^{m} p_i^* e^{-q^+ \cdot \frac{1}{q^+}} \\ &= -\alpha^* \xi + \sum_{i=1}^{m} p_i^* e^{-1} > 0. \end{aligned}$$

Which is a contradiction. This proves that $l > \xi$. □

Remark 3.3. In Theorem 3.1 and Theorem 3.2, the conditions that ensure the existence, uniqueness and globally exponential stability of the positive $\omega-$periodic solution for delayed Lasota-Wazewska model with and without impulsive effects are simple and easily to test, which is less conservative than the conditions that are required in some previous works, for example [11] and [12].

4. AN EXAMPLE AND ITS SIMULATION

In this section, a numerical simulation is performed to verify the feasibility and effectiveness of the above criteria.

Example 4.1. Consider the following impulsive Lasota-Wazewska model with delays

$$\begin{cases} y'(t) = -(11 + \sin^2 \pi t)y(t) + (\frac{5}{8} + \frac{1}{2}|\sin \pi t|) \times e^{-(\frac{3}{2}+\frac{1}{2}|\sin \pi t|)y(t-1)} \\ +(\frac{5}{8} - \frac{1}{2}|\cos \pi t|)e^{-(\frac{3}{2}-\frac{1}{2}|\cos \pi t|)y(t-1)}, \\ y(t_k^+) = (1 + b_k)y(t_k), \quad k = 1, 2, \cdots, \end{cases} \tag{4.1}$$

with initial condition

$$y(s) = \ln(3+t)) = \varphi(t), \ t \in [-1, 0]. \tag{4.2}$$

where $b_k = 2^{\sin\frac{\pi}{2}k} - 1$, and $t_k = k, k = 1, 2, \cdots$.

Let $f(t) = \prod\limits_{0<t_k<t} (1+b_k) = \prod\limits_{0<t_k<t} 2^{\sin\frac{\pi}{2}k}$, then

$$\begin{aligned} f(t+4) &= \prod_{0<t_k<t+4} 2^{\sin\frac{\pi}{2}k} = \prod_{0<t_k\leq 4} 2^{\sin\frac{\pi}{2}k} \cdot \prod_{4<t_k<t+4} 2^{\sin\frac{\pi}{2}k} \\ &= 2^{\sum\limits_{k=1}^{4} \sin\frac{\pi}{2}k} \cdot \prod_{0<t_k<t} 2^{\sin\frac{\pi}{2}(k-4)} = 2^0 \cdot \prod_{0<t_k<t} 2^{\sin\frac{\pi}{2}k} = f(t). \end{aligned}$$

which implies that $f(t)$ is periodic function with periodic 4.

Since

$$\alpha(t) = 11 + \sin^2 \pi t, \quad \beta_1(t) = \frac{5}{8} + \frac{1}{2}|\sin \pi t|,$$

$$\beta_2(t) = \frac{5}{8} - \frac{1}{2}|\cos \pi t|, \quad \gamma_1(t) = \frac{3}{2} + \frac{1}{2}|\sin \pi t|, \quad \gamma_2(t) = \frac{3}{2} - \frac{1}{2}|\cos \pi t|,$$

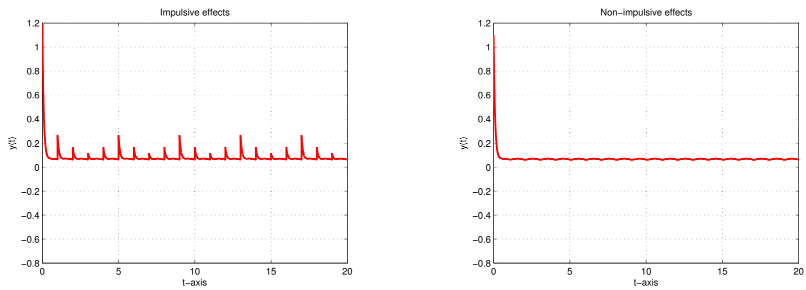
it is obvious that

$$\begin{aligned} p_1(t) &= \prod_{t-1\leq t_k<t} 2^{\sin\frac{\pi}{2}k}(\frac{5}{8} + \frac{1}{2}|\sin \pi t|), \\ p_2(t) &= \prod_{t-1\leq t_k<t} 2^{\sin\frac{\pi}{2}k}(\frac{5}{8} - \frac{1}{2}|\cos \pi t|), \\ q_1(t) &= \prod_{0<t_k<t-1} 2^{\sin\frac{\pi}{2}k}(\frac{3}{2} - |\sin \pi t|), \\ q_2(t) &= \prod_{0<t_k<t-1} 2^{\sin\frac{\pi}{2}k}(\frac{3}{2} - |\sin \pi t|). \end{aligned}$$

Therefore,we have

$$\xi = \frac{1}{96e} \leq \frac{\sum\limits_{i=1}^{2} p_i(t)}{e\alpha(t)} \leq \frac{\sum\limits_{i=1}^{2} p_i(t)}{\alpha(t)} \leq \frac{5}{22} < \frac{1}{3} = \frac{1}{q^+}.$$

It follows from Theorem 3.2 that equation (4.1) with initial condition (4.2) has a unique 4-periodic solution $y^*(t)$, which is globally exponentially stable with convergent rate $\lambda \approx 0.5$ (it is sufficient to solve inequality $-(11 + sin^2\pi t - \lambda) + 3(p_1(t) + p_2(t))e^{\lambda} \leq 0$) The numerical simulation in Fig.1.



(a) (b)

Figure 1: (a)Time-series of the y of system (4.1) with impulsive effects. (b)Time-series of the y of system (4.1) without impulsive effects.

Remark 4.2. System (4.1) is a simple form of impulsive Lasota-Wazewska model with delays. Since $q_1^- = q_2^- = \frac{5}{4} > 1$, it is clear that the condition of in [11] and [12] are not satisfied. Therefore, all the results obtained in [11,12] and the references therein cannot be applicable to system (4.1). This implies that the results of this paper are essentially new.

5. CONCLUSION

In this paper, a class of impulsive Lasota-Wazewska model with delays are investigated. First, the equivalent relation between the solution (or positive periodic solution) of the impulsive Lasota-Wazewska model with delays and that of a corresponding nonimpulsive Lasota-Wazewska model with delays is established. Then, by applying cone fixed point theorem, some criteria are established for existence, uniqueness and globally exponential stability of positive periodic solution of the given systems. Our results imply that under the appropriate linear periodic impulsive perturbations, the impulsive Lasota-Wazewska model with delays preserve the original periodic property of the nonimpulsive Lasota-Wazewska model with delays. Finally, an example and its simulation are given to illustrate the main results. It is worthwhile to note that there are only very few results on impulsive Lasota-Wazewska model with delays, and our results extend and improve greatly some earlier works reported in the literature. Furthermore, our results are important in applications of periodic oscillatory delayed Lasota-Wazewska model with impulsive control.

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